COMPLEXITY OF SHORT RECTANGLES AND PERIODICITY

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Abstract. The Morse-Hedlund Theorem states that a bi-infinite sequence η in a finite alphabet is periodic if and only if there exists $n \in \mathbb{N}$ such that the block complexity function $P_\eta(n)$ satisfies $P_\eta(n) \leq n$. In dimension two, Nivat conjectured that if there exist $n, k \in \mathbb{N}$ such that the $n \times k$ rectangular complexity $P_\eta(n,k)$ satisfies $P_\eta(n,k) \leq nk$, then η is periodic. Sander and Tijdeman showed that this holds for $k \leq 2$. We generalize their result, showing that Nivat's Conjecture holds for $k \leq 3$. The method involves translating the combinatorial problem to a question about the nonexpansive subspaces of a certain $\mathbb{Z}^2$ dynamical system, and then analyzing the resulting system.

1. Nivat's Conjecture for colorings of height 3

1.1. **Background and statement of the theorem.** The Morse-Hedlund Theorem [8] gives a classic relation between the periodicity of a bi-infinite sequence taking values in a finite alphabet $\mathcal{A}$ and the complexity of the sequence. For higher dimensional sequences $\eta = \big(\eta(\vec{n})\colon \vec{n} \in \mathbb{Z}^d\big)$ with $d \geq 1$ taking values in the finite alphabet $\mathcal{A}$, a possible generalization is the Nivat Conjecture [9]. To state this precisely, we define $\eta\colon \mathbb{Z}^d \to \mathcal{A}$ to be *periodic* if there exists $\vec{m} \in \mathbb{Z}^d$ with $\vec{m} \neq \vec{0}$ such that $\eta(\vec{n} + \vec{m}) = \eta(\vec{n})$ for all $\vec{n} \in \mathbb{Z}^d$ and define the *rectangular complexity* $P_\eta(n_1, \dots, n_d)$ to be the number of distinct $n_1 \times \dots \times n_d$ rectangular colorings that occur in η. Nivat conjectured that for $d = 2$, if there exist $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \leq nk$, then η is periodic. This is a two dimensional phenomenon, as counterexamples for the corresponding statement in dimension $d \geq 3$ were given in [11]. There are numerous partial results, including for example [11, 6, 10] (see also related results in [2, 3, 5]). In [4] we showed that under the stronger hypothesis that there exist $n, k \in \mathbb{N}$ such that $P_\eta(n,k) \leq nk/2$, then η is periodic.

We prove that Nivat's Conjecture holds for rectangular colorings of height at most 3:

Theorem 1.1. *Suppose $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, where $\mathcal{A}$ denotes a finite alphabet. Assume that there exists $n \in \mathbb{N}$ such that $P_\eta(n,3) \leq 3n$. Then η is periodic.*

If there exists $n \in \mathbb{N}$ such that $P_\eta(n,1) \leq n$, periodicity of η follows quickly from the Morse-Hedlund Theorem [8]: each row is horizontally periodic of period at most n and so $n!$ is an upper bound for the minimal horizontal period of η. When there exists $n \in \mathbb{N}$ such that $P_\eta(n,2) \leq 2n$, periodicity of η was established by Sander and Tijdeman [12]. The extension to colorings of height 3 is the main result of this

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article. By the obvious symmetry, the analogous result holds if there exists $n \in \mathbb{N}$ such that $P_\eta(3, n) \leq 3n$.

1.2. **Generalized complexity functions.** To study rectangular complexity, we need to consider the complexity of more general shapes. As introduced by Sander and Tijdeman [11], if $\mathcal{S} \subset \mathbb{Z}^2$ is a finite set, we define $P_\eta(\mathcal{S})$ to be the number of distinct colorings in η that can fill the shape $\mathcal{S}$. For example, $P_\eta(n, k) = P_\eta(R_{n,k})$, where $R_{n,k} = \{(x, y) \in \mathbb{Z}^2 \colon 0 \leq x < n, 0 \leq y < k\}$. Similar to methods introduced in [4], we find subsets of $R_{n,3}$ (the *generating sets*) that can be used to study periodicity. Using the restrictive geometry imposed by colorings of height 3, we derive stronger properties that allow us to prove periodicity only using the complexity bound $3n$, rather than $3n/2$ as relied upon in [4].

1.3. **Translation to dynamics.** As in [4], we translate the problem to a dynamical one. We define a dynamical system associated with $\eta \colon \mathbb{Z}^2 \to \mathcal{A}$ in a standard way: endow $\mathcal{A}$ with the discrete topology, $X = \mathcal{A}^{\mathbb{Z}^2}$ with the product topology, and define the $\mathbb{Z}^2$-action by translations on X by $(T^{\vec{u}}\eta)(\vec{x}) := \eta(\vec{x}+\vec{u})$ for $\vec{u} \in \mathbb{Z}^2$. With respect to this topology, the maps $T^{\vec{u}} \colon X \to X$ are continuous. Let $\mathcal{O}(\eta) := \{T^{\vec{u}}\eta \colon \vec{u} \in \mathbb{Z}^2\}$ denote the $\mathbb{Z}^2$-orbit of $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ and set $X_\eta := \overline{\mathcal{O}(\eta)}$. When we refer to the dynamical system X_η, we implicitly assume that this means the space X_η endowed with the $\mathbb{Z}^2$-action by the translations $T^{\vec{u}}$, where $\vec{u} \in \mathbb{Z}^2$. Note that in general $\overline{\mathcal{O}(\eta)} \setminus \mathcal{O}(\eta)$ is nonempty.

The dynamical system X_η reflects the properties of η. An often used fact is that if $F \subset \mathbb{Z}^2$ is finite and $f \in X_\eta$, then there exists $\vec{u} \in \mathbb{Z}^2$ such that $(T^{\vec{u}}\eta)\restriction_F = f\restriction_F$, where by $\cdot\restriction_F$ we mean the restriction to the region F. So, for example, if η satisfies some complexity bound, such as the existence of a finite set $\mathcal{S} \subset \mathbb{Z}^2$ satisfying $P_\eta(\mathcal{S}) \leq N$ for some $N \geq 1$, then every $f \in X_\eta$ satisfies the same complexity bound. Moreover, if η is periodic with some period vector, then every $f \in X_\eta$ is also periodic with the same period vector. Similarly, if $\vec{u} \in \mathbb{Z}^2$ and $F \subset \mathbb{Z}^2$, there is a natural correspondence between a coloring of the form $(T^{-\vec{u}}f)\restriction_F$ and a coloring $f\restriction_{F+\vec{u}}$.

Characterizing periodicity of $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ amounts to studying properties of its orbit closure X_η. In particular, note that η is doubly periodic if and only if it has two non-commensurate period vectors, or equivalently X_η is finite.

1.4. **Expansive and nonexpansive lines.** Restricting a more general definition given by Boyle and Lind [1] to a dynamical system X with a continuous $\mathbb{Z}^2$-action $(T^{\vec{u}} \colon \vec{u} \in \mathbb{R}^2)$ on X, we say that a line $\ell \subset \mathbb{R}^2$ is an *expansive line* if there exist $r > 0$ and $\delta > 0$ such that whenever $f, g \in X$ satisfy $d(T^{\vec{u}}f, T^{\vec{u}}g) < \delta$ for all $\vec{u} \in \mathbb{Z}^2$ with $d(\vec{u}, \ell) < r$, then $f = g$. Any line that is not expansive is called a *nonexpansive line*.

For the system $X = \mathcal{A}^{\mathbb{Z}^2}$ with the continuous $\mathbb{Z}^2$-action on X by translation (sometimes called the *full $\mathcal{A}$-shift*), it is easy to see that there are no expansive lines. However, more interesting behavior arises when we restrict to X_η.

Boyle and Lind [1] proved a general theorem that nonexpansive lines (and, more generally, subspaces) are abundant. In the context of X_η with the continuous $\mathbb{Z}^2$-action on X_η by translation, this theorem implies that for infinite X_η, there exists at least one nonexpansive line. Rephrased in our context the Boyle and Lind result becomes:

Theorem 1.2 (Boyle and Lind [1]). *For $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, η is doubly periodic if and only if there are no nonexpansive lines for the $\mathbb{Z}^2$-action by translation on X_η.*

In [4], we further characterized the situation with a single nonexpansive line:

Theorem 1.3 (Cyr and Kra [4]). *Let $\eta \in \mathcal{A}^{\mathbb{Z}^2}$. If there exists $R_{n,k}$ such that $P_\eta(R_{n,k}) \leq nk$ and there is a unique nonexpansive line for the $\mathbb{Z}^2$-action by translation on X_η, then η is periodic but not doubly periodic.*

Thus Theorem 1.1 follows once we show that there can not be more than a single nonexpansive line, making its proof equivalent to showing:

Theorem 1.4. *If $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ and there exists $R_{n,k}$ such that $P_\eta(R_{n,k}) \leq nk$ for some $k \leq 3$, then there is at most one nonexpansive line for the dynamical system X_η.*

The proof of this result occupies the remainder of the paper.

1.5. **Conventions.** Throughout the paper, we assume that $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, where $\mathcal{A}$ denotes a finite alphabet with $|\mathcal{A}| \geq 2$ and $X_\eta = \overline{O(\eta)}$ denotes the associated dynamical system, endowed with the continuous transformations $T^{\vec{u}}$ for $\vec{u} \in \mathbb{Z}^2$. We do not explicitly mention this hypothesis again. However, each time we make an assumption on the complexity, in particular the existence of $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$, we make this explicit.

2. Generating and balanced sets

2.1. **Generating sets.** We review some definitions from [4], adapted to our current problem.

If $\mathcal{S} \subset \mathbb{R}^2$, we denote the convex hull of $\mathcal{S}$ by $\mathrm{conv}(\mathcal{S})$. We say $\mathcal{S} \subset \mathbb{Z}^2$ is *convex* if $\mathcal{S} = \mathrm{conv}(\mathcal{S}) \cap \mathbb{Z}^2$ and in this case we set $\partial\mathcal{S}$ to be the boundary of $\mathrm{conv}(\mathcal{S})$. A *boundary edge* of $\mathcal{S}$ is an edge of the convex polygon $\mathrm{conv}\mathcal{S}$ and a *boundary vertex* is a vertex of $\mathrm{conv}\mathcal{S}$. We denote the set of boundary edges by $E(\mathcal{S})$ and the set of boundary vertices by $V(\mathcal{S})$. Our convention is that if $conv(\mathcal{S})$ has zero area, then $E(\mathcal{S}) = \emptyset$.

If the area of $conv(\mathcal{S})$ is positive, we orient the boundary of $\mathcal{S}$ positively. We also consider infinite, convex sets $\mathcal{S}$ with an associated interior, and we orient the boundary of $\mathrm{conv}(\mathcal{S})$ such that the interior is on the left. This allows us to refer to a directed line as being parallel to a boundary edge of $\mathcal{S}$. We say two directed lines are *parallel* if the (undirected) lines they determine are parallel and, as directed lines, they have the same orientation. We say they are *antiparallel* if they determine parallel (undirected) lines, but are endowed with opposite orientations.

Definition 2.1. If $\mathcal{S} \subseteq \mathbb{Z}^2$ is convex and if $u, v \in E(\mathcal{S})$ and $u \cap v \neq \emptyset$ then the positive orientation on $\partial\mathcal{S}$ gives an ordering to the set $\{u, v\}$. If v is the larger (with respect to this ordering) of the two edges, we say that v is the *successor edge* to u, and that u is the *predecessor edge* to v. In this case we define $\mathrm{succ}(u) := v$ and $\mathrm{pred}(v) := u$. If $u_1, u_2, \dots, u_n \in E(\mathcal{S})$ are distinct, we say that $\{u_1, u_2, \dots, u_n\}$ is *connected* if $u_1 \cup u_2 \cup \cdots \cup u_n$ is a connected subset of $\mathbb{R}^2$.

If $\mathcal{S} \subset \mathbb{Z}^2$, then $|\mathcal{S}|$ denotes the number of elements of $\mathcal{S}$. We define the *$\mathcal{S}$-words of η* to be

$$\mathcal{W}_\eta(\mathcal{S}) := \left\{(T^{\vec{u}}\eta)\restriction_{\mathcal{S}} \colon \vec{u} \in \mathbb{Z}^2\right\}.$$

Following Sander and Tijdeman [11], we define the *η-complexity* of a set $\mathcal{S} \subset \mathbb{Z}^2$ by

$$P_\eta(\mathcal{S}) := |\mathcal{W}_\eta(\mathcal{S})|.$$

As in [4], we define the *η-discrepancy function* D_η on the set of nonempty, finite subsets of $\mathbb{Z}^2$ by

$$D_\eta(\mathcal{S}) := P_\eta(\mathcal{S}) - |\mathcal{S}|.$$

For $W \subset \mathbb{Z}^2$, by an *η-coloring of W* we mean $(T^{\vec{u}}\eta)\restriction_W$ for some $\vec{u} \in \mathbb{Z}^2$, and when η is clear from the context, we omit it from the terminology and refer to a coloring of W.

Definition 2.2. If $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathbb{Z}^2$ are sets and $\alpha \in X_\eta$, we say that $\alpha\restriction_{\mathcal{S}_1}$ *extends uniquely to an η-coloring of $\mathcal{S}_2$* if for all $\beta \in X_\eta$ such that $\alpha\restriction_{\mathcal{S}_1} = \beta\restriction_{\mathcal{S}_1}$, we have that $\alpha\restriction_{\mathcal{S}_2} = \beta\restriction_{\mathcal{S}_2}$. Otherwise, we say that the coloring $\alpha\restriction_{\mathcal{S}_1}$ *extends non-uniquely to an η-coloring of $\mathcal{S}_2$*.

Definition 2.3. If $\mathcal{S} \subset \mathbb{Z}^2$ is a finite set, then $x \in \mathcal{S}$ is *η-generated* by $\mathcal{S}$ if every η-coloring of $\mathcal{S} \setminus \{x\}$ extends uniquely to an η-coloring of $\mathcal{S}$. A nonempty, finite, convex subset of $\mathbb{Z}^2$ for which every boundary vertex is η-generated is called an *η-generating set*.

We note that if $\mathcal{S}$ is an η-generating set and $\vec{v} \in \mathbb{Z}^2$, then $\mathcal{S} + \vec{v}$ is also an η-generating set. Similarly if $\mathcal{S}$ is an η-generating set and $\alpha \in X_\eta$, then $\mathcal{S}$ is also an α-generating set.

Lemma 2.4. *Suppose $\mathcal{S} \subset \mathbb{Z}^2$ is finite, $|\mathcal{S}| \geq 2$, and $x \in \mathcal{S}$. If x is η-generated by $\mathcal{S}$, then $D_\eta(\mathcal{S} \setminus \{x\}) = D_\eta(\mathcal{S}) + 1$. If x is not η-generated by $\mathcal{S}$, then $D_\eta(\mathcal{S} \setminus \{x\}) \leq D_\eta(\mathcal{S})$.*

Proof. If x is η-generated by $\mathcal{S}$, then $P_\eta(\mathcal{S} \setminus \{x\}) = P_\eta(\mathcal{S})$. Then

$$D_\eta(\mathcal{S} \setminus \{x\}) = P_\eta(\mathcal{S} \setminus \{x\}) - |\mathcal{S}| + 1 = P_\eta(\mathcal{S}) - |\mathcal{S}| + 1 = D_\eta(\mathcal{S}) + 1.$$

If x is not η-generated by $\mathcal{S}$, then $P_\eta(\mathcal{S} \setminus \{x\}) < P_\eta(\mathcal{S})$. Thus

$$D_\eta(\mathcal{S} \setminus \{x\}) = P_\eta(\mathcal{S} \setminus \{x\}) - |\mathcal{S}| + 1 < P_\eta(\mathcal{S}) - |S| + 1 = D_\eta(\mathcal{S}) + 1.$$

Since $D_\eta(\mathcal{S} \setminus \{x\})$ and $D_\eta(\mathcal{S})$ are both integers, $D_\eta(\mathcal{S} \setminus \{x\}) \leq D_\eta(\mathcal{S})$. □

Corollary 2.5. *Suppose $\mathcal{S} \subset \mathbb{Z}^2$ is finite and $p_1, \dots, p_j \in \mathcal{S}$. Then $D_\eta(\mathcal{S} \setminus \{p_1, \dots, p_j\}) \leq D_\eta(\mathcal{S}) + j$.*

2.2. **Nonexpansiveness.** We reformulate the definition of expansive, and more importantly nonexpansive, in the context of a particular configuration η. While this is a priori weaker than Boyle and Lind's definition of expansiveness introduced in Section 1.4, it is easy to check that they are equivalent in the symbolic setting:

Definition 2.6. A line $\ell \subset \mathbb{R}^2$ is a *nonexpansive line for η* (or just a *nonexpansive line* when η is clear from the context) if for all $r \in \mathbb{R}$, there exist $f_r, g_r \in X_\eta$ such that $f_r \neq g_r$, but

$$f_r(\vec{u}) = g_r(\vec{u}) \text{ for all } \vec{u} \in \mathbb{Z}^2 \text{ such that } d(\vec{u}, \ell) < r.$$

We say that ℓ is an *expansive line for η* (or just an *expansive line*) if it is not a nonexpansive line.

If ℓ is a directed line, let $H(\ell) \subset \mathbb{R}^2$ be the half-plane whose (positively oriented) boundary passes through the origin and is parallel to ℓ. We say that a directed line ℓ is a *nonexpansive direction for η* (or just a *nonexpansive direction* when η is clear from the context) if there exist $f, g \in X_\eta$ such that $f \neq g$ but $f{\restriction}H(\ell) = g{\restriction}H(\ell)$. We say ℓ is an *expansive direction for* η (or just an *expansive direction*) if it is not a nonexpansive direction for η.

Remark 2.7. The set of expansive lines (similarly expansive directions, nonexpansive lines, and nonexpansive directions) is invariant under translations in $\mathbb{R}^2$.

More generally, the same definitions apply in an arbitrary subshift, and not just the subshift generated by a single η. We use this to give an example to illustrate the difference between expansive lines and expansive directions:

Example 2.8. Let X be the *Ledrappier* 3*-dot system* [7]:

$$X := \{\eta \in \{0,1\}^{\mathbb{Z}^2} : \eta(x,y) + \eta(x+1,y) + \eta(x,y-1) \equiv 0 \pmod 2\}.$$

Then X is a closed subshift of $\{0,1\}^{\mathbb{Z}^2}$.

Let $\eta \in X$, $a \le b$ be integers, and let

$$S := \{(x,y) \in \mathbb{Z}^2 \colon a \le y \le b\}$$

be a horizontal strip in $\mathbb{Z}^2$. By the definition of X, the restriction of η to S extends uniquely to the half-plane

$$\{(x,y) \in \mathbb{Z}^2 \colon y \le b\}.$$

It does not, however, extend uniquely to all of $\mathbb{Z}^2$, meaning that we cannot recover η from its restriction to the strip S. As this holds for any horizontal strip (see [1, Example 2.8]), the x-axis is a nonexpansive line for X.

Set

$$\begin{aligned} H^- &:= \{(x,y) \in \mathbb{Z}^2 \colon y \le 0\}; \\ H^+ &:= \{(x,y) \in \mathbb{Z}^2 \colon y \ge 0\}. \end{aligned}$$

Taking the orientation on each of H^- and H^+ such that the interior is on the left, the boundary of H^- is a leftward-oriented horizontal line and the boundary of H^+ is a rightward-oriented horizontal line. Fixing $\eta \in X$, we can recover η from its restriction to H^+ (using the rules defining X) but cannot recover η from its restriction to H^-. Therefore the leftward orientation on the x-axis is a non-expansive direction for X, while the rightward orientation on the x-axis is an expansive direction for X.

Expansive (and nonexpansive) lines are closely related to expansive (and nonexpansive) directions; this is clarified in Proposition 2.11.

We summarize properties of generating sets proved in [4] that we use here. As the setting is slightly different, for completeness we include proofs:

Proposition 2.9 ([4], Lemmas 2.3 and 3.3)**.** *Suppose there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \le 3n$. Then there exists an η-generating set $\mathcal{S} \subset R_{n,3}$ with the property that*

$$\text{(1)} \qquad \text{if } \mathcal{S}' \subset \mathcal{S} \text{ is nonempty and convex, then } D_\eta(\mathcal{S}') \ge D_\eta(\mathcal{S}) + 1.$$

Moreover, for any nonexpansive direction ℓ, there is a boundary edge $w_\ell \in E(\mathcal{S})$ that is parallel to ℓ.

(In fact this proposition holds for η such that there exist $n, k \ge 1$ with $P_\eta(n,k) \le nk$, but we do not need this more general result in our setting.)

Proof. By assumption, $D_\eta(R_{n,3}) \leq 0$. Let $\mathcal{S} \subset R_{n,3}$ be a convex set which is minimal (with respect to the partial ordering by inclusion) among all convex subsets of $R_{n,3}$ whose discrepancy is nonpositive. Since $|\mathcal{A}| \geq 2$, the discrepancy of a set with a single element is $|\mathcal{A}| - 1 > 0$, and so $\mathcal{S}$ contains at least two elements. In particular for any $x \in V(\mathcal{S})$, the set $\mathcal{S} \setminus \{x\}$ is nonempty and convex. If $x \in V(\mathcal{S})$ is not η-generated by $\mathcal{S}$, then $D_\eta(\mathcal{S} \setminus \{x\}) \leq D_\eta(\mathcal{S})$ by Lemma 2.4. Therefore, by minimality of $\mathcal{S}$, if $x \in V(\mathcal{S})$ then x is η-generated by $\mathcal{S}$. This establishes that $\mathcal{S}$ is an η-generating set. Claim (1) follows from the minimality of $\mathcal{S}$.

Finally, suppose ℓ is a directed line that is not parallel to any of the edges of $\mathcal{S}$. Without loss of generality, we can assume that ℓ points either southwest or south (all other cases are similar). We claim that ℓ is expansive for η, thereby establishing the second part of the proposition.

Suppose this does not hold. Let $H \subset \mathbb{R}^2$ be a half-plane whose (positively oriented) boundary edge is parallel to ℓ. Let ℓ_0 be the translation of ℓ that passes through $(0,0)$ and for all $t \in \mathbb{R}$, set $\ell_t := \ell_0 + (t, 0)$. Since ℓ is nonexpansive for η, there exist $f, g \in X_\eta$ such that $f \neq g$ but $f\restriction_H = g\restriction_H$. Let $A := \{\vec{u} \in \mathbb{Z}^2 \colon f(\vec{u}) \neq g(\vec{u})\}$ and set

$$t_{\max} := \sup\{t \in \mathbb{R} \colon \ell_t \cap A \neq \emptyset\}.$$

Since $f\restriction_H = g\restriction_H$ and ℓ points southwest or south, we have that $t_{\max} < \infty$. Since ℓ is not parallel to any of the edges of $\mathcal{S}$, there is a vertex $x_\ell \in V(\mathcal{S})$ and a half-plane whose boundary is parallel to ℓ such that $\mathcal{S} \setminus \{x_\ell\}$ is contained in this half-plane but x_ℓ is not. If $\ell_{t_{\max}} \cap A \neq \emptyset$, let $\vec{u}_{\max} \in \ell_{t_{\max}} \cap A$. There is a translation of $\mathcal{S}$ that takes x_ℓ to $\vec{u}_{\max}$ and $\mathcal{S} \setminus \{x_\ell\}$ is translated to the region on which f and g coincide. But this is a contradiction of the fact that $\mathcal{S}$ is η-generating, as x_ℓ is η-generated by $\mathcal{S}$. If instead $\ell_{t_{\max}} \cap A = \emptyset$ let d be the distance from x_ℓ to the half-plane separating x_ℓ from $\mathcal{S} \setminus \{x_\ell\}$. Let $\vec{u} \in A$ be a point such that $d(\vec{u}, \ell_{t_{\max}}) < d/2$. Then there is again a translation of $\mathcal{S}$ taking x_ℓ to $\vec{u}$ and $\mathcal{S} \setminus \{x_\ell\}$ is translated to the region on which f and g coincide. Once again, this is a contradiction of x_ℓ being η-generated. Thus ℓ is an expansive direction for η, completing the proof. $\square$

Corollary 2.10. *Suppose there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$ and $\mathcal{S}$ is the η-generating set constructed in Proposition 2.9. Then for any $w \in E(\mathcal{S})$, we have*

$$D_\eta(\mathcal{S} \setminus w) \geq D_\eta(\mathcal{S}) + 1.$$

Proof. If $E(\mathcal{S}) \neq \emptyset$, then $\mathrm{conv}(\mathcal{S})$ has positive area (recall our convention that if $\mathrm{conv}(\mathcal{S})$ has zero area then the edge set is empty), and so by (1) we are done. $\square$

Proposition 2.11. *Suppose there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. If ℓ is a nonexpansive line for η, then at least one of the orientations on ℓ determines a nonexpansive direction for η. If $\tilde{\ell}$ is an expansive line for η, then both orientations on $\tilde{\ell}$ determine expansive directions for η.*

Proof. Assume ℓ is a nonexpansive line. For each $r \in \mathbb{N}$, let $f_r, g_r \in X_\eta$ be such that $f_r \neq g_r$ but $f_r(\vec{u}) = g_r(\vec{u})$ for all $d(\vec{u}, \ell) < r$. Let

$$R(r) := \sup\{R > 0 \colon f_r(\vec{u}) = g_r(\vec{u}) \text{ for all } d(\vec{u}, \ell) < R\}.$$

Let $\vec{v}_r \in \mathbb{Z}^2$ be such that $f_r(\vec{v}_r) \neq g_r(\vec{v}_r)$ and such that the distance $d(\vec{v}_r, \ell) \leq R(r)+1$. We have $r \leq R(r) \leq d(\vec{v}_r, \ell) \leq R(r)+1 < \infty$. By passing to a subsequence, choose $r_1, r_2, \ldots$ such that $\vec{v}_{r_i}$ all lie in the same connected component of $\mathbb{R}^2 \setminus \ell$. Define $\tilde{f}_{r_i} = f_{r_i} \circ T^{\vec{v}_{r_i}}$ and similarly $\tilde{g}_{r_i} = g_{r_i} \circ T^{\vec{v}_{r_i}}$. Then $\tilde{f}_{r_i}(0,0) \neq \tilde{g}_{r_i}(0,0)$

but $\tilde{f}_{r_i}(\vec{u}) = \tilde{g}_{r_i}(\vec{u})$ for all $\{\vec{u} \in \mathbb{Z}^2 \colon d(\vec{u}, \ell - \vec{v}_{r_i}) < R(r_i)\}$. The distance from the set $\{\vec{u} \in \mathbb{Z}^2 \colon d(\vec{u}, \ell - \vec{v}_{r_i}) < R(r_i)\}$ to the origin is at most 1, because $d(\vec{v}_{r_i}, \ell) \leq R(r_i) + 1$. By compactness of X_η, and passing again to a subsequence (which, by abuse of notation, we continue to call $r_1, r_2, \dots$), we can assume that the colorings $\tilde{f}_{r_i}$ and $\tilde{g}_{r_i}$ both converge. By construction, the limits disagree at $(0,0)$ but agree on the set

$$\bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \{\vec{u} \in \mathbb{Z}^2 \colon d(\vec{u}, \ell - \vec{v}_{r_j}) < R(r_j)\},$$

which is a halfplane (of distance at most 1 from the origin) bordered by a translation of ℓ. This implies that at least one orientation on ℓ makes it into a nonexpansive direction. This establishes the first part of the proposition.

Since half-planes contain arbitrarily wide strips, the second part of the proposition is immediate. □

Corollary 2.12. *Suppose there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. If ℓ is a nonexpansive line for η, then ℓ has rational slope.*

Proof. Let ℓ be a nonexpansive line for η. By Proposition 2.11, at least one of the orientations on ℓ determines a nonexpansive direction for η. By Proposition 2.9, there exists an η-generating set $\mathcal{S} \subset R_{n,3}$ and there is a (positively oriented) edge $w_\ell \in E(\mathcal{S})$ parallel to ℓ. The two endpoints of w_ℓ are both boundary vertices of $\mathcal{S}$, and so in particular are integer points in $R_{n,3}$. It follows that the line determined by ℓ has rational slope. □

Proposition 2.11 shows that if ℓ is a nonexpansive line for η, then there is an orientation on ℓ that determines a nonexpansive direction for η. We do not know, a priori, that both orientations on ℓ determine nonexpansive directions for η. In the sequel, this is a significant hurdle: we put considerable effort into the construction of particular sets (Proposition 2.16) which can be used to show (Proposition 2.19) that when there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$, it is indeed the case that *both* orientations of a nonexpansive line for η determine nonexpansive directions.

Proposition 2.13. *Suppose there exists a finite, convex set $\mathcal{S} \subset \mathbb{Z}^2$ and an edge $w \in E(\mathcal{S})$ such that*

$$D_\eta(\mathcal{S} \setminus w) > D_\eta(\mathcal{S}).$$

Then there are at most $|w \cap \mathcal{S}| - 1$ η-colorings of $\mathcal{S} \setminus w$ that do not extend uniquely to an η-coloring of $\mathcal{S}$.

Proof. Since $|\mathcal{S} \setminus w| = |\mathcal{S}| - |w \cap \mathcal{S}|$,

$$P_\eta(\mathcal{S} \setminus w) - |\mathcal{S}| + |w \cap \mathcal{S}| = D_\eta(\mathcal{S} \setminus w) > D_\eta(\mathcal{S}) = P_\eta(\mathcal{S}) - |\mathcal{S}|.$$

Therefore $P_\eta(\mathcal{S}) \leq P_\eta(\mathcal{S} \setminus w) + |w \cap \mathcal{S}| - 1$. On the other hand, defining $\pi \colon \mathcal{W}_\eta(\mathcal{S}) \to \mathcal{W}_\eta(\mathcal{S} \setminus w)$ to be the natural restriction, the number of η-colorings of $\mathcal{S} \setminus w$ that extend non-uniquely to an η-coloring of $\mathcal{S}$ is the number of points in $\mathcal{W}_\eta(\mathcal{S} \setminus w)$ whose preimage under π contains more than one element. Since π is surjective, this is at most $|\mathcal{W}_\eta(\mathcal{S})| - |\mathcal{W}_\eta(\mathcal{S} \setminus w)|$. In other words, it is at most $P_\eta(\mathcal{S}) - P_\eta(\mathcal{S} \setminus w)$. □

Proposition 2.14. *Suppose there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. If ℓ is a nonexpansive line for η, $\mathcal{S} \subset \mathbb{Z}^2$ is a finite set, and $x \in V(\mathcal{S})$ is η-generated by $\mathcal{S}$, then there is no translation of ℓ that separates x from* $\operatorname{conv}(\mathcal{S} \setminus \{x\})$.

Proof. The argument is a straightforward modification of the proof of (1) in Proposition 2.9. □

2.3. **Balanced sets.** We define the types of sets that are used to show that under the complexity assumption, both orientations of a nonexpansive line for η determine nonexpansive directions:

Definition 2.15. Suppose ℓ is a directed line. A finite, convex set $\mathcal{S} \subset \mathbb{Z}^2$ is *ℓ-balanced* if

(i) There is an edge $w \in E(\mathcal{S})$ parallel to ℓ;
(ii) Both endpoints of w are η-generated by $\mathcal{S}$;
(iii) The set $\mathcal{S}$ satisfies $D_\eta(\mathcal{S} \setminus w) > D_\eta(\mathcal{S})$;
(iv) Every line parallel to ℓ that has nonempty intersection with $\mathcal{S}$ intersects $\mathcal{S}$ in at least $|w \cap \mathcal{S}| - 1$ integer points.

Note that an ℓ-balanced set is not necessarily an η-generating set.

Definition 2.15 is slightly less general than the definition of an ℓ-balanced set used in [4], where an ℓ-balanced set is not necessarily assumed to contain an edge parallel to ℓ (the first condition).

The main result of this section is Proposition 2.18, where we use balanced sets to deduce the periodicity of certain elements of X_η. In [4], we relied on the stronger assumption that $P_\eta(R_{n,k}) \leq \frac{nk}{2}$ to show the existence of balanced sets (as well as other uses related to the existence of generating sets with further properties). Due to the simplified geometry available in rectangles of height 3, we are able to avoid the stronger assumption.

We start by showing the existence of balanced sets:

Proposition 2.16. *Suppose there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$ and suppose that $\ell \subset \mathbb{R}^2$ is a nonexpansive direction for η. If η is aperiodic, then there exists an ℓ-balanced subset.*

Proof. Suppose ℓ is a nonexpansive direction for η. We make some simplifying assumptions. First, if $n = 1$ then by the Morse-Hedlund Theorem [8], η is periodic and so we can assume that $n > 1$. Second, if $P_\eta(R_{n,2}) \leq 2n$, then by Sander and Tijdeman's Theorem [12], η is periodic and so we can assume that $P_\eta(R_{n,2}) > 2n$, meaning that

$$D_\eta(R_{n,3}) \leq 0 < D_\eta(R_{n,2}). \tag{2}$$

Finally, we can assume that $P_\eta(R_{(n-1),3}) > 3n - 3$, meaning that n is chosen to be the minimal integer satisfying $P_\eta(R_{n,3}) \leq 3n$.

We consider three cases depending on the direction of ℓ: vertical, horizontal, and neither vertical nor horizontal.

By Proposition 2.9, there exists an η-generating set $\mathcal{S} \subset R_{n,3}$ and there is an edge $w \in E(\mathcal{S})$ parallel to ℓ. If $|w \cap \mathcal{S}| = 2$, then $\mathcal{S}$ is ℓ-balanced and we are done. Thus it suffices to assume that $|w \cap \mathcal{S}| \geq 3$.

Assume ℓ is vertical. Suppose that ℓ points downward (the case that ℓ points upward is similar). Then since a vertical line cannot intersect a subset of $R_{n,3}$ in more than three places, $|w \cap \mathcal{S}| = 3$. Observe that $(0, 0)$ and $(0, 2)$ are both η-generated by $R_{n,3}$ since $\mathcal{S}$ can be translated into $R_{n,3}$ in such a way that w is translated to the set $\{(0, 0), (0, 1), (0, 2)\}$. In this case $R_{n,3}$ is ℓ-balanced.

Assume ℓ is horizontal. Suppose that ℓ points leftward (the case that ℓ points rightward is similar). For $0 \le a \le b \le n-1$, set

$$\mathcal{S}_{[a,b]} := R_{n,2} \cup \{(x,2)\colon a \le x \le b\}\,.$$

Let $\widetilde{\mathcal{S}}$ be a minimal set of this form (with respect to the partial ordering by inclusion) satisfying $D_\eta(\widetilde{\mathcal{S}}) \le D_\eta(R_{n,3})$; say $\widetilde{\mathcal{S}} = \mathcal{S}_{[a_0,b_0]}$ for some $a_0 \le b_0$. Suppose first that $a_0 = b_0$. If $(a_0,2)$ is η-generated by $\widetilde{\mathcal{S}}$, Proposition 2.14 contradicts the fact that the leftward horizontal is a nonexpansive direction for η. If $(a_0,2)$ is not η-generated by $\widetilde{\mathcal{S}}$, then $D_\eta(R_{n,2}) \le D_\eta(\widetilde{\mathcal{S}}) \le D_\eta(R_{n,3})$; a contradiction of (2). Therefore we can assume $a_0 < b_0$ and $D_\eta(\widetilde{\mathcal{S}}) \le D_\eta(R_{n,3}) < D_\eta(R_{n,2})$. By minimality and Lemma 2.4, the points $(a_0,2)$ and $(b_0,2)$ must both be η-generated by $\widetilde{\mathcal{S}}$. In this case $\widetilde{\mathcal{S}}$ is an ℓ-balanced set.

Assume ℓ is neither vertical nor horizontal. Making a coordinate change of the form $(x,y) \mapsto (\pm x, \pm y)$ if necessary, we can assume that ℓ points southwest. A line parallel to ℓ cannot intersect $R_{n,3}$ in more than three places and so $|w \cap \mathcal{S}| = 3$. Since ℓ is not horizontal, $w \cap \mathcal{S}$ can have at most one integer point at any y-coordinate and thus $w \cap \mathcal{S}$ has exactly one integer point at each of the three y-coordinates in $R_{n,3}$. Therefore there exists an integer $a > 0$ such that $(-a,-1)$ is parallel to ℓ. It follows immediately that $a \le n/2$. Since a translation of any η-generating set is also η-generating, without loss of generality we can assume the bottom-most integer point on w is $(0,0)$.

We claim that any η-coloring of $R_{n,3}$ extends uniquely to an η-coloring of the set $R_{n,3} \cup \{(-1,0),(-2,0),\dots,(-a,0)\}$. Set $T_0 := R_{n,3}$ and for $0 < i \le a$, define

$$T_i := R_{n,3} \cup \{(-1,0),(-2,0),\dots,(-i,0)\}.$$

Then the set $\mathcal{S}-(i,0)$ is contained in T_i and $(\mathcal{S}\setminus\{(0,0)\})-(i,0)$ is contained in T_{i-1}. Since $\mathcal{S}-(i,0)$ is an η-generating set, the color of vertex $(-i,0)$ can be deduced from the coloring of $\mathcal{S} \setminus \{(0,0)\} - (i,0)$. Thus for $0 < i \le a$, every η-coloring of T_{i-1} extends uniquely to an η-coloring of T_i. Inductively, every η-coloring of $R_{n,3}$ extends uniquely to an η-coloring of T_a and the claim follows (see Figure 1).

Therefore, $P_\eta(T_a) = P_\eta(R_{n,3})$ and we obtain

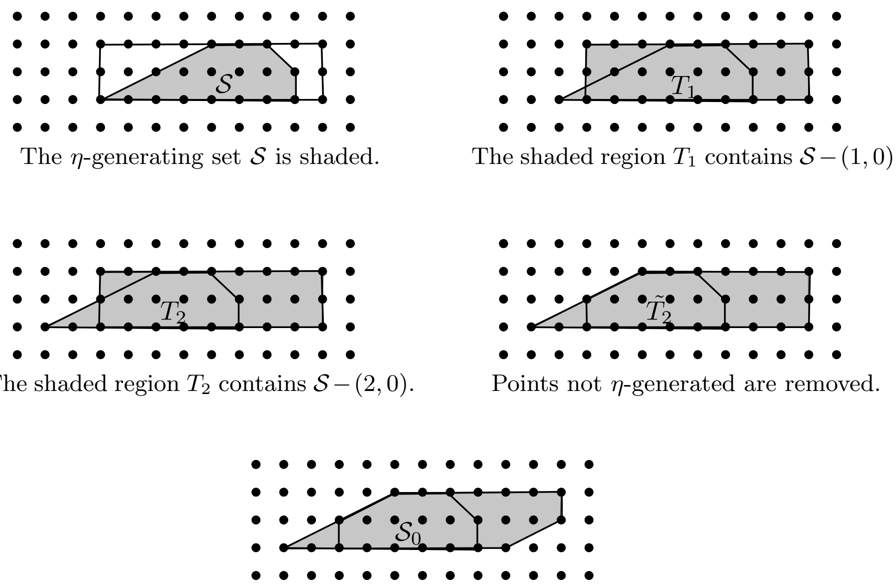
$$D_\eta(T_a) = D_\eta(R_{n,3}) - a \le -a.$$

Observe that any line parallel to ℓ that intersects $\{(0,2),(1,2),\dots,(a-1,2)\}$ must intersect T_a in precisely one integer point. Inductively applying Proposition 2.14, we have that for each $0 \le i < a$, the point $(i,2)$ is not η-generated by the set $T_a \setminus \{(0,2),\dots,(i-1,2)\}$ and so $D_\eta(T_a \setminus \{(0,2),\dots,(i-1,2)\}) \le D_\eta(T_a)$. Setting $\tilde{T}_a := T_a \setminus \{(0,2),(1,2),\dots,(a-1,2)\}$, it follows that $D_\eta(\tilde{T}_a) \le -a$. Define

$$\mathcal{S}_0 := \tilde{T}_a \setminus \{(0,n-a),(0,n-a+1),\dots,(0,n-1)\}.$$

By Corollary 2.5, $D_\eta(\mathcal{S}_0) \le D_\eta(T_a) + a \le 0$. (See Figure 1). Moreover, every line parallel to ℓ that has nonempty intersection with $\mathcal{S}_0$ intersects it in at least two places.

We claim that $\mathcal{S}_0$ contains an ℓ-balanced subset. Let $w_0 \in E(\mathcal{S}_0)$ be the edge of $\mathcal{S}_0$ that is parallel to ℓ, and let ℓ_0 be the translation of ℓ that has nonempty intersection with w_0.



The η-generating set $\mathcal{S}$ is shaded. The shaded region T_1 contains $\mathcal{S}-(1,0)$.

The shaded region T_2 contains $\mathcal{S}-(2,0)$. Points not η-generated are removed.

Rational lines parallel to ℓ intersecting the shaded set $\mathcal{S}_0$ contain at least 2 integer points.

FIGURE 1. Steps in the proof of Proposition 2.16 when ℓ is neither vertical nor horizontal.

For $0 < i \leq n-1$, let $\ell_i := \ell_0 + (i,0)$. Then for all i, $\ell_i \cap \mathcal{S}_0 \neq \emptyset$ and every element of $\mathcal{S}_0$ is contained in exactly one of $\ell_0, \ldots, \ell_{n-1}$. Let

$$U_i := \bigcup_{j=i}^{n-1} \ell_j \cap \mathcal{S}_0$$

and observe that $U_0 = \mathcal{S}_0$. Thus $D_\eta(U_0) \leq 0$. If $D_\eta(U_{n-1}) \leq 0$, then U_{n-1} contains an η-generating set. Since U_{n-1} is a convex subset of a single line, the Morse-Hedlund Theorem [8] implies that η is periodic, a contradiction. Therefore we have that $D_\eta(U_{n-1}) > 0$ and there is a maximal index $0 \leq i_{\max} < n-1$ such that $D_\eta(U_{i_{\max}}) \leq 0$.

Write $\ell_{i_{\max}} \cap \mathcal{S}_0 = \{q_1, q_2, q_3\}$, where q_1 is the bottom-most element and q_3 is the top-most. (Note that we have reduced to the case that $|w \cap \mathcal{S}| = 3$.) If both q_1 and q_3 are η-generated by $U_{i_{\max}}$, then $U_{i_{\max}}$ is ℓ-balanced and we are done (here we are using the fact that every line parallel to ℓ that has nonempty intersection with $\mathcal{S}_0$ intersects it in at least two places). Otherwise, suppose q_3 is not η-generated by $U_{i_{\max}}$ (we argue similarly if q_1 is not η-generated). Set

$$\mathcal{S}_1 := U_{i_{\max}} \setminus \{q_3\}.$$

Since this removes a non-generated vertex from a set of nonpositive discrepancy, it follows that $D_\eta(\mathcal{S}_1) \leq D_\eta(U_{i_{\max}}) \leq 0$. We claim that both q_1 and q_2 are η-generated by $\mathcal{S}_1$. Say, for example, that q_2 is not η-generated by $\mathcal{S}_1$. Then $D_\eta(\mathcal{S}_1 \setminus \{q_2\}) \leq 0$ and q_1 is η-generated by $\mathcal{S}_1 \setminus \{q_2\}$, as otherwise $D_\eta(U_{i_{\max}+1}) \leq D_\eta(\mathcal{S}_1) \leq D_\eta(U_{i_{\max}})$ contradicting maximality of $i_{\max}$. By Proposition 2.14, this contradicts the fact that

ℓ is a nonexpansive direction for η. The same argument holds if q_1 is not η-generated and so we conclude that both q_1 and q_2 are η-generated by $\mathcal{S}_1$. Therefore $\mathcal{S}_1$ is an ℓ-balanced set. □

Definition 2.17. Given a nonexpansive direction ℓ and an ℓ-balanced set $\mathcal{S}^\ell$, define the associated *border* $B_\ell(\mathcal{S}^\ell)$ to be the thinnest strip with edges parallel to ℓ that contains $\mathcal{S}^\ell$. If $w_\ell \in E(\mathcal{S}^\ell)$ is the edge of $\mathcal{S}^\ell$ that is parallel to ℓ, then $B_\ell(S^\ell \setminus w_\ell)$ denotes the thinnest strip with edges parallel to ℓ that contains $S^\ell \setminus w_\ell$.

Note that if η is aperiodic and there exists $n \in \mathbb{N}$ satisfying $P_\eta(R_{n,3}) \leq 3n$, then Proposition 2.16 guarantees the existence of the set $\mathcal{S}^\ell$ and the boundary edge w_ℓ.

Proposition 2.18. *Suppose η is aperiodic, there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$, ℓ is a nonexpansive direction for η, and H is a half-plane whose boundary is parallel to ℓ. Then if $f, g \in X_\eta$ are such that $f \neq g$ but $f\restriction_H = g\restriction_H$, then both f and g are periodic with period vector parallel to ℓ.*

Furthermore, if in addition there exists an ℓ-balanced set $\mathcal{S}^\ell$, $w_\ell \in E(\mathcal{S}^\ell)$ is the edge of $\mathcal{S}^\ell$ parallel to ℓ, and $B_\ell(S^\ell)$ and $B_\ell(S^\ell \setminus w_\ell)$ are the associated borders, then for any $\vec{u} \in \mathbb{Z}^2$:

(i) If the restriction $(T^{\vec{u}}f)\restriction_{B_\ell(\mathcal{S}^\ell \setminus w_\ell)}$ does not extend uniquely to an η-coloring of $B_\ell(\mathcal{S}^\ell)$, then the period of $(T^{\vec{u}}f)\restriction_{B_\ell(\mathcal{S}^\ell \setminus w_\ell)}$ is at most $|w_\ell \cap \mathbb{Z}^2| - 1$;

(ii) If the restriction $(T^{\vec{u}}f)\restriction_{B_\ell(\mathcal{S}^\ell \setminus w_\ell)}$ extends uniquely to an η-coloring of $B_\ell(\mathcal{S}^\ell)$, then the period of $(T^{\vec{u}}f)\restriction_{B_\ell(\mathcal{S}^\ell \setminus w_\ell)}$ is at most $2|w_\ell \cap \mathbb{Z}^2| - 2$.

Proof. We assume that ℓ is a nonexpansive direction and there exists $n \in \mathbb{N}$ with $P_\eta(R_{n,3}) \leq 3n$. Let $\mathcal{S}^\ell$ be an ℓ-balanced set (which exists by Proposition 2.16), $w_\ell \in E(\mathcal{S}^\ell)$ be the edge of $\mathcal{S}^\ell$ parallel to ℓ and let $B_\ell(S^\ell)$ and $B_\ell(S^\ell \setminus w_\ell)$ be the associated borders. By definition, $\mathcal{S}^\ell \setminus w_\ell$ is contained in $B_\ell(\mathcal{S}^\ell \setminus w_\ell)$. Find $A \in SL_2(\mathbb{Z})$ such that $A(\ell)$ points vertically downward and define $\tilde{\eta}\colon \mathbb{Z}^2 \to \mathcal{A}$ by $\tilde{\eta} = \eta \circ A^{-1}$ and $\tilde{\mathcal{S}}^\ell = A(\mathcal{S}^\ell)$. Observe that η is aperiodic if and only if $\tilde{\eta}$ is aperiodic, and that $\tilde{\mathcal{S}}^\ell$ is $A(\ell)$-balanced for $\tilde{\eta}$.

Let $f, g \in X_\eta$ be as in the statement of the proposition. Let $\tilde{f} := f \circ A^{-1}$, $\tilde{g} := g \circ A^{-1}$, and $\tilde{w}_\ell := A(w_\ell)$. It suffices to show that for any $\vec{u} \in \mathbb{Z}^2$, $\tilde{f}, \tilde{g}$ are vertically periodic and that $(T^{\vec{u}}\tilde{f})\restriction_{A(B_\ell(\mathcal{S}^\ell \setminus w_\ell))}$ satisfies the claimed bounds on its period.

The proof proceeds in three steps. First we show that the restriction of f to the strip $B_\ell(\mathcal{S}^\ell \setminus w_\ell)$ is periodic. Next we use this fact to show that f itself is periodic. Finally we use the periodicity of f (with some as yet unknown period) to establish the claimed bounds on the period of $(T^{\vec{u}}f)\restriction_{B_\ell(\mathcal{S}^\ell \setminus w_\ell)}$.

Step 1: Showing $f\restriction_{B_\ell(\mathcal{S}^\ell \setminus w_\ell)}$ is periodic. For $i \in \mathbb{Z}$, let

$$H_i := \{(x, y) \in \mathbb{Z}^2 \colon x \geq i\}.$$

By translating the coordinate system if necessary and using the nonexpansiveness of ℓ, we can assume that $A(H) = H_0$. Furthermore, there exists a translation $(i, 0)$ such that $(T^{-(i,0)}\tilde{f})\restriction_{H_0} = (T^{-(i,0)}\tilde{g})\restriction_{H_0}$, but $(T^{-(i,0)}\tilde{f})\restriction_{H_{-1}} \neq (T^{-(i,0)}\tilde{g})\restriction_{H_{-1}}$. Without loss of generality, we can assume that $i = 0$. Set $B := A(B_\ell(\mathcal{S}^\ell \setminus w_\ell))$ and without loss of generality, assume that $B \subset H_0$ and $B \not\subset H_1$. Choose $L \in \mathbb{N}$ such that

$$B = \left\{(x, y) \in \mathbb{Z}^2 \colon 0 \leq x < L\right\}. \tag{3}$$

For $i \in \mathbb{Z}$, set

$$C_i := \tilde{\mathcal{S}}^\ell + (0, i) \text{ and } D_i := \tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell + (0, i).$$

We claim that for all $i \in \mathbb{Z}$, the $\tilde{\eta}$-coloring $\tilde{f}\restriction_{D_i}$ does not extend uniquely to an $\tilde{\eta}$-coloring of C_i. If, on the other hand, it does extend uniquely, then $\tilde{f}\restriction_B$ extends uniquely to an $\tilde{\eta}$-coloring of $B \cup C_i$ for some $i \in \mathbb{Z}$. Since any translation of an ℓ-balanced set is also ℓ-balanced, the top-most vertex of the edge of C_{i+1} parallel to $A(\ell)$ is $\tilde{\eta}$-generated by C_{i+1}. This is the only element of C_{i+1} that is not contained in $B \cup C_i$, and so $\tilde{f}\restriction_B$ extends uniquely to an η-coloring of $B \cup C_i \cup C_{i+1}$. By induction, $\tilde{f}\restriction_B$ extends uniquely to an η-coloring of $B \cup \bigcup_{j \geq i} C_j$. The bottom-most vertex of the edge of C_i parallel to $A(\ell)$ is also η-generated by C_i, and so a similar induction argument shows that $\tilde{f}\restriction_B$ extends uniquely to an $\tilde{\eta}$-coloring of $B \cup \bigcup_{j \in \mathbb{Z}} C_j$. This contradicts the fact that $\tilde{f}\restriction_{H_0} = \tilde{g}\restriction_{H_0}$ but $\tilde{f}\restriction_{H_{-1}} \neq \tilde{g}\restriction_{H_{-1}}$ and so the claim follows. Equivalently, for all $j \in \mathbb{Z}$, the $\tilde{\eta}$-coloring $(T^{(0,j)}\tilde{f})\restriction_{D_0}$ does not extend uniquely to an $\tilde{\eta}$-coloring of C_0.

By Proposition 2.13, there are at most $|\tilde{w}_\ell \cap \tilde{\mathcal{S}}^\ell| - 1 = |w_\ell \cap \mathcal{S}^\ell| - 1$ many colorings of D_0 that extend non-uniquely to an $\tilde{\eta}$-coloring of C_0. Thus

$$\left|\left\{(T^{(0,i)}\tilde{f})\restriction_{D_0} \colon i \in \mathbb{Z}\right\}\right| \leq |w_\ell \cap \mathcal{S}^\ell| - 1.$$

For each integer $0 \leq x < L$, where L is defined as in (3), let p_x be the bottom-most element of $\tilde{\mathcal{S}}^\ell \cap \{(x, j) \colon j \in \mathbb{Z}\}$. Set

$$V := \{p_x \colon 0 \leq x < L\} \text{ and } U := \bigcup_{y=0}^{|w_\ell \cap \mathcal{S}^\ell| - 2} V + (0, y).$$

Since $\tilde{\mathcal{S}}^\ell$ is $A(\ell)$-balanced, $U \subset D_0$. (See Figure 2). Define $\alpha \colon \mathbb{Z} \to \mathcal{W}_\eta(V)$ by $\alpha(j) := (T^{(0,j)}\tilde{f})\restriction_V$. Patterns of the form $\alpha\restriction\{m, m+1, \ldots, m + |w_\ell \cap \mathcal{S}^\ell| - 2\}$ are in one-to-one correspondence with colorings of the form $(T^{(0,m)}\tilde{f})\restriction_U$. The number of such coloringss is at most the number of coloringss of the form $(T^{(0,m)}\tilde{f})\restriction_{D_0}$, which is at most $|w_\ell \cap \mathcal{S}^\ell| - 1$. By the Morse-Hedlund Theorem [8], α is periodic with period at most $|w_\ell \cap \mathcal{S}^\ell| - 1$. Therefore $\tilde{f}\restriction_B$ is vertically periodic with period at most $|w_\ell \cap \mathcal{S}^\ell| - 1$ as well.

Step 2: Showing f is periodic. For $i \in \mathbb{Z}$, set

$$B_i := B + (i, 0).$$

We claim that for any $i \geq 0$, we have that $\tilde{f}\restriction_{B_{-i}}$ is vertically periodic and the periods satisfy the bounds in the statement of the proposition. For $i = 0$, we have already shown that $\tilde{f}\restriction_{B_0}$ is vertically periodic of period at most $|w_\ell \cap \mathcal{S}^\ell| - 1$. We proceed by induction and suppose that for all $0 \leq i < k$, we have that $\tilde{f}\restriction_{B_{-i}}$ is periodic and

(i) The period of $\tilde{f}\restriction_{B_{-i}}$ is at most $2|w_\ell \cap \mathcal{S}^\ell| - 2$;
(ii) If for all $j \in \mathbb{Z}$, the $\tilde{\eta}$-coloring $(T^{-(-i,j)}\tilde{f})\restriction_{\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell}$ does not extend uniquely to an $\tilde{\eta}$-coloring of $\widetilde{\mathcal{S}}^\ell$, then the period of $\tilde{f}\restriction_{B_{-i}}$ is at most $|w_\ell \cap \mathcal{S}^\ell| - 1$.

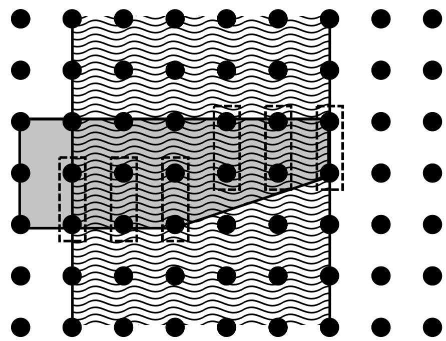


FIGURE 2. The shaded region represents $\mathcal{S}$, the union of the boxes is U, and the union of the bottom most elements of the boxes is the set V. Step 1 of the proof shows that the wavy region B is periodic.

First we show that $\tilde{f}\restriction B_{-k}$ is vertically periodic of period at most $2|w_\ell \cap \mathcal{S}^\ell| - 2$. Suppose there exists $j \in \mathbb{Z}$ such that

$$(T^{-(-k+1,j)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell \text{ extends uniquely to an } \eta\text{-coloring of } \widetilde{\mathcal{S}}^\ell. \tag{4}$$

Let $p \leq 2|w_\ell \cap \mathcal{S}^\ell| - 2$ be the minimal vertical period of $\tilde{f}\restriction B_{-k+1}$. Then for all $m \in \mathbb{Z}$, $(T^{-(-k+1,j+mp)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell$ extends uniquely to an $\tilde{\eta}$-coloring of $\widetilde{\mathcal{S}}^\ell$ and in particular all the colorings of the form $(T^{-(-k+1,j+mp)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell$ coincide. By periodicity of $\tilde{f}\restriction B_{-k+1}$, all of the colorings $(T^{-(-k+1,j+mp+1)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell$ coincide and so all of the colorings $(T^{-(-k+1,j+mp+1)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell$ coincide except possibly on the top-most element of $\tilde{w}_\ell$. Since $\tilde{\mathcal{S}}^\ell$ is $A(\ell)$-balanced, the top-most element of $\tilde{w}_\ell$ is η-generated by $\tilde{S}^\ell$, and so the colorings coincide on the top-most element of $\tilde{w}_\ell$ as well. By induction, for any q with $0 \leq q < p$ and all $m \in \mathbb{Z}$, all colorings of the form $(T^{-(-k+1,j+mp+q)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell$ coincide. This implies that $\tilde{f}\restriction B_{-k}$ is periodic and that its period divides the period of $\tilde{f}\restriction B_{-k+1}$.

Otherwise, if (4) does not hold, we can suppose that for all $j \in \mathbb{Z}$, the coloring $(T^{-(-k+1,j)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell$ does not extend uniquely to an $\tilde{\eta}$-coloring of $\tilde{S}^\ell$. Then by applying the Morse-Hedlund Theorem as in Step 1, the vertical period of $\tilde{f}\restriction B_{-k+1}$ is at most $|w_\ell \cap \mathcal{S}^\ell| - 1$. As above, let $0 < p \leq |w_\ell \cap \mathcal{S}^\ell| - 1$ be the minimal vertical period of $\tilde{f}\restriction B_{-k+1}$. Let $\pi\colon \mathcal{W}_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell) \to \mathcal{W}_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell)$ be the natural restriction map. As in Proposition 2.13, there are at most $P_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell) - P_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell)$ elements of $\mathcal{W}_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell)$ whose pre-image under π contains more than one element; say the number of such elements is Q. There are at most $Q + P_{\tilde{\eta}}(\tilde{S}^\ell) - P_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell)$ elements of $\mathcal{W}_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell)$ where π is not one-to-one. That is, there are at most

$$2(P_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell) - P_{\tilde{\eta}}(\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell)) \leq 2|w_\ell \cap \mathcal{S}^\ell| - 2$$

many η-colorings of $\tilde{\mathcal{S}}^\ell$ whose restrictions to $\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell$ do not extend uniquely to an η-coloring of $\tilde{\mathcal{S}}^\ell$.

Each of the colorings $(T^{-(-k+1,j)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell$ is such a coloring. By the pigeonhole principle, there exist $0 \le j_1 < j_2 < 2|w_\ell \cap \mathcal{S}^\ell| - 2$ such that

$$(T^{-(-k+1,j_1)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell = (T^{-(-k+1,j_2)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell. \tag{5}$$

In particular, this implies that

$$(T^{-(-k+1,j_1)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell = (T^{-(-k+1,j_2)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell \setminus \tilde{w}_\ell. \tag{6}$$

Since $\tilde{\mathcal{S}}^\ell$ is $A(\ell)$-balanced, every vertical line with nonempty intersection with $\tilde{\mathcal{S}}^\ell$ contains at least $|w_\ell \cap \mathcal{S}^\ell| - 1 \ge p$ integer points. Therefore, it follows from (6) that $j_2 - j_1$ is a multiple of p. Using induction as previously, it follows from (5) that we have

$$(T^{-(-k+1,j_1+j)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell = (T^{-(-k+1,j_2+j)}\tilde{f})\restriction\tilde{\mathcal{S}}^\ell$$

for all $j \in \mathbb{Z}$. In particular $\tilde{f}\restriction B_{-k+1} \cup B_{-k}$ is vertically periodic of period $j_2 - j_1 \le 2|w_\ell \cap \mathcal{S}^\ell| - 2$.

By induction, for all $k > 0$ we have that $\tilde{f}\restriction B_{-k}$ is vertically periodic with the bounds claimed in the proposition. We consider two cases, depending if the direction antiparallel to ℓ is nonexpansive or is expansive. If this direction is nonexpansive, let $\mathcal{T}^\ell \subset R_{n,3}$ be a set which is balanced in the direction antiparallel to ℓ (such a set exists by Proposition 2.16). Since the restriction of $\tilde{f}$ to the vertical half-plane $\{(x,y) \in \mathbb{Z}^2 : x \le 0\}$ is periodic, a similar induction argument (using $\mathcal{T}^\ell$ in place of $\mathcal{S}^\ell$) shows that $\tilde{f}$ is vertically periodic on all of $\mathbb{Z}^2$, where the precise bounds on the period are yet to be determined. (*A priori*, these bounds depend on the number of integer points on the edge of $\mathcal{T}^\ell$ that is antiparallel to ℓ.) If the directional antiparallel to ℓ is expansive, then there exist $a, b \in \mathbb{N}$ such that every $\tilde{\eta}$-coloring extends the rectangle $[-a,-1] \times [-b,b]$ uniquely to an $\tilde{\eta}$-coloring of this rectangle union $\{(0,0)\}$. Thus every $\tilde{\eta}$-coloring of the strip $[-a,-1] \times \mathbb{Z}$ extends uniquely to the right. It is easy to check that any vertically periodic coloring of this strip with period p extends uniquely to the right to a periodic coloring, with period dividing p.

Step 3: Showing that the period of f satisfies the claimed bounds. We are left with showing that $\tilde{f}\restriction B_k$ satisfies the claimed bounds for all $k \in \mathbb{Z}$. We remark that the argument showing that $\tilde{f}\restriction B_{-k}$ is vertically periodic with the claimed bounds relied only the fact that $\tilde{f}\restriction B_0$ was vertically periodic of period at most $|w_\ell \cap \mathcal{S}^\ell| - 1$. Thus it suffices to show that for infinitely many $k > 0$, the vertical period of $\tilde{f}\restriction B_k$ is at most $|w_\ell \cap \mathcal{S}^\ell| - 1$, since then the previous argument shows that the half-plane to the left of such a B_k satisfies the claimed bounds. As before, it further suffices to show that for infinitely many $k > 0$, the η-coloring $\tilde{f}\restriction B_k$ does not extend uniquely to an η-coloring of $B_k \cup B_{k-1}$.

Since $\tilde{f}\restriction B_k$ is vertically periodic for all k and there is a global common period (each vertical arises from f), there are only finitely many colorings B_0 that are of the form $(T^{-(k,0)}\tilde{f})\restriction B_0$ for some $k \in \mathbb{Z}$. Say there exists an integer $k_{\min} \ge 0$ such that $(T^{-(k,0)}\tilde{f})\restriction B_0$ extends uniquely to an η-coloring of $B_0 \cup B_{-1}$ for all $k > k_{\min}$ and without loss of generality assume that $k_{\min}$ is the minimal integer with this property. Let $K \geq k_{\min}$ be the smallest integer for which there exists $i \in \mathbb{N}$ such that

$$(T^{-(K+i,0)}\tilde{f})\restriction B_0 = (T^{-(K,0)}\tilde{f})\restriction B_0$$

(K exists by the pigeonhole principle). Then by definition of $k_{\min}$, there is a unique extension of $(T^{-(K+i,0)}\tilde{f})\restriction B_0$ to an $\tilde{\eta}$-coloring of $B_0 \cup B_{-1}$. In particular,

$$(T^{-(K-1,0)}\tilde{f})\restriction B_0 = (T^{-(K+i-1,0)}\tilde{f})\restriction B_0. \tag{7}$$

If $K > k_{\min}$, then (7) contradicts the minimality of K. If $K = k_{\min}$ the fact that $(T^{-(K+i,0)}\tilde{f})\restriction B_0 = (T^{-(K,0)}\tilde{f})\restriction B_0$ extends uniquely to an $\tilde{\eta}$-coloring of $B_0 \cup B_1$ contradicts the definition of $k_{\min}$. Either case leads to a contradiction, and so we conclude that no such integer $k_{\min}$ exists. The bounds on $\tilde{f}\restriction B_k$ claimed in the proposition follow.

The analogous argument applied to g implies the periodicity of g. □

Proposition 2.19. *Assume η is aperiodic and there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. If ℓ is a nonexpansive direction for η, then the direction antiparallel to ℓ is also nonexpansive for η. In particular, if $\mathcal{S}$ is an η-generating set, there is an edge $\widehat{w}_\ell \in E(\mathcal{S})$ antiparallel to ℓ.*

Proof. We proceed by contradiction. Suppose ℓ is nonexpansive but the antiparallel direction $\widehat{\ell}$ is expansive for η. By Corollary 2.12, ℓ is a rational direction. Recall (see Definition 2.6) that since ℓ is a nonexpansive direction for η, there exist $f, g \in X_\eta$ and a half-plane H whose (positively oriented) border is parallel to ℓ such that $f\restriction H = g\restriction H$ but $f \neq g$. Without loss of generality, we assume that the border of H passes through the origin. By Proposition 2.18, f and g are both periodic η-colorings of $\mathbb{Z}^2$ and they both have (nonzero) period vectors parallel to ℓ.

Choose $A \in SL_2(\mathbb{Z})$ such that $A(\ell)$ points vertically downward so that $A(H) = \{(x,y) \in \mathbb{Z}^2 \colon x \geq 0\}$. Note that since $\widehat{\ell}$ is an expansive direction for η, $A(\widehat{\ell})$ is an expansive direction for $(\eta \circ A^{-1})$. The $(\eta \circ A^{-1})$-expansiveness of $A(\widehat{\ell})$ means there exist $a, b \in \mathbb{N}$ such that every $(\eta \circ A^{-1})$-coloring of $[-a+1, 0] \times [-b+1, b-1]$ extends uniquely to an $(\eta \circ A^{-1})$-coloring of the larger set $[-a+1, 0] \times [-b+1, b-1] \cup \{(1,0)\}$. (Otherwise, we can define rectangles $Q_R = [-R+1, 0] \times [-R+1, R-1]$ and for every $R \geq 1$ there exist functions $f_R, g_R \in X_{\eta \circ A^{-1}}$ such that $f_R\restriction Q_R = g_R\restriction Q_R$ and $f_R(1,0) \neq g_R(1,0)$. Passing to a limit we obtain $f_\infty, g_\infty \in X_{\eta \circ A^{-1}}$ that agree on the half plane $\{(x,y) \in \mathbb{Z}^2 \colon x \leq 0\}$ but disagree at $(1,0)$, contradicting expansiveness.)

Then both $\tilde{f} = f \circ A^{-1}$ and $\tilde{g} = g \circ A^{-1}$ are vertically periodic and agree on the vertical half plane $A(H)$. At most one of $\tilde{f}$ and $\tilde{g}$ can be horizontally periodic, so without loss of generality assume that $\tilde{f}$ is not horizontally periodic. Let C be the set of $\tilde{f}$-colorings of the strip $V := [-a+1, 0] \times (-\infty, \infty)$. Vertical periodicity of $\tilde{f}$ guarantees that C is finite. We produce a coloring $\alpha \colon \mathbb{Z} \to C$ by coloring the integer i with the color $(T^{(-i,0)}\tilde{f})\restriction V$. Since every η-coloring of $[-a+1, 0] \times [-b+1, b-1]$ extends uniquely to an η-coloring of $[-a+1, 0] \times [-b+1, b-1] \cup \{(1,0)\}$, we also have that every η-coloring of V extends uniquely to an η-coloring of $V \cup (V + (1,0))$. Therefore for any $i \in \mathbb{Z}$, the α-color of $\{i, i+1, \ldots, i+a-1\}$ uniquely determines the α-color of $i+a$. Therefore α is periodic and hence $\tilde{f}$ is horizontally periodic; a contradiction. Thus $\widehat{\ell}$ must be nonexpansive for η.

By Proposition 2.9, there is an edge $\widehat{w}_\ell \in E(\mathcal{S})$ antiparallel to ℓ. □

Corollary 2.20. *Assume that η is aperiodic and there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. Let $\mathcal{S} \subset R_{n,3}$ be an η-generating set satisfying* (1)*. Then for every nonhorizontal, nonexpansive direction ℓ, $\mathcal{S}$ is ℓ-balanced.*

If ℓ is horizontal and nonexpansive, then $\mathcal{S}$ is either ℓ-balanced or $\widehat{\ell}$-balanced, where $\widehat{\ell}$ is the antiparallel direction.

Proof. Assume that ℓ is a nonhorizontal and nonexpansive direction. We check the four conditions of Definition 2.15. The first condition follows from Proposition 2.9, the second is immediate from the definition of an η-generating set and the third follows since $\mathcal{S}$ satisfies (1). If $|w \cap \mathcal{S}| = 2$, then the fourth condition follows since every line with nonempty intersection with $\mathcal{S}$ intersects in at least one point. If $|w \cap \mathcal{S}| = 3$, then ℓ is either vertical or determines a line with slope of the form $1/a$ for some integer $a > 0$. By Proposition 2.19, there exists $w_{\widehat{\ell}} \in E(\mathcal{S})$ antiparallel to ℓ. Since both endpoints of $w_{\widehat{\ell}}$ are boundary vertices of $\mathcal{S}$, $|w_{\widehat{\ell}} \cap \mathcal{S}| \geq 2$. Therefore any line parallel to ℓ that has nonempty intersection with $\mathcal{S}$, intersects $\mathcal{S}$ in at least two integer points.

If ℓ is horizontal, let n be the smaller of the number of integer points on the top and bottom edges of $\mathcal{S}$. By convexity of $\mathcal{S}$, the middle line has length $r \geq n$ for some $r \in \mathbb{R}$. Thus the middle line contains at least $\lfloor r \rfloor \geq n$ integer points, and so $\mathcal{S}$ is balanced for either ℓ or $\widehat{\ell}$. □

Corollary 2.21. *Assume there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. Suppose ℓ is an oriented rational line in $\mathbb{R}^2$, $\widehat{\ell}$ is the antiparallel line, $\mathcal{S}^\ell$ is an ℓ-balanced set, $\widehat{\mathcal{S}^\ell}$ is an $\widehat{\ell}$-balanced set, $w_\ell \in E(\mathcal{S})$ is the edge parallel to ℓ and $B \subset \mathbb{Z}^2$ is the thinnest bi-infinite strip with edges parallel and antiparallel to ℓ that contains $\mathcal{S}^\ell \setminus w_\ell$. If $\eta{\restriction}_B$ is periodic, then η is periodic with period vector parallel to ℓ.*

Proof. Let $\mathcal{S}^\ell$ be an ℓ-balanced set and let $w_\ell \in E(\mathcal{S})$ be the associated edge and B the associated strip. The argument is nearly identical to the proof of Step 2 of Proposition 2.18 and so we just summarize the differences. Maintaining the notation in that proof, if there exists $i \in \mathbb{Z}$ such that $\tilde{\eta}{\restriction}_{B_i}$ does not extend uniquely to an η-coloring of $B_i \cup B_{i-1}$, then $\tilde{\eta}{\restriction}_{B_i}$ is periodic of period at most $|w_\ell \cap \mathcal{S}^\ell| - 1$ and the remainder of the induction is identical. Otherwise, for every $i \in \mathbb{Z}$, the coloring $\tilde{\eta}{\restriction}_{B_i}$ extends uniquely to an η-coloring of $B_i \cup B_{i-1}$. By the pigeonhole principle and the fact that $\mathcal{S}^\ell$ is ℓ-balanced, as in Step 2 of Proposition 2.18, it follows that whenever $\tilde{\eta}{\restriction}_{B_i}$ is vertically periodic, $\tilde{\eta}{\restriction}_{B_{i-1}}$ is vertically periodic of period dividing that of $\tilde{\eta}{\restriction}_{B_i}$. This establishes the result for the restriction of $\tilde{\eta}$ to $\bigcup_{j=0}^\infty B_{i-j}$. The restriction to the other half-plane follows a similar argument using the antiparallel line $\widehat{\ell}$ and associated $\widehat{\ell}$-balanced set $\widehat{\mathcal{S}}^\ell$ instead of $\mathcal{S}^\ell$. □

Corollary 2.22. *Suppose there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$ and $f \in X_\eta$. Suppose ℓ is a nonexpansive direction for η, $\vec{u} \in \mathbb{Z}^2$ is the shortest integer vector parallel to ℓ, $\mathcal{S}$ is an ℓ-balanced set, and $w \in E(\mathcal{S})$ is the edge parallel to ℓ. Let $B_\ell(\mathcal{S} \setminus w)$ be the intersection of $\mathbb{Z}^2$ with all lines parallel to ℓ that have nonempty intersection with $\mathcal{S} \setminus w$. Finally, suppose there exists $R \in \mathbb{N}$ such that for all $r \geq R$, $(T^{r \cdot \vec{u}} f){\restriction}_{\mathcal{S} \setminus w}$ does not extend uniquely to an η-coloring of $\mathcal{S}$. Then the restriction of f to the semi-infinite strip*

$$\bigcup_{r \geq R} T^{-r \cdot \vec{u}}(\mathcal{S} \setminus w)$$

is eventually periodic with period vector parallel to $\vec{u}$ and period at most $|w \cap \mathcal{S}| - 1$. Moreover, there exists $0 \leq I < |w \cap \mathcal{S}| - 1$ such that the restriction of f to the semi-infinite strip

$$\bigcup_{r \geq R+I} T^{-r \cdot \vec{u}}(\mathcal{S} \setminus w)$$

is periodic.

Proof. The proof is almost identical to Step 1 of Proposition 2.18. Define

$$\alpha \colon \mathbb{N} \to \{(T^{r \cdot \vec{u}} f)\restriction_{\mathcal{S} \setminus w} \colon r \geq R\}$$

by setting $\alpha(i) := (T^{(R+i-1) \cdot \vec{u}} f)\restriction_{\mathcal{S} \setminus w}$. As in Proposition 2.18, we have that the number of colorings of the form $\alpha\restriction_{\{m, m+1, \ldots, m+|w \cap \mathcal{S}|-2\}}$ is at most $|w \cap \mathcal{S}| - 1$. The one-sided version of the Morse-Hedlund Theorem [8] shows that α is eventually periodic with period at most $|w \cap \mathcal{S}| - 1$ and is such that the initial portion has length at most $|w \cap \mathcal{S}| - 1$. □

Corollary 2.23. *Assume there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. Suppose ℓ is an oriented rational line and there exists an ℓ-balanced set $\mathcal{S}^\ell$. Let $w_\ell \in E(\mathcal{S}^\ell)$ be the edge parallel to ℓ and suppose $\mathcal{T} \subset \mathbb{Z}^2$ is an infinite convex set with a semi-infinite edge W parallel to ℓ. Let*

$$U := \left\{\vec{u} \in \mathbb{Z}^2 \colon (\mathcal{S}^\ell \setminus w_\ell) + \vec{u} \subset \mathcal{T} \text{ and } w_\ell + \vec{u} \not\subset \mathcal{T}\right\}.$$

If $\eta\restriction_{(\mathcal{S} \setminus w_\ell) + U}$ is periodic with period vector parallel to ℓ, then $\eta\restriction_{\mathcal{S} + U}$ is periodic with period vector parallel to ℓ. Moreover if for all $\vec{u} \in U$ the coloring $(T^{\vec{u}}\eta)\restriction_{\mathcal{S} \setminus w_\ell}$ does not extend uniquely to an η-coloring of $\mathcal{S}$, then the period of $\eta\restriction_{(\mathcal{S} \setminus w_\ell) + U}$ is at most $|w_\ell \cap \mathcal{S}^\ell| - 1$ and the period of $\eta\restriction_{\mathcal{S} + \vec{u}}$ is at most $2|w_\ell \cap \mathcal{S}^\ell| - 2$. Otherwise the period of $\eta\restriction_{\mathcal{S} + \vec{u}}$ is equal to the period of $\eta\restriction_{(\mathcal{S} \setminus w_\ell) + U}$.

Proof. This follows from the Morse-Hedlund Theorem and the pigeonhole principle, as in Steps 2 and 3 of Proposition 2.18, and in Corollary 2.22. □

3. Complexity with multiple nonexpansive lines

In this section, we show that the complexity assumption of the existence of $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$ is incompatible with the existence of more than one nonexpansive line for η.

We assume throughout this section that:

(H1) X_η has at least two nonexpansive lines.

(H2) There exists $n \in \mathbb{N}$ such that $P_\eta(n, 3) \leq 3n$.

If η is periodic, let $\vec{u} \in \mathbb{Z}^2$ be a period vector and consider any line ℓ that is not parallel to $\vec{u}$. By taking a neighborhood of ℓ wide enough to include $\ell \pm \vec{u}$, we have that ℓ is expansive. Thus every line apart from possibly the direction determined by $\vec{u}$ is expansive, so there is at most one nonexpansive line. Thus Hypothesis (H1) implies that

(8) η is aperiodic.

We begin with some general facts about the shape of an η-generating set. By Proposition 2.9, if $\mathcal{S}$ is an η-generating set, then the boundary $\partial\mathcal{S}$ contains an edge parallel to each nonexpansive direction. By Proposition 2.19, whenever ℓ

is a nonexpansive direction, the direction antiparallel to ℓ is also a nonexpansive direction. Since $\mathcal{S} \subset R_{n,3}$, $\partial\mathcal{S}$ cannot consist of more than six edges (at most two edges are horizontal and the others connect integer points in $R_{n,3}$ with different y-coordinates). Thus there are at most three nonexpansive lines for η, and each orientation on each line determines a nonexpansive direction.

We start with a construction of a large convex set that is used in Propositions 3.4 and 3.7 to show that η cannot have multiple nonexpansive lines while also having low complexity.

As noted, we have at most three nonexpansive lines for η. Let

$$\ell_1, \ell_2 \subset \mathbb{R}^2 \text{ or } \ell_1, \ell_2, \ell_3 \subset \mathbb{R}^2 \text{ denote the nonexpansive lines for } \eta, \tag{9}$$

depending if there are 2 or 3 nonexpansive lines. We write all statements for three nonexpansive lines, with the implicit understanding that when there are only 2 nonexpansive lines, we remove any reference to ℓ_3.

Without loss of generality, we can assume that all ℓ_i pass through the origin. By Corollary 2.12, we can assume that the nonexpansive lines are rational lines and without loss of generality we can assume that ℓ_1, ℓ_3 are not horizontal. By Proposition 2.19, any choice of orientation on ℓ_1, ℓ_2, ℓ_3 determines nonexpansive directions for η. For the remainder of this construction, we make a slight abuse of notation and view ℓ_1, ℓ_2, ℓ_3 as directed lines that determine nonexpansive directions.

Let $\mathcal{S} \subset R_{n,3}$ be an η-generating set. By Proposition 2.9, there exist edges $w_1, w_2, w_3 \in E(\mathcal{S})$ parallel to ℓ_1, ℓ_2, ℓ_3, respectively. By Proposition 2.19, there exist $\widehat{w}_1, \widehat{w}_2, \widehat{w}_3 \in E(\mathcal{S})$ such that w_i is antiparallel to $\widehat{w}_i$, for $i = 1, 2, 3$. By Corollary 2.20, since w_1 and w_3 are not horizontal, we have that $\mathcal{S}$ is $w_1, \widehat{w}_1, w_3$ and $\widehat{w}_3$-balanced. If w_2 is not horizontal, then again applying Corollary 2.20, we have that $\mathcal{S}$ is both w_2 and $\widehat{w}_2$-balanced. If w_2 is horizontal, then $\mathcal{S}$ is balanced for at least one of w_2 and $\widehat{w}_2$. So, without loss of generality, we can assume that

$$\mathcal{S} \text{ is } w_1, \widehat{w}_1, w_3, \widehat{w}_3 \text{ and } w_2\text{-balanced.}$$

Let $H_0' = H(\ell_1)$, and we recall that this denotes the half-plane with boundary parallel to ℓ_1 whose boundary passes through the origin. Let H_{-1}' be the smallest half-plane strictly containing H_0' whose boundary contains an integer point (this is well-defined for any rational line ℓ_1). Since ℓ_1 is a nonexpansive direction, there exist $f, g \in X_\eta$ such that $f{\upharpoonright}H_0' = g{\upharpoonright}H_0'$ but $f{\upharpoonright}H_{-1}' \neq g{\upharpoonright}H_{-1}'$. Since ℓ_2 is not parallel to ℓ_1 and $f{\upharpoonright}H_0' = g{\upharpoonright}H_0'$, at most one of $f{\upharpoonright}H_{-1}'$ and $g{\upharpoonright}H_{-1}'$ extends to a $\mathbb{Z}^2$-coloring that is periodic with period vector parallel to ℓ_2. Without loss of generality, suppose $f{\upharpoonright}H_{-1}'$ is an η-coloring of H_{-1}' which cannot be extended to a periodic η-coloring of $\mathbb{Z}^2$ with a period vector parallel to ℓ_2. By Proposition 2.18, f is periodic with period vector parallel to ℓ_1. Translating if needed, we can assume that $(w_1 \cap \mathbb{Z}^2) \subset H_{-1}' \setminus H_0'$. It follows that $\mathcal{S} \subset H_{-1}'$ (recall that the boundaries of both $\mathcal{S}$ and H_{-1}' are positively oriented).

To make the constructions clearer, it is convenient to make a change of coordinates such that ℓ_1 points vertically downward. Since ℓ_1 has rational slope, we can choose $A \in SL_2(\mathbb{Z})$ such that $A(\ell_1)$ points vertically downward. Define

$$\tilde{\eta} := \eta \circ A^{-1}; \tilde{f} := f \circ A^{-1}; \widetilde{\mathcal{S}} := A(\mathcal{S}), \tag{10}$$

and

$$\tilde{\ell}_i := A(\ell_i),\ \tilde{w}_i := A(w_i) \text{ and } \widehat{\widehat{w}_i} := A(\widehat{w}_i),\ \text{ for } i = 1, 2, 3. \tag{11}$$

Then for any finite, nonempty set $\mathcal{T} \subset \mathbb{Z}^2$, we have $D_\eta(\mathcal{T}) = D_{\tilde{\eta}}(A(\mathcal{T}))$. It follows that $\tilde{\eta}$ is aperiodic and

(12)
$\tilde{f}$ is vertically periodic (say with minimal period p) and is not doubly periodic.

Further,

$$\widetilde{\mathcal{S}} \text{ is an } \tilde{\eta}\text{-generating set} \tag{13}$$

and

(14) $\widetilde{\mathcal{S}}$ is $\tilde{w}_1, \widehat{\tilde{w}_1}, \tilde{w}_3, \widehat{\tilde{w}_3}$-balanced and is balanced for at least one of $\tilde{w}_2$ and $\widehat{\tilde{w}_2}$.

For $i \in \mathbb{Z}$, define

$$H_i := \left\{(x,y) \in \mathbb{Z}^2 \colon x \geq i\right\}.$$

Note that $H_0 = A(H_0')$ and $H_{-1} = A(H_{-1}')$. For $i \in \mathbb{N}$, let B_i be a vertical strip of width i defined by

$$B_i := H_{-1} \setminus H_{i-1} \tag{15}$$

and $\bar{B}_i$ be the vertical sub-strip of width $i-1$ defined by

$$\bar{B}_i := H_0 \setminus H_{i-1}.$$

Let

(16) $d \in \mathbb{N}$ be the number of distinct vertical lines passing through $\widetilde{\mathcal{S}}$

and note that $\widetilde{\mathcal{S}} \subset B_d$ and $(\widetilde{\mathcal{S}} \setminus \tilde{w}_1) \subset \bar{B}_d$.

We claim there are infinitely many integers $x \geq 0$ such that

(17) $\tilde{f}\restriction_{\bar{B}_d + (x,0)}$ does not extend uniquely to an $\tilde{\eta}$-coloring of $B_d + (x,0)$.

By construction, $x = 0$ is such an integer. If there are not infinitely many such integers, let $x_{\max}$ denote the largest such integer. By (12), $\tilde{f}$ is vertically periodic and there are only finitely many colorings of the form $(T^{(x,0)}\tilde{f})\restriction_{\bar{B}_d}$; say there are P such colorings. By the pigeonhole principle, there are distinct integers $x_1, x_2 \in \{x_{\max}+1, \dots, x_{\max}+P+1\}$ such that

$$(T^{(x_1,0)}\tilde{f})\restriction_{\bar{B}_d} = (T^{(x_2,0)}\tilde{f})\restriction_{\bar{B}_d};$$

without loss of generality assume that $x_1 > x_{\max}$ is the smallest integer for which there exists x_2 with this property. Since $(T^{(x_2,0)}\tilde{f})\restriction_{\bar{B}_d}$ extends uniquely to an $\tilde{\eta}$-coloring of B_d, so does $(T^{(x_1,0)}\tilde{f})\restriction_{\bar{B}_d}$. Therefore

$$(T^{(x_1-1,0)}\tilde{f})\restriction_{\bar{B}_d} = (T^{(x_2-1,0)}\tilde{f})\restriction_{\bar{B}_d}.$$

Since $x_2 - 1 > x_{\max}$, we have that $(T^{(x_2-1,0)}\tilde{f})\restriction_{\bar{B}_d}$ extends uniquely to an $\tilde{\eta}$-coloring of B_d. Thus so does $(T^{(x_1-1,0)}\tilde{f})\restriction_{\bar{B}_d}$, and since $(T^{(x_{\max},0)}\tilde{f})\restriction_{\bar{B}_d}$ does not have this property, we must have that $x_1 - 1 > x_{\max}$. However, this contradicts the choice of x_1 as the smallest integer with this property and the claim follows.

Let $0 = x_1 < x_2 < x_3 < \dots$ be a sequence integers satisfying (17). Then since $\widetilde{\mathcal{S}}$ is $\tilde{w}_1$-balanced by (14) and for all $i \in \mathbb{N}$, $f\restriction_{A^{-1}(\bar{B}_d + (x_i,0))}$ satisfies condition (i) in Proposition 2.18, we have that $f\restriction_{A^{-1}(\bar{B}_d + (x_i,0))}$ has period at most $|w_1 \cap \mathcal{S}| - 1 = |\tilde{w}_1 \cap \widetilde{\mathcal{S}}| - 1$. It follows that for all $i \in \mathbb{N}$, $\tilde{f}\restriction_{\bar{B}_d + (x_i,0)}$ is vertically periodic of period at most $|\tilde{w}_1 \cap \widetilde{\mathcal{S}}| - 1$.

Claim 3.1. *For all $i \geq d$ and $j \in \mathbb{Z}$, there is no finite set $F_i \subset B_i$ such that $(T^{(0,j)}\tilde{f})\restriction_{F_i}$ extends uniquely to an $\tilde{\eta}$-coloring of B_i.*

Proof. We proceed by contradiction. If not, suppose $F_i \subset B_i$ is a finite set and for some $j \in \mathbb{Z}$ the coloring $(T^{(0,j)}\tilde{f})\restriction_{F_i}$ extends uniquely to an $\tilde{\eta}$-coloring of B_i. Since $\tilde{f} \in X_{\tilde{\eta}}$, there exists $\vec{u} \in \mathbb{Z}^2$ such that

$$\tilde{f}\restriction_{F_i} = (T^{\vec{u}}\tilde{\eta})\restriction_{F_i},$$

where the existence of $\vec{u}$ follows from the fact that every finite coloring occurring in an element of $X_{\tilde{\eta}}$ also occurs in $\tilde{\eta}$. Therefore $(T^{\vec{u}}\tilde{\eta})\restriction_{B_i} = \tilde{f}\restriction_{B_i}$ is vertically periodic by (12). By Corollary 2.21, we have that $\tilde{\eta}$ is periodic and thus that η is periodic, a contradiction of (8). $\square$

We now continue with the construction of the large set needed for the proofs of Propositions 3.4 and 3.7. We define:

Definition 3.2. If $\mathcal{S} \subset \mathbb{Z}^2$ is a convex set, then $\mathcal{T} \subset \mathbb{Z}^2$ is $E(\mathcal{S})$-*enveloped* if

(i) $\mathcal{T}$ is convex;
(ii) For all $w \in E(\mathcal{T})$, there exists $u \in E(\mathcal{S})$ such that w is parallel to u and $|w| \geq |u|$;
(iii) Either the set

$$\{u \in E(\mathcal{S}) \colon \exists w \in E(\mathcal{T}) \text{ such that } w \text{ is parallel to } u\}$$

is connected (recall Definition 2.1) or $\mathcal{T}$ is the set of integer points in a bi-infinite strip in $\mathbb{R}^2$.

Maintaining notation of $\tilde{f}$ and $\tilde{S}$ defined in (10) and B_i defined in (15), we inductively define a convex set G_∞ on which we can control periodicity. For each $i \in \mathbb{N}$, let

$$F_i \subset B_{d+i-1} \text{ be a finite, } E(\widetilde{\mathcal{S}})\text{-enveloped set containing } [-1, d+i-2] \times [-d-i-2, d+i+2]. \tag{18}$$

and let

$$G_i \subset B_{d+i-1} \text{ be a largest (with respect to the partial ordering by inclusion) } E(\widetilde{\mathcal{S}})\text{-enveloped subset of } B_{d+i-1} \text{ to which } \tilde{f}\restriction_{F_i} \text{ extends uniquely} \tag{19}$$

(we allow the possibilities that $G_i = F_i$ or that G_i is infinite).

By Claim 3.1, $G_j \neq B_{d+j-1}$ and so the set

$$G_j \cap \{(-1, y) \colon y \in \mathbb{Z}\} \tag{20}$$

is not bi-infinite. This (finite or semi-infinite) line either has an element of maximal y-coordinate or of minimal y-coordinate (or both). Therefore there is either a subsequence $\{j_k\}_{k=0}^{\infty}$ such that $G_{j_k} \cap \{(-1, y) \colon y \in \mathbb{Z}\}$ has an element of maximal y-coordinate for all k or there is a subsequence such that $G_{j_k} \cap \{(-1, y) \colon y \in \mathbb{Z}\}$ has an element of minimal y-coordinate for all k. Without loss of generality (the other case being similar), suppose that there are infinitely many $j \in \mathbb{N}$ such that the set $G_j \cap \{(-1, y) \colon y \in \mathbb{Z}\}$ has an element of maximal y-coordinate. Without loss of generality (passing to a subsequence if necessary), we assume $G_j \cap \{(-1, y) \colon y \in \mathbb{Z}\}$ has an element of maximal y-coordinate for all $j \in \mathbb{N}$ and let $y_j^{\max}$ be this y-coordinate. By (12), $\tilde{f}$ is vertically periodic with minimal period p. There exists

$0 \leq J_{\max} < p$ such that for infinitely many j, $y_j^{\max} \equiv J_{\max} \pmod p$. Passing to this subsequence and maintaining the same notation on indices j, for each such j, let $k_j \in \mathbb{Z}$ be such that $y_j^{\max} = k_j \cdot p + J_{\max}$. By periodicity, $T^{(0,k_j \cdot p)}\tilde{f}\restriction G_j = \tilde{f}\restriction G_j$ and so by (19) $\tilde{f}\restriction G_j - (0, k_j \cdot p)$ does not extend uniquely to an $\tilde{\eta}$-coloring of any larger $E(\widetilde{\mathcal{S}})$-enveloped set in B_{d+j-1}. Our construction yields the following:

(a) For all $j \in \mathbb{N}$, the point $(-1, J_{\max})$ is the top-most element of
$$\{(-1, y) \colon y \in \mathbb{Z}\} \cap (G_j - (0, k_j \cdot p)).$$

(b) The set $G_j - (0, k_j \cdot p)$ contains the set
$$([-1, d+j-1] \times [-d-j-1, d+j+1]) - (0, k_j \cdot p),$$
which is a subset of $\{(x, y) \colon x \geq -1, y \leq J_{\max}\}$.

(c) The set $G_j - (0, k_j \cdot p)$ is $E(\widetilde{\mathcal{S}})$-enveloped.

Set
$$G_\infty := \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} \big(G_j - (0, k_j \cdot p)\big). \tag{21}$$

Claim 3.3. *The set G_∞ is an $E(\widetilde{\mathcal{S}})$-enveloped set which contains the semi-infinite line $\{(-1, y) \in \mathbb{Z}^2 \colon y \leq 0\}$ and is such that $\tilde{f}\restriction G_\infty$ does not extend uniquely to an $\tilde{\eta}$-coloring of any larger $E(\widetilde{\mathcal{S}})$-enveloped subset of $\{(x, y) \in \mathbb{Z}^2 \colon x \geq -1\}$.*

Proof. We order the edges of $E(\widetilde{\mathcal{S}})$ by setting $u_0 \in E(\widetilde{\mathcal{S}})$ to be the edge that points vertically downward and defining $u_{i+1} := \mathrm{pred}(u_i)$ (recall Definition (2.1)) for all $i = 0, 1, \ldots, |E(\widetilde{\mathcal{S}})| - 2$. For $0 \leq i \leq |E(\widetilde{\mathcal{S}})| - 1$, set $L_i(j)$ to be the length of the edge of $G_j - (0, k_j \cdot p)$ parallel to u_i. Since $G_j - (0, k_j \cdot p)$ is $E(\widetilde{\mathcal{S}})$-enveloped, $L_i(j) > 0$ for all i, j. Define
$$L_i(\infty) := \limsup_j L_i(j).$$
Since u_0 determines a nonexpansive direction for $\tilde{\eta}$, $\widetilde{\mathcal{S}}$ has an edge antiparallel to u_0 (by Corollary 2.19). Let $i_0 \in \{1, 2, \ldots, |E(\widetilde{\mathcal{S}})| - 1\}$ be such that $u_{i_0} \in E(\widetilde{\mathcal{S}})$ is antiparallel to u_0. Since $G_j - (0, k_j \cdot p)$ contains the set
$$([-1, d+j-1] \times [-d-j-1, d+j+1]) - (0, k_j \cdot p),$$
there exists $1 \leq i_1 < i_0$ such that $L_{i_1}(\infty) = \infty$; without loss of generality let i_1 be the smallest positive index with this property. It follows that for all $1 \leq k < i_1$, $E(G_\infty)$ has an edge parallel to u_k of length $L_k(\infty)$, as well as a semi-infinite edge parallel to u_{i_1}. On the other hand, since $G_j - (0, k_j \cdot p)$ contains
$$([-1, d+j-1] \times [-d-j-1, d+j+1]) - (0, k_j \cdot p),$$
it contains the line segment $\{(-1, y) \colon -d-j-1-k_j \cdot p \leq y \leq 0\}$ and so G_∞ contains the line segment $\{(-1, y) \in \mathbb{Z}^2 \colon y \leq 0\}$. There cannot be any other edges in $E(G_\infty)$, since $G_j - (0, k_j \cdot p)$ only has edges parallel to those that appear in $E(\widetilde{\mathcal{S}})$ for all j. Finally we observe that if $\tilde{f}\restriction G_\infty$ extended uniquely to a larger $E(\widetilde{\mathcal{S}})$-enveloped subset of $\{(x, y) \in \mathbb{Z}^2 \colon x \geq -1\}$, then by compactness there would exist j such that $\tilde{f}\restriction G_j - (0, k_j \cdot p)$ extends uniquely to a larger $E(\widetilde{\mathcal{S}})$-enveloped subset of B_{d+j-1}, a contradiction of (19). □

It follows from the construction that G_∞ is an infinite $E(\widetilde{\mathcal{S}})$-enveloped set. Moreover, there are infinitely many distinct vertical lines that have nonempty intersection with G_∞ (by (b)). If necessary, we again make a change of coordinates and assume that $J_{\max} = 0$. Thus by Claim 3.3,

$$G_\infty \text{ is an } E(\widetilde{\mathcal{S}})\text{-enveloped set that intersects every vertical line in } H_{-1}. \tag{22}$$

By construction, $E(G_\infty)$ has a semi-infinite edge that points vertically downward from $(-1, 0)$. By (22),

$$G_\infty \text{ has a nonvertical, semi-infinite edge } u \in E(G_\infty) \tag{23}$$

and u is parallel to some edge in $E(\tilde{\mathcal{S}})$. This edge determines a nonexpansive direction for $\tilde{\eta}$, since by the claim, $\tilde{f}{\restriction}G_\infty$ cannot be uniquely extended to any larger $E(\tilde{\mathcal{S}})$-enveloped set.

Define $K \supset G_\infty$ such that

$$K \text{ is the smallest } E(\widetilde{\mathcal{S}})\text{-enveloped set containing } G_\infty \text{ with } u \notin \partial K, \tag{24}$$

meaning that K is the set obtained by extending the successor edge to u backwards until it intersects an integer point and then taking the convex hull (note that successor edge is meant with respect to positive orientation on the boundary). By construction,

$$\text{there exists } \tilde{h} \in X_{\tilde{\eta}} \text{ such that } \tilde{f}{\restriction}G_\infty = \tilde{h}{\restriction}G_\infty \text{ and } \tilde{f}{\restriction}K \neq \tilde{h}{\restriction}K. \tag{25}$$

By (12), $\tilde{f}$ is vertically periodic and so

$$\tilde{h}{\restriction}G_\infty \text{ is vertically periodic (with minimal period p) but } \tilde{h}{\restriction}K \text{ is not.} \tag{26}$$

We use the construction of G_∞ to eliminate the case of 2 nonexpansive lines:

Proposition 3.4. *Suppose there are exactly two nonexpansive lines for X_η. Then for all $n \in \mathbb{N}$, $P_\eta(R_{n,3}) > 3n$.*

Proof. We proceed by contradiction and assume that η has exactly two nonexpansive directions and that there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. Thus hypotheses (H1) and (H2) are satisfied. In particular, by (8), η is aperiodic.

We maintain the notation of the nonexpansive lines in (9) (where we assume only two), the quantities in (10) and (11), and of the construction of the set G_∞ defined in (21) satisfying (22). Since there are only two nonexpansive lines for η, the edge u defined in (23) must either be parallel or antiparallel to $\tilde{\ell}_2$. Let $K \supset G_\infty$ be defined as in (24) and $\tilde{h}$ as in (25). Then $K \setminus G_\infty$ can be written as

$$K \setminus G_\infty = \bigcup_{k=1}^{k_0} (l_k \cap K),$$

where $l_1, l_2, \dots, l_{k_0}$ are (undirected) lines parallel to $\tilde{\ell}_2$ and k_0 is the number of lines produced in the construction of K. By (26), $\tilde{h}{\restriction}K$ cannot be extended to a vertically periodic η-coloring of H_{-1}. Let $u_0 := u$ and label the edges of G_∞ by $u_{i+1} := \mathrm{succ}(u_i)$ for $i = 0, \dots, |E(G_\infty)| - 1$, where $\mathrm{succ}(\cdot)$ denotes the successor edge taken with positive orientation.

Suppose $u_I \in E(G_\infty)$ is the edge parallel to $\tilde{\ell}_1$, meaning that u_I points vertically downward. Define a sequence of sets

$$G_\infty = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_I,$$

where L_{i+1} is obtained from L_i by extending the edge of L_i parallel to u_{I-i-1} to be semi-infinite and taking the intersection of $\mathbb{Z}^2$ with the convex hull of the resulting shape (see Figure 3). Then $E(L_{i+1}) = E(L_i) \setminus \{u_{I-i}\}$.

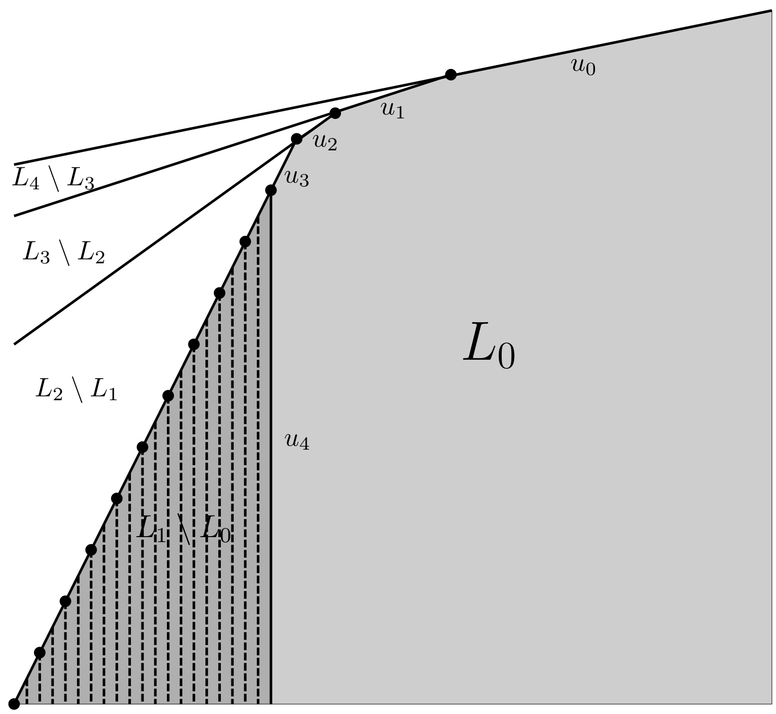


FIGURE 3. The construction of the nested sets $L_0 \subset L_1 \cdots$. Integer points at the intersection of two lines are marked with a dot and the dotted lines show $L_1 \setminus L_0 = s_1 \cup s_2 \cup \cdots$.

Claim 3.5. *For $0 \le i \le I$, $\tilde{h}\restriction_{L_i}$ is vertically periodic, but possibly of larger period than that of $\tilde{h}\restriction_{L_0}$.*

Proof. For $i = 0$, this follows directly from the construction of G_∞. For $i = 1$, write

$$L_1 \setminus L_0 = s_1 \cup s_2 \cup \cdots$$

where s_j is the semi-infinite line defined by $s_j := \{(-j-1, y) \colon y \in \mathbb{Z}\} \cap L_1$. For integers $0 \le a \le b$, write $s_{[a,b]} = s_a \cup s_{a+1} \cup \ldots \cup s_b$.

$$\text{Suppose that } \tilde{h}\restriction_{L_0 \cup s_{[1,j]}} \text{ is vertically periodic.} \tag{27}$$

Let $\vec{v}_j(i) \in \mathbb{Z}^2$ be the translation of $\widetilde{\mathcal{S}}$ such that the top-most element of $\tilde{w}_1 + \vec{v}_j(i)$ is the point (j, i). If for all $R < 0$ there exists $r \le R$ such that $(T^{\vec{v}_j(r)}\tilde{h})\restriction_{\widetilde{\mathcal{S}} \setminus \tilde{w}_1}$ extends uniquely to an η-coloring of $\widetilde{\mathcal{S}}$, then there is a unique extension of $\tilde{h}\restriction_{L_0 \cup s_{[1,j]}}$ to an $\tilde{\eta}$-coloring of $L_0 \cup s_{[1,j+1]}$ by (13). In this case, arguing as in Step 1 in the proof of Proposition 2.18, the restriction of $\tilde{h}$ to $L_0 \cup s_{[1,j+1]}$ is vertically periodic of the same period as $\tilde{h}\restriction L_0 \cup s_{[1,j]}$. Otherwise there exists $R < 0$ such that for all $r \leq R$ the coloring $(T^{\vec{v}_j(r)}\tilde{h})\restriction\widetilde{\mathcal{S}} \setminus \tilde{w}_1$ does not extend uniquely to an η-coloring of $\widetilde{\mathcal{S}}$. Then by Corollary 2.22 the restriction of $\tilde{h}$ to $L_1 \cap s_{[j-m+1,j]}$ is eventually periodic of period at most $|\tilde{w}_1 \cap \widetilde{\mathcal{S}}| - 1$ and the initial portion which may not be periodic has length at most $|\tilde{w}_1 \cap \widetilde{\mathcal{S}}| - 1$, where m is the number of vertical lines in L_1 which have nonempty intersection with $\widetilde{\mathcal{S}} \setminus \tilde{w}_1$. So by Corollary 2.23, we have that $\tilde{h}\restriction L_1 \cap s_{j+1}$ is eventually vertically periodic of period at most $2|\tilde{w}_1 \cap \widetilde{\mathcal{S}}| - 2$ and the initial portion which may not be periodic again has length at most $|\tilde{w}_1 \cap \widetilde{\mathcal{S}}| - 1$. Moreover, $\tilde{h}\restriction L_0 \cup s_{[1,j]}$ is periodic by (27), $\mathcal{S}$ is a generating set, the restriction of $\tilde{h}$ to the union of $L_0 \cup s_{[1,j]}$, and any semi-infinite portion of s_{j+1} uniquely extends to an η-coloring of $L_0 \cup s_{[1,j+1]}$ and so there is no initial portion on which $\tilde{h}$ is not periodic. It follows by induction that $\tilde{h}\restriction L_1$ is vertically periodic and $(T^{(0,|\tilde{w}\cap\widetilde{\mathcal{S}}|+1)}\tilde{h})\restriction L_1$ is vertically periodic. If $L_1 = L_I$ then the claim follows. Otherwise the semi-infinite edge of L_1 parallel to u_{I-1} determines an expansive direction for $\tilde{\eta}$. Write

$$L_2 \setminus \big(L_0 \cup (L_1 - (0, |\tilde{w} \cap \widetilde{\mathcal{S}}| - 1))\big) = \bigcup_{i=1}^{k_1} \tilde{s}_i$$

where the $\tilde{s}_i$ are semi-infinite lines parallel to u_{I-1}.

Since u_{I-1} is expansive, there is a unique extension of $\tilde{h}\restriction L_1 - (0, |\tilde{w} \cap \widetilde{\mathcal{S}}| - 1)$ to an η-coloring of $(L_1 - (0, |\tilde{w} \cap \widetilde{\mathcal{S}}| - 1)) \cup \tilde{s}_1$ (the uniqueness of this extension to a semi-infinite portion of $\tilde{s}_1$ follows from the block map guaranteed by expansiveness and since $\mathcal{S}$ is $\tilde{\eta}$-generating, this coloring extebds uniquely to all of $\tilde{s}_1$). Since $L_1 - (0, |\tilde{w} \cap \widetilde{\mathcal{S}}| - 1)$ is colored in the same way as $L_1 - (0, |\tilde{w} \cap \widetilde{\mathcal{S}}| - 1 - q)$, where q is the vertical period, and there is a unique way to extend this coloring to an $\tilde{\eta}$-coloring of $(L_1 - (0, |\tilde{w} \cap \widetilde{\mathcal{S}}| - 1)) \cup \tilde{s}_1$, we have that the vertical periodicity of $\tilde{h}\restriction L_1 - (0, |\tilde{w} \cap \widetilde{\mathcal{S}}| - 1)$ implies that $\tilde{h}\restriction\big(L_1 - (0, |\tilde{w} \cap \widetilde{\mathcal{S}}| - 1)\big) \cup \tilde{s}_1$ is also vertically periodic. Inductively it follows that $\tilde{h}\restriction L_2$ is vertically periodic. More generally, suppose that $\tilde{h}\restriction L_i$ is vertically periodic for $i < I$. Then L_i has two semi-infinite edges, one of which it shares with L_0 and the other determines an expansive direction for $\tilde{\eta}$. Write

$$L_{i+1} \setminus L_i = t_1 \cup t_2 \cup \cdots$$

where $L_i \cup t_{[1,j]}$ is convex for all $j = 1, 2, \ldots$, each t_j is the intersection of $\mathbb{Z}^2$ with a semi-infinite line parallel to u_{I-i} and contained in L_{i+1}, and $t_{[a,b]} = t_a \cup t_{a+1} \cup \ldots \cup t_b$. Suppose that $\tilde{h}\restriction L_i \cup t_{[1,j]}$ is vertically periodic. Since u_{I-i} determines an $\tilde{\eta}$-expansive direction, again using the same reasoning, there is a unique extension of $L_i \cup t_{[1,j]}$ to an $\tilde{\eta}$-coloring of $L_i \cup t_{[1,j+1]}$. By vertical periodicity, $\tilde{h}\restriction L_i \cup t_{[1,j]} = (T^{(0,-q)}\tilde{h})\restriction L_i \cup t_{[1,j]}$, where q denotes the smallest vertical period of $\tilde{h}\restriction L_i \cup t_{[1,j]}$. By uniqueness, $\tilde{h}\restriction L_i \cup t_{[1,j+1]} = (T^{(0,-q)}\tilde{h})\restriction L_i \cup t_{[1,j+1]}$ and hence is also vertically periodic. By induction, this holds for all j and hence $\tilde{h}\restriction L_{i+1}$ is vertically periodic. □

Let C denote the smallest bi-infinite strip whose edges are parallel to $\tilde{\ell}_2$ that contains $\widetilde{\mathcal{S}} \setminus \tilde{w}_2$. Let $J \in \mathbb{Z}$ be the maximal integer such that $C + (0, J)$ is a subset of the region in $\mathbb{Z}^2$ on which $\tilde{h}$ is vertically periodic, let $C_j := C + (0, j)$, and let

$Q \in \mathbb{N}$ be the smallest vertical period of $\tilde{h}\restriction_{L_I - (0,J)}$. The integer J is well-defined by (26). Then for all $j \leq J$, we have that $\tilde{h}\restriction_{C_j} = (T^{(0,-Q)}\tilde{h})\restriction_{C_j}$.

We claim that for all $j \leq J$, $\tilde{h}\restriction_{C_j}$ is not periodic with period vector parallel to $\tilde{\ell}_2$. By the preceding remark, it suffices to show that this holds for all sufficiently small values of j. For all $j \in \mathbb{Z}$ sufficiently negative that the only edge of L_0 that C_j intersects is the edge parallel to $\tilde{\ell}_1$ (all but finitely many C_j have this property), recall that $\tilde{h}\restriction_{L_0} = \tilde{f}\restriction_{L_0}$. By the construction of $\tilde{f}$, we have that $\tilde{f}\restriction_{H_{-1}}$ cannot be extended to an $\tilde{\eta}$-coloring of $\mathbb{Z}^2$ which is periodic with period vector parallel to $\tilde{\ell}_2$. If $\tilde{h}\restriction_{C_j}$ is $\tilde{\ell}_2$-periodic, then by Corollary 2.21 it follows that $\tilde{h}$ itself is $\tilde{\ell}_2$-periodic. But the sequence $(T^{(0,-k)}\tilde{h})$ has an accumulation point, and any such accumulation point is also $\tilde{\ell}_2$-periodic. Moreover, the restriction of any such accumulation point to H_{-1} is one of the functions $\tilde{f}\restriction_{H_{-1}}, (T^{(0,-1)}\tilde{f})\restriction_{H_{-1}}, \dots, (T^{(0,-p+1)}\tilde{f})\restriction_{H_{-1}}$ (where again $p \in \mathbb{N}$ is the minimal vertical period of $\tilde{f}$). This contradicts the fact that $\tilde{f}\restriction_{H_{-1}}$ does not extend to an $\tilde{\ell}_2$-periodic coloring of $\mathbb{Z}^2$, and the claim follows.

If ℓ_2 is not horizontal, then $\widetilde{\mathcal{S}}$ is u-balanced, where u is the edge defined in (23). In this case every line parallel to u that has nonempty intersection with $\widetilde{\mathcal{S}}$ contains at least $|\tilde{w}_2 \cap \widetilde{\mathcal{S}}| - 1$ integer points. Since $\tilde{h}\restriction_{C_j}$ is not $\tilde{\ell}$-periodic, the Morse-Hedlund Theorem implies that there are at least $|\tilde{w}_2 \cap \widetilde{\mathcal{S}}|$ distinct $\tilde{\eta}$-colorings of $\widetilde{\mathcal{S}} \setminus \tilde{w}_2$ that occur in C_j (otherwise the coloring would be periodic). But there are at most $|\tilde{w} \cap \widetilde{\mathcal{S}}| - 1$ η-colorings of $\widetilde{\mathcal{S}} \setminus w_2$ that extend non-uniquely to an η-coloring of $\widetilde{\mathcal{S}}$, and so by Corollary 12, the coloring of C_j extends uniquely to an $\tilde{\eta}$-coloring of $C_j \cup C_{j+1}$ for all $j \leq J$, via the same reasoning used to show that $\tilde{h}\restriction_{L_i}$ is vertically periodic. Since the restriction of $\tilde{h}$ to the region $\bigcup_{j \leq J} C_j$ is vertically periodic and $\tilde{h}\restriction_{C_J}$ extends uniquely to an $\tilde{\eta}$-coloring of $C_J \cup C_{J+1}$, the restriction of $\tilde{h}$ to the region $\bigcup_{j \leq J+1} C_j$ is vertically periodic. But this contradicts the definition of J. If ℓ_2 is horizontal, then the same argument applies to $\mathcal{S}^{\ell_2}$ in place of $\mathcal{S}$, where $\mathcal{S}^{\ell_2}$ is an ℓ_2-balanced subset of $R_{n,3}$ constructed by Proposition 2.16. □

Following standard terminology in the literature (e.g. [6]) we make the following definition:

Definition 3.6. Suppose $\mathcal{T} \subset \mathbb{Z}^2$ and $\vec{u} \in \mathbb{Z}^2$. We say that $\alpha \colon \mathcal{T} \to \mathcal{A}$ is *periodic when restricted to the region* $\mathcal{T}$ *with period vector* $\vec{u}$ if $\alpha(\vec{x}) = \alpha(\vec{x} + \vec{u})$ for all $\vec{x} \in \mathcal{T}$ such that $\vec{x} + \vec{u} \in \mathcal{T}$.

Finally we show that the low complexity assumption is not compatible with more nonexpansive lines. While it may seem, a priori, like this should be a simpler setting to rule out, it turns out that the more complicated generating shape introduces new complications.

Proposition 3.7. *Suppose there are exactly three nonexpansive lines for* η*. Then for all* $n \in \mathbb{N}$*,* $P_\eta(R_{n,3}) > 3n$*.*

Proof. We proceed by contradiction and assume that η has exactly three nonexpansive directions and that there exists $n \in \mathbb{N}$ such that $P_\eta(R_{n,3}) \leq 3n$. Thus hypotheses (H1) and (H2) are satisfied. In particular, η is aperiodic (8).

By Proposition 2.9, there exists an η-generating set $\mathcal{S} \subset R_{n,3}$ which satisfies (1) and every nonexpansive direction for η is parallel to one of the edges of $\mathcal{S}$. By

Proposition 2.19, the direction antiparallel to any nonexpansive direction is also nonexpansive. Since there are exactly three nonexpansive lines for η, $\mathcal{S}$ has precisely six edges, all of which determine nonexpansive directions. Since $\mathcal{S} \subset R_{n,3}$, two of these edges must be horizontal and the remaining four edges each contain exactly two integer points. Again by Proposition 2.19, every edge of $\mathcal{S}$ is antiparallel to another edge of $\mathcal{S}$, and so $\partial\mathcal{S}$ is a hexagon comprised of three pairs of parallel edges. It follows that the two horizontal edges contain the same number of integer points and this number is at most $n-1$. Let $w_1 \in E(\mathcal{S})$ be the predecessor edge to the top horizontal edge in $E(\mathcal{S})$ and recursively define $w_{i+1} := \mathrm{succ}(w_i)$ for $i = 1, 2, 3, 4, 5$ (see Figure 4). Then w_{i+3} is antiparallel to w_i for all i, where the indices are understood to be taken $\pmod 6$.

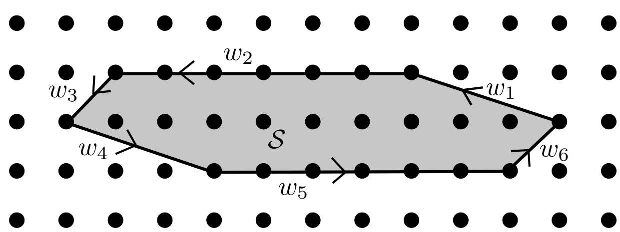


FIGURE 4. The set $\mathcal{S}$ with oriented edges labeled.

We summarize: $|w_2 \cap \mathbb{Z}^2| = |w_5 \cap \mathbb{Z}^2| \leq n - 1$ and $|w_i \cap \mathbb{Z}^2| = |w_{i+3} \cap \mathbb{Z}^2| = 2$ for $i = 1, 3$. It follows that

$$\mathcal{S} \text{ is balanced in every nonexpansive direction.} \tag{28}$$

For convenience, define $a_1, a_3, a_4, a_6 \in \mathbb{Z}$ such that

$$w_i \text{ is parallel to } (a_i, 1) \text{ for } i = 1, 6 \text{ and } w_i \text{ is parallel to } (a_i, -1) \text{ for } i = 3, 4.$$

By convexity, one of the statements:

$$\begin{aligned} &a_1, a_3 \leq 0; \\ &a_1 \leq 0, a_3 \geq 0, |a_1| > a_3; \\ &a_1 \geq 0, a_3 \leq 0, |a_3| > a_1; \end{aligned}$$

holds. In each case, every horizontal line that has nonempty intersection with $\mathcal{S}$ contains at least

$$|w_2 \cap \mathcal{S}| \text{ integer points} \tag{29}$$

(e.g. in the first case the middle horizontal line in $\mathcal{S}$ contains $|w_2 \cap \mathcal{S}| + |a_1| + |a_3|$ integer points, and the other cases are similar).[1]

For $j \in \mathbb{Z}$, let V_j be the horizontal half-plane defined by

$$V_j := \{(x, y) \colon x \in \mathbb{Z},\ y \leq j\}.$$

Since the direction of w_2 is nonexpansive for η, by Proposition 2.18 there exist $f, g \in X_\eta$ such that $f\restriction_{V_0} = g\restriction_{V_0}$ but $f\restriction_{V_1} \neq g\restriction_{V_1}$. At most one of f and g is

[1]This bound is stronger than our usual bound that every horizontal line that has nonempty intersection with $\mathcal{S}$ intersects in at least $|w_2 \cap \mathcal{S}| - 1$ integer points.

periodic with period vector parallel to w_1, and so we can suppose without loss of generality that f is not. Furthermore, without loss of generality we can assume that

(30)
$f{\restriction}_{V_1}$ does not extend to a periodic η-coloring of $\mathbb{Z}^2$ with period vector parallel to w_1.

Since $\mathcal{S}$ is w_2-balanced by (28), it follows from Proposition 2.18 that f is horizontally periodic and the restriction of f to any horizontal strip of height two has period at most $2|w_2 \cap \mathcal{S}| - 2$. Set

$$B := \big\{(x,y) \in \mathbb{Z}^2 \colon y \in \{-1,0\}\big\} \text{ and } C := \{(x,y) \colon y \in \{-1,0,1\}\}\,.$$

For any $j \in \mathbb{Z}$ such that $(T^{-(0,j)}f){\restriction}_B$ does not extend uniquely to an η-coloring of C, we have that $(T^{-(0,j)}f){\restriction}_B$ is horizontally periodic of period at most $|w_1 \cap \mathcal{S}| - 1$. In particular, this holds for $j = 0$.

We claim that there are infinitely many integers $j \leq 0$ such that

$$(T^{-(0,j)}f){\restriction}_B \text{ does not extend uniquely to an } \eta\text{-coloring of } C. \tag{31}$$

The proof of the claim is similar to that of (17). We proceed by contradiction. Suppose that there exists an integer $J \leq 0$ such that for all $j < J$, the coloring $(T^{-(0,j)}f){\restriction}_B$ extends uniquely to an η-coloring of C and assume that $|J|$ is minimal. Since $f{\restriction}_{V_0}$ is horizontally periodic, there are only finitely many η-colorings of the form $(T^{-(0,j)}f){\restriction}_B$ for $j \leq 0$. Say there are M such colorings. Then by the pigeonhole principle, there exist $1 \leq j_1 < j_2 \leq M+2$ such that $(T^{-(0,J-j_1)}f){\restriction}_B = (T^{-(0,J-j_2)}f){\restriction}_B$. Choose j_1 to be the smallest integer such that there exists j_2 with this property. Then by construction, $(T^{-(0,J-j_1)}f){\restriction}_C = (T^{-(0,J-j_2)}f){\restriction}_C$ and hence $(T^{-(0,J-j_1+1)}f){\restriction}_B = (T^{-(0,J-j_2+1)}f){\restriction}_B$. If $j_1 > 1$, this contradicts the minimality of j_1. If $j_1 = 1$, then the fact that $(T^{-(0,J-j_2+1)}f){\restriction}_B = (T^{-(0,J)}f){\restriction}_B$ extends uniquely to an η-coloring of C contradicts the minimality of $|J|$. The claim follows.

Let

$$\mathcal{S}_R := \mathcal{S} \text{ with the rightmost element of every row removed;} \tag{32}$$

$$\mathcal{S}_L := \mathcal{S} \text{ with the leftmost element of every row removed.} \tag{33}$$

Claim 3.8. *There do not exist integers $y_1, y_2 \in \mathbb{Z}$ such that both of the following hold simultaneously:*

$$\textit{for all } x \in \mathbb{Z},\ (T^{(x,y_1)}f){\restriction}_{\mathcal{S}_R} \textit{ extends uniquely to an } \eta\textit{-coloring of } \mathcal{S}; \tag{34}$$

$$\textit{for all } x \in \mathbb{Z},\ (T^{(x,y_2)}f){\restriction}_{\mathcal{S}_L} \textit{ extends uniquely to an } \eta\textit{-coloring of } \mathcal{S}. \tag{35}$$

Proof. We proceed by contradiction. Suppose instead that such integers $y_1, y_2 \in \mathbb{Z}$ exist and assume $y_1 \leq y_2$ (the other case being similar). Define $F := [0, |\mathcal{S}|] \times [y_1, y_2 + 2]$ and observe that since $f \in X_\eta$, there exists $\vec{u} \in \mathbb{Z}^2$ such that $f{\restriction}_F = (T^{\vec{u}}\eta){\restriction}_F$. By (34) and (35), $T^{\vec{u}}\eta$ coincides with f on the set

$$F \cup ([0,\infty) \times [y_1, y_1+2]) \cup ((-\infty, 0] \times [y_2, y_2+2])\,, \tag{36}$$

and so $T^{\vec{u}}\eta$ is horizontally periodic on this set. Let $v \in V(\mathcal{S})$ be the vertex of intersection of the edges w_1 and w_2. There is a translation of $\mathcal{S}$ that takes v to the point $(|\mathcal{S}|+1, y_1+3)$ and takes $\mathcal{S} \setminus v$ to a subset of $F \cup ([0,\infty) \times [y_1, y_1+2])$. Since

$\mathcal{S}$ is η-generating and $T^{\vec{u}}\eta$ coincides with f on $F \cup ([0,\infty) \times [y_1, y_1+2])$, we have that

$$(T^{\vec{u}}(\eta))(|\mathcal{S}|+1, y_1+3) = f(|\mathcal{S}|+1, y_1+3).$$

It follows by induction that $(T^{\vec{u}}(\eta))(|\mathcal{S}|+k, y_1+3) = f(|\mathcal{S}|+k, y_1+3)$ for all $k \geq 1$. A similar induction argument shows that

$$(T^{\vec{u}}(\eta))(|\mathcal{S}|+k, y_1+2+j) = f(|\mathcal{S}|+k, y_1+2+j)$$

for all $k \geq 1$ and all $1 \leq k \leq y_2 - y_1$. Therefore $T^{\vec{u}}\eta$ and f coincide on the set larger than in (36), defined by:

$$F \cup ([0,\infty) \times [y_1, y_2+2]) \cup ((-\infty, 0] \times [y_2, y_2+2]).$$

A similar argument, using the vertex $v' \in V(\mathcal{S})$ that is the intersection of the edges w_4 and w_5 in place of v, shows that $T^{\vec{u}}\eta$ and f coincide on the set

$$(-\infty, \infty) \times [y_1, y_2+2],$$

and so $T^{\vec{u}}\eta$ is horizontally periodic on this set. Since $\mathcal{S}$ is horizontally balanced by (28) it follows from Corollary 2.21 that $T^{\vec{u}}\eta$ is horizontally periodic and hence η is periodic. This is a contradiction of (8). $\square$

Thus henceforth we assume that for all $y \in \mathbb{Z}$, there exists $x_y \in \mathbb{Z}$ such that

(37) $\quad (T^{(x_y,y)}f)\restriction_{\mathcal{S}_R}$ does not extend uniquely to an η-coloring of $\mathcal{S}$.

(The remainder of the proof is analogous if instead, for all $y \in \mathbb{Z}$, there exists $x_y \in \mathbb{Z}$ such that $(T^{(x_y,y)}f)\restriction_{\mathcal{S}_L}$ does not extend uniquely to an η-coloring of $\mathcal{S}$.)

Claim 3.9. *There exists a nonpositive integer y such that $f\restriction_{V_y}$ is doubly periodic, $f\restriction_{V_{y+1}}$ is not doubly periodic, and either $(-a_1,-1)$ or $(-a_6,-1)$ is a period vector for $f\restriction_{V_y}$.*

Proof. As V_y is a half plane, double periodicity is interpreted in the sense of Definition 3.6. Recall that $B = \{(x,y) \in \mathbb{Z}^2 \colon y \in \{-1,0\}\}$. Let B' be the thinnest strip with edges parallel and antiparallel to w_1 which contains $\mathcal{S} \setminus w_1$. For $x \in \mathbb{Z}$, let

$$B'_x := B' + (x, 0).$$

If there exists $x_0 \in \mathbb{Z}$ such that $f\restriction_{B'_{x_0} \cap V_0}$ does not extend uniquely to an η-coloring of $(B'_{x_0} \cup B'_{x_0+1}) \cap V_0$, then for any $\vec{u} \in \mathbb{Z}^2$ such that $(\mathcal{S} \setminus w_1 + \vec{u}) \subset B'_{x_0} \cap V_0$, since $\mathcal{S}$ is η-generating we have that $(T^{-\vec{u}}f)\restriction_{\mathcal{S} \setminus w_1}$ extends non-uniquely to an η-coloring of $\mathcal{S}$. Since $\mathcal{S}$ satisfies (1), by Corollary 2.10 we have that $D_\eta(\mathcal{S} \setminus w_1) > D_\eta(\mathcal{S})$. Since $|w_1 \cap \mathcal{S}| = 2$, there is precisely one coloring of $\mathcal{S} \setminus w_1$ that extends non-uniquely to an η-coloring of $\mathcal{S}$. In particular, since

$$B'_{x_0} \cap V_0 = \bigcup_{k=2}^{\infty} ((\mathcal{S} \setminus w_1) + (x_0 - k a_1, -k))$$

it follows that f restricted to $B'_{x_0} \cap V_0$ is periodic with period vector $(-a_1, -1)$. Since $f\restriction_{V_1} \neq g\restriction_{V_1}$, we have that $f\restriction_B$ is horizontally periodic of period at most $|w_2 \cap \mathcal{S}| - 1$. The region $(B'_{x_0} \cap V_0) \cap B$ is convex and both $\{(x,-1) \colon x \in \mathbb{Z}\}$ and $\{(x,0) \colon x \in \mathbb{Z}\}$ intersect it in at least $|w_2 \cap \mathcal{S}| - 1$ integer points by (29), as the strip B'_{x_0} is only wide enough to contain $\mathcal{S} \setminus w_1$. Therefore $f(x,0) = f(x - a_1, -1)$ for all $x \in \mathbb{Z}$.

Recall that $\mathcal{S} \subset R_{n,3}$, and we assume that the bottom most row of $R_{n,3}$ lies on the x-axis. We have that $\mathcal{S} + (x_0 - 2a_1, -2)$ is contained in the set $(B'_{x_0} \cap V_0) \cup B$. If $v \in V(\mathcal{S})$ is the vertex incident to w_6 and w_1, observe that $(\mathcal{S} \setminus v) + (x_0 - 3a_1, -3)$ is also contained in $(B'_{x_0} \cap V_0) \cup B$ and moreover that

$$(T^{-(x_0-2a_1,-2)} f)\restriction_{\mathcal{S} \setminus v} = (T^{-(x_0-3a_1,-3)} f)\restriction_{\mathcal{S} \setminus v}.$$

Since v is η-generated by $\mathcal{S}$,

$$(T^{-(x_0-2a_1,-2)} f)\restriction_{\mathcal{S}} = (T^{-(x_0-3a_1,-3)} f)\restriction_{\mathcal{S}}.$$

It follows by induction that the coloring $f\restriction_{(B'_{x_0} \cup B'_{x_0+1}) \cap V_0}$ is periodic with period vector $(-a_1, -1)$. Inductively it follows that the restriction of f to $V_0 \cap \bigcup_{k=0}^{\infty} B'_{x_0+k}$ is periodic with period vector $(-a_1, -1)$ as well. A final induction, where the vertex v is replaced by the vertex v' incident to w_4 and w_5, shows that $f\restriction_{V_0}$ is doubly periodic and that $(-a_1, -1)$ is a period vector. A similar argument applies if there exists $x_0 \in \mathbb{Z}$ such that $f\restriction_{B_{x_0} \cap V_0}$ does not extend uniquely to an η-coloring of $(B'_{x_0-1} \cup B'_{x_0}) \cap V_0$. Thus we are finished unless for every $x \in \mathbb{Z}$ the coloring $f\restriction_{B'_x \cap V_0}$ extends uniquely to an η-coloring of $(B'_{x-1} \cup B'_x \cup B'_{x+1}) \cap V_0$. If

$$D(r) := \bigcup_{k=2}^{r} (\mathcal{S} \setminus w_1) + (-ka_1, -k),$$

it follows that for all $x \in \mathbb{Z}$ there exists $R_x \in \mathbb{N}$ such that for all $r > R_x$, the coloring $(T^{(x,0)} f)\restriction_{D(r)}$ extends uniquely to an η-coloring of

$$\bar{D}(r) := \bigcup_{k=2}^{r} \mathcal{S} + (-ka_1, -k) \cup \bigcup_{k=2}^{r} \mathcal{S} + (-ka_1 - 1, -k)$$

(we assume R_x is minimal with this property). Since $f\restriction_{V_1}$ is horizontally periodic, the set $\{R_x : x \in \mathbb{Z}\}$ is finite so $R := \max_{x \in \mathbb{Z}} R_x$ is well-defined. It also follows that $f\restriction_{V_0 \setminus V_{-R}}$ is doubly periodic where $(-a_1, -1)$ is one period vector and the horizontal period is at most $|w \cap \mathcal{S}| - 1$. For $s \in \mathbb{N}$, set

$$E(s) := \bigcup_{k=0}^{s} (\mathcal{S} \setminus w_6) + (-Ra_1 - ka_3, -R - k).$$

As above, if there exists $x \in \mathbb{Z}$ such that $(T^{(x,0)} f)\restriction_{E(s)}$ extends non-uniquely to an η-coloring of

$$\bar{E}(s) := \bigcup_{k=0}^{s} \mathcal{S} + (-Ra_1 - ka_3, -R - k) \cup \bigcup_{k=0}^{s} \mathcal{S} + (-Ra_1 - ka_3 - 1, -R - k),$$

then $f\restriction_{V_{R-1}}$ is doubly periodic and $(-a_6, -1)$ is a period for it. Otherwise, for all $x \in \mathbb{Z}$, there exists $R'_x \in \mathbb{N}$ such that for all $r > R'_x$, the coloring $(T^{(x,0)} f)\restriction_{E(s)}$ extends uniquely to an η-coloring of $\bar{E}(s)$ and again $R' := \max_{x \in \mathbb{Z}} R'_x$ is well-defined. The claim has been shown unless this last case occurs. In that case, for all $x \in \mathbb{Z}$ the coloring $(T^{(x,0)} f)\restriction_{D(R) \cup E(R')}$ extends uniquely to an η-coloring of $\bar{D}(R) \cup \bar{E}(R')$. It follows that for all $x \in \mathbb{Z}$, the coloring $(T^{(x,0)} f)\restriction_{D(R) \cup E(R')}$ extends uniquely to an η-coloring of $V_0 \setminus V_{R+R'}$. Since $f \in X_\eta$, there exists $\vec{u} \in \mathbb{Z}^2$ such that $(T^{\vec{u}} \eta)\restriction_{D(R) \cup E(R')} = f\restriction_{D(R) \cup E(R')}$ and therefore $f\restriction_{V_0 \setminus V_{R+R'}} = (T^{\vec{u}} \eta)\restriction_{R + R'}$ is horizontally periodic. By Corollary 2.21, η itself is horizontally periodic; a contradiction of (8). Therefore either $f\restriction V_0$ is doubly periodic with period vector $(-a_1,-1)$ or there exists $R\in\mathbb{N}$ such that $f\restriction V_R$ is doubly periodic with period vector $(-a_6,-1)$. $\square$

Thus we can define $y_0\le 0$ to be the integer of least absolute value for which Claim 3.9 holds. Recalling that $f\restriction V_1$ is not doubly periodic, we have shown:

$$f\restriction V_{y_0} \text{ is doubly periodic and } f\restriction V_{y_0+1} \text{ is not doubly periodic,} \tag{38}$$

and either $(-a_1,-1)$ or $(-a_6,-1)$ is a period vector for $f\restriction V_{y_0}$. Henceforth we assume that $i\in\{1,6\}$ is chosen such that $(-a_i,-1)$ is a period vector for $f\restriction V_{y_0}$.

By (31), there exists $j<y_0$ such that $(T^{(0,j)}f)\restriction B$ is horizontally periodic of period at most $|w_2\cap\mathcal{S}|-1$. Since $(-a_i,-1)$ is a period vector for $f\restriction V_{y_0}$, it follows that the horizontal period of $f\restriction V_{y_0}$ is at most $|w_2\cap\mathcal{S}|-1$. By (30), $f\restriction V_1$ cannot be extended to a periodic coloring of $\mathbb{Z}^2$ with period vector parallel to w_i. It follows that $f\restriction V_{y_0+1}$ is not doubly periodic (if $y_0<0$ this follows from the definition of y_0 and if $y_0=0$ from (30)). Let $p_1\in V(\mathcal{S})$ be the vertex at the intersection of the edges w_1 and w_2 and let $p_2\in V(\mathcal{S})$ be the vertex at the intersection of the edges w_2 and w_3. Since $\mathcal{S}$ is η-generating, if there exists $j\in\{1,2\}$ and $x\in\mathbb{Z}$ such that $(T^{-(x,y_0-1)}f)\restriction\mathcal{S}\setminus p_j$ coincides with $(T^{-(x-a_i,y_0-2)}f)\restriction\mathcal{S}\setminus p_j$, then $f\restriction V_{y_0+1}$ is doubly periodic, a contradiction. It follows that for all $m\in\mathbb{Z}$, there exists $x\in\{m,m+1,\dots,m+|w_2\cap\mathcal{S}|-2\}$ such that

$$f(x,y_0+1)\neq f(x-a_i,y_0). \tag{39}$$

Let $\mathcal{S}_R$ be as in (32). By (29), every horizontal line that has nonempty intersection with $\mathcal{S}$ intersects in at least $|w_1\cap\mathcal{S}|$ integer points, and so every such line intersects $\mathcal{S}_R$ in at least $|w_1\cap\mathcal{S}|-1$ integer points.

We claim that there are at least three distinct η-colorings of $\mathcal{S}_R$ which extend non-uniquely to an η-coloring of $\mathcal{S}$.

First by (37), there exists $x\in\mathbb{Z}$ such that $(T^{-(x,y_0-2)}f)\restriction\mathcal{S}_R$ does not extend uniquely to an η-coloring of $\mathcal{S}$ and by (38) this coloring of $\mathcal{S}_R$ is periodic with period vector $(-a_i,-1)$. Thus there is an η-coloring of $\mathcal{S}_R$ that does not extend uniquely to an η-coloring of $\mathcal{S}$ and this coloring is periodic with period vector $(-a_i,-1)$.

Second, consider the set of colorings of $\mathcal{S}_R$ of the form $(T^{-(x,y_0-1)}f)\restriction\mathcal{S}_R$. By (37), there exists $x_{y_0-1}\in\mathbb{Z}$ such that $(T^{-(x_{y_0-1},y_0-1)}f)\restriction\mathcal{S}_R$ does not extend uniquely to an η-coloring of $\mathcal{S}$. By (39), there exists a integer point $(x,2)\in w_2$ such that $(T^{-(x_{y_0-1},y_0-1)}f)\restriction\mathcal{S}_R(x,2)\neq(T^{-(x_{y_0-1},y_0-1)}f)\restriction\mathcal{S}_R(x-a_i,1)$ but the bottom two horizontal lines of $\mathcal{S}$ are periodic with period vector $(-a_i,-1)$ by (38). Therefore this coloring is distinct from the first coloring of $\mathcal{S}_R$.

Third, consider the set of colorings of $\mathcal{S}_R$ of the form $(T^{-(x,y_0)}f)\restriction\mathcal{S}_R$. Again by (37), there exists $x_{y_0}\in\mathbb{Z}$ such that $(T^{-(x_{y_0},y_0)}f)\restriction\mathcal{S}_R$ does not extend uniquely to an η-coloring of $\mathcal{S}$. By (39), there exists an integer point $(x,0)\in w_5$ such that $(T^{-(x_{y_0},y_0)}f)\restriction\mathcal{S}_R(x,0)\neq(T^{-(x_{y_0},y_0)}f)\restriction\mathcal{S}_R(x+a_i,1)$. Therefore this coloring is distinct from the first two colorings. Thus we have three distinct η-colorings of $\mathcal{S}_R$ which extend non-uniquely to an η-coloring of $\mathcal{S}$.

But since $\mathcal{S}$ satisfies (1), we have $D_\eta(\mathcal{S}_R)>D_\eta(\mathcal{S})$. By definition, $|\mathcal{S}_R|=|\mathcal{S}|-3$, and so we have $P_\eta(\mathcal{S})\le P_\eta(\mathcal{S}_R)+2$. Therefore there are at most two colorings of $\mathcal{S}_R$ that extend non-uniquely to an η-coloring of $\mathcal{S}$, a contradiction. $\square$

4. Completing the proof of the main theorem

We recall the statement of Theorem 1.1:

Theorem. *Suppose $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$ and there exists $n \in \mathbb{N}$ such that $P_\eta(n,3) \leq 3n$. Then η is periodic.*

Proof. Suppose there exists $n \in \mathbb{N}$ such that $P_\eta(n,3) \leq 3n$. By Proposition 2.9 there exists an η-generating set $\mathcal{S} \subset R_{n,3}$. Since $\mathcal{S}$ is convex and the endpoints of any edge of $\partial\mathcal{S}$ are integer points in $R_{n,3}$, $E(\mathcal{S})$ has at most six edges. Also by Proposition 2.9 every nonexpansive direction is parallel to an edge in $E(\mathcal{S})$, and so there are at most six nonexpansive directions for η. By Proposition 2.11, every nonexpansive line has an orientation that determines a nonexpansive direction. By Proposition 2.19, the direction antiparallel to any nonexpansive direction is also nonexpansive (i.e. if ℓ is a nonexpansive line then both orientations on ℓ determine nonexpansive directions). Therefore there are at most three nonexpansive lines for η.

There are four cases to consider. If there are no nonexpansive lines for η, then η is doubly periodic by Theorem 1.2. If there is exactly one nonexpansive line for η, then η is singly (but not doubly) periodic by Theorem 1.3. If there are exactly two nonexpansive lines for η, then Proposition 3.4 implies that $P_\eta(R_{n,3}) > 3n$, a contradiction. If there are exactly three nonexpansive lines for η, then Proposition 3.7 implies that $P_\eta(R_{n,3}) > 3n$, again a contradiction. The theorem follows. □

5. Further Directions

Sander and Tijdeman [11] conjectured that for $\eta\colon \mathbb{Z}^2 \to \mathcal{A}$, if there exists a finite and convex set $\mathcal{S} \subset \mathbb{Z}^2$ such that $P_\eta(\mathcal{S}) \leq |\mathcal{S}|$, then η is periodic. Their result in [12] shows that this conjecture holds for rectangles $R_{n,2}$ of height 2. More generally, rephrasing their arguments in our language, their proof also covers more convex shapes of height 2. Namely, if $\mathcal{S} \subset \mathbb{Z}^2$ is a finite set that is the restriction of a convex set in $\mathbb{R}^2$ to $\mathbb{Z}^2$ satisfying $P_\eta(\mathcal{S}) \leq |\mathcal{S}|$ and such that $\mathcal{S}$ is contained in the union of two adjacent parallel rational lines, then η is periodic. The construction of a generating set works in the more general setting of such a shape $\mathcal{S}$, and results in a generating set with 3 or 4 edges, and with the possible exception of a single direction (the analog of horizontal) it is balanced. There can be at most 2 nonexpansive directions, and we eliminate the case of 2 in a similar manner to that done for rectangular shapes.

However, in height 3, we are unable to generalize our result of Theorem 1.1 to prove the analog for more general convex shapes with a restriction on the height, meaning a convex shape contained in a strip of width 3. While the construction of generating sets passes through, resulting in generating sets with at most 6 edges, we are not able to show that they are balanced in all (but perhaps the analog of the horizontal) directions. This is the only hurdle remaining for completing a more general result for configurations of height 3.

For more general rectangles $R_{n,k}$ with $k \geq 4$, the construction of generating sets, once again, is general. Again, a problem arises with proving the existence of balanced sets. Furthermore, the counting of configurations seems to be significantly more difficult in the absence of the simple geometry available in height 3.

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