NIL-BOHR SETS OF INTEGERS

BERNARD HOST AND BRYNA KRA

ABSTRACT. We study relations between subsets of integers that are large, where large can be interpreted in terms of size (such as a set of positive upper density or a set with bounded gaps) or in terms of additive structure (such as a Bohr set). Bohr sets are fundamentally abelian in nature and are linked to Fourier analysis. Recently it has become apparent that a higher order, nonabelian, Fourier analysis plays a role both in additive combinatorics and in ergodic theory. Here we introduce a higher order version of Bohr sets and give various properties of these objects, generalizing results of Bergelson, Furstenberg, and Weiss.

1. INTRODUCTION

1.1. **Additive combinatorics and Bohr sets.** Additive combinatorics is the study of structured subsets of integers, concerned with questions such as what one can say about sets of integers that are large in terms of size or about sets that are large in terms of additive structure. An interesting problem is finding various relations between classes of large sets.

Sets with positive upper Banach density or syndetic sets[1] are examples of sets that are large in terms of size. A simple result relates these two notions: if $A \subset \mathbb{Z}$ has positive upper Banach density, then the set of differences $\Delta(A) = A - A = \{a - b \colon a, b \in A\}$ is syndetic.

An example of a structured set is a Bohr set. Following a modification of the traditional definition introduced in [2], we say that a subset $A \subseteq \mathbb{Z}$ is a Bohr set if there exist $m \in \mathbb{N}$, $\alpha \in \mathbb{T}^m$, and an open set

The first author was partially supported by the Institut Universitaire de France and the second by NSF grant 0900873 and by the Clay Mathematics Institute. This work was begun during the visit of the authors to MSRI and completed while the second author was a visitor at Institut Henri Poincaré; we thank the institutes for their hospitality.

[1]If $A \subset \mathbb{Z}$, the *upper Banach density* $d^*(A)$ is defined to be

$$\limsup_{b_n - a_n \to \infty} \frac{|A \cap [a_n, b_n]|}{b_n - a_n} .$$

A set $A \subset \mathbb{Z}$ is said to be *syndetic* if it intersects every sufficiently large interval.

$U \subset \mathbb{T}^m$ such that

$$\{n \in \mathbb{Z} \colon n\alpha \in U\}$$

is contained in A (see Definition 2.1). It is easy to check that the class of Bohr sets is closed under translations.

Most of the notions of a large set that are defined solely in terms of size are also closed under translation. However, we have important classes of structured sets that are not closed under translation. One particular example is that of a Bohr_0-set ([2], [4]): a subset $A \subseteq \mathbb{Z}$ is a Bohr_0-set if it is a Bohr set such that the set U in the previous definition contains 0.

A simple application of the pigeonhole principle gives that if S is an infinite set of integers, then $S-S$ has nontrivial intersection with every Bohr_0-set. This is another example of largeness: a set is large if it has nontrivial intersection with every member of some class of sets. Such notions of largeness are generally referred to as dual notions and are denoted with a star. For example, a Δ^*-set is a set that has nontrivial intersection with the set of differences $\Delta(A)$ from any infinite set A.

Here we study converse results. If a set intersects every set of a given class, then our goal is to show that it has some sort of structure. Such theorems are not, in general, exact converses of the direct structural statements. For example, there exist Δ^*-sets that are not Bohr_0-sets (see [2]). But, this statement is not far from being true. Strengthening a result of [2], we show (Theorem 2.8) that a Δ^*-set is a piecewise Bohr_0-set, meaning that it agrees with a Bohr_0-set on a sequence of intervals whose lengths tend to infinity.

1.2. **Nil-Bohr sets.** Bohr sets are fundamentally linked to abelian groups and Fourier analysis. In the past few years, it has become apparent in both ergodic theory and additive combinatorics that nilpotent groups and a higher order Fourier analysis play a role (see, for example, [6], [8], and [7]). As such, we define a d-step nil-Bohr_0-set, analogous to the definition of a Bohr_0-set, but with a nilmanifold replacing the role of an abelian group (see Definition 2.3). For $d = 1$, the abelian case, this is exactly the object studied in [2]. Here we generalize their results for $d \geq 1$.

We obtain a generalization of Theorem 2.8 on different sets, introducing the idea of a set of sums with gaps. For an integer $d \geq 1$ and an infinite sequence $P = (p_i \colon i \geq 1)$ in $\mathbb{N}$, the *set of sums with gaps of length* $< d$ of P is defined to be the set $\mathrm{SG}_d(P)$ of all integers of the form

$$\epsilon_1 p_1 + \epsilon_2 p_2 + \cdots + \epsilon_n p_n \ ,$$

where $n \geq 1$ is an integer, $\epsilon_i \in \{0,1\}$ for $1 \leq i \leq n$, the ϵ_i are not all equal to 0, and the blocks of consecutive 0's between two 1's have length $< d$.

We remark that P is considered as a sequence, and not a set of integers. We do not assume that the p_i are distinct, nor do we assume that the sequence $(p_i : i \geq 1)$ is increasing.

Our main result (Theorem 2.6) is that a set with nontrivial intersection with any SG_d-set is a piecewise d-step nilpotent Bohr_0-set.

Acknowledgment. Hillel Furstenberg introduced us to this problem and we thank him for his encouragement, as well as for helpful comments on a preliminary version of this article.

2. Precise statements of definitions and results

2.1. **Bohr sets and Nil-Bohr sets.** We formally define the objects described in the introduction:

Definition 2.1. A subset $A \subseteq \mathbb{Z}$ is a Bohr set if there exist $m \in \mathbb{N}$, $\alpha \in \mathbb{T}^m$, and an open set $U \subset \mathbb{T}^m$ such that

$$\{n \in \mathbb{Z} : n\alpha \in U\}$$

is contained in A; the set A is a Bohr_0-set if additionally $0 \in U$.

Note that these sets can also be defined in terms of the topology induced on $\mathbb{Z}$ by embedding the integers into the Bohr compactification: a subset of $\mathbb{Z}$ is Bohr if it contains a nonempty open set in the induced topology and is Bohr_0 if it contains an open neighborhood of 0 in the induced topology.

We can generalize the definition of a Bohr_0-set for return times in a nilsystem, rather than just in a torus. We first give a short definition of a nilsystem and refer to Section 3.2 for further properties.

Definition 2.2. If G is a d-step nilpotent Lie group and $\Gamma \subset G$ is a discrete and cocompact subgroup, the compact manifold $X = G/\Gamma$ is a *d-step nilmanifold.* The *Haar measure* μ of X is the unique probability measure that is invariant under the action $x \mapsto g \cdot x$ of G on X by left translations.

If T denotes left translation on X by a fixed element of G, then (X, μ, T) is a *d-step nilsystem.*

Using neighborhoods of a point, we define a generalization of a Bohr set:

Definition 2.3. A subset $A \subseteq \mathbb{Z}$ is a $\mathrm{Nil}_d\,\mathrm{Bohr}_0$-set if there exist a d-step nilsystem (X, μ, T), $x_0 \in X$, and an open set $U \subset X$ containing x_0 such that

$$\{n \in \mathbb{Z} \colon T^n x_0 \in U\}$$

is contained in A.

Similar to the Bohr compactification of $\mathbb{Z}$ that can be used to define the Bohr sets, there is a *d-step nilpotent compactification* of $\mathbb{Z}$ that can be used to define the $\mathrm{Nil}_d\,\mathrm{Bohr}_0$-sets. This compactification is a non-metric compact space $\widehat{Z}$, endowed with a homeomorphism T and a particular point $\widehat{x_0}$ with dense orbit, and is characterized by the following properties:

i) Given any d-step nilsystem (Z, T) and a point $x_0 \in Z$, there is a unique factor map $\pi_Z \colon \widehat{Z} \to Z$ with $\pi_Z(\widehat{x_0}) = x_0$.
ii) The topology of $\widehat{Z}$ is spanned by these factor maps π_Z.

Remark. A Bohr_0-set can be defined in terms of almost periodic sequences. In the same way, a $\mathrm{Nil}_d\,\mathrm{Bohr}_0$-set can be defined in terms of some particular sequences, the d-step *nilsequences*. Since $\mathrm{Nil}_d\,\mathrm{Bohr}_0$-sets are defined locally, it seems likely that they can be defined by certain particular types of nilsequences, namely those arising from generalized polynomials without constant terms. We do not address this issue here.

2.2. **Piecewise versions.** If $\mathcal{F}$ denotes a class of subsets of integers, various authors, for example Furstenberg in [5] and Bergelson, Furstenberg, and Weiss in [2], define a subset A of integers to be a *piecewise-$\mathcal{F}$ set* if A contains the intersection of a sequence of arbitrarily long intervals and a member of $\mathcal{F}$. For example, the notions of piecewise-Bohr set, a piecewise-Bohr_0-set, and a piecewise-$\mathrm{Nil}_d\,\mathrm{Bohr}_0$-set, can be defined in this way.

However, the notion of a piecewise set is rather weak: for example, a piecewise-Bohr set defined in this manner is not necessarily syndetic. The properties that we can prove are stronger than the traditional piecewise statements, and in particular imply the traditional piecewise versions. For this, we introduce a stronger definition of piecewise:

Definition 2.4. Given a class $\mathcal{F}$ of subsets of integers, the set $A \subset \mathbb{Z}$ is said to be *strongly piecewise-$\mathcal{F}$*, written PW-$\mathcal{F}$, if for every sequence $(J_k \colon k \geq 1)$ of intervals whose lengths $|J_k|$ tend to ∞, there exists a sequence $(I_j \colon j \geq 1)$ of intervals satisfying:

i) For each $j \geq 1$, there exists some $k = k(j)$ such that the interval I_j is contained in J_k;

ii) The lengths $|I_j|$ tend to infinity;
iii) There exists a set $\Lambda \in \mathcal{F}$ such that $\Lambda \cap I_j \subset A$ for every $j \geq 1$.

Note that Λ depends on the sequence $(J_k \colon k \geq 1)$. With this definition of strongly piecewise, if the class $\mathcal{F}$ consists of syndetic sets then every PW-$\mathcal{F}$-set is syndetic. In particular, a strongly piecewise-Bohr set, denoted PW-Bohr, is syndetic. Similarly, we denote a strongly piecewise-Bohr_0- set by PW-Bohr_0 and a strongly piecewise-$\mathrm{Nil}_d\,\mathrm{Bohr}_0$-set by PW-$\mathrm{Nil}_d\,\mathrm{Bohr}_0$ and these sets are also syndetic.

2.3. **Sumsets and Difference Sets.**

Definition 2.5. Let $E \subset \mathbb{N}$ be a set of integers. The *sumset of E* is the set $\mathrm{S}(E)$ consisting of all nontrivial finite sums of distinct elements of E.

A subset A of $\mathbb{N}$ is a S^*_r*-set* if $A \cap \mathrm{S}(E) \neq \emptyset$ for every set $E \subset \mathbb{N}$ with $|E| = r$.

We have:

Theorem 2.6. *Every S^*_{d+1}-set is a* PW-$\mathrm{Nil}_d\,\mathrm{Bohr}_0$*-set.*

For clarity, we include some examples of these objects.

Example (An S^*_2-set)**.** Let $r \in (1/3, 1/2)$ be real, $\alpha \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ be irrational, and

$$A = \{n \in \mathbb{N} \colon n\alpha \in (-r, r) \bmod 1\}\ .$$

Then we claim that A is a S^*_2-set. Any S_2-set is a set of the form $\{m, n, m+n\}$ for some distinct, positive integers m and n. If $m \notin A$ and $m + n \notin A$, then

$$n\alpha \bmod 1 \in \big(\mathbb{T} \setminus (-r, r)\big) - \big(\mathbb{T} \setminus (-r, r)\big) = [2r-1, 1-2r] \subset (-r, r)$$

and so $n \in A$.

Example (A $\mathrm{Nil}_2 - \mathrm{Bohr}_0$ set which is not an S^*_2-set)**.** Let $r \in (0, 1/2)$ be real, $\alpha \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ be irrational, and

$$B = \{n \in \mathbb{N} \colon n^2\alpha \in (-r, r) \bmod 1\}\ .$$

Then B is a $\mathrm{Nil}_2\,\mathrm{Bohr}_0$-set, as can be checked by considering the transformation on $\mathbb{T}^2$ defined by $(x, y) \mapsto (x + \alpha, y + x)$. On the other hand, by the Weyl Equidistribution Theorem, the set

$$\big\{(m^2\alpha, n^2\alpha, (m+n)^2\alpha) \colon 1 \leq m < n\big\}$$

is dense in $\mathbb{T}^3$ and so there exist distinct $m, n \in \mathbb{N}$ such that $m^2\alpha \notin (-r, r)$, $n^2\alpha \notin (-r, r)$, and $(m+n)^2\alpha \notin (-r, r)$. Therefore B is not an S^*_2-set.

Example (An S_3^*-set which is not an S_2^*-set)**.** We claim that for $r \in (3/7, 1/2)$, the set B defined above is an S_3^*-set. Indeed, if the integers $m, n, p, m+n, m+p$, and $n+p$ do not belong to B, then $(m+n)^2\alpha - n^2\alpha$, $(n+p)^2\alpha - p^2\alpha$, and $(m+p)^2\alpha - m^2\alpha$ belong to $[2r-1, 1-2r]$. Using the identity

$$(m+n+p)^2 = \big((m+n)^2 - n^2\big) + \big((n+p)^2 - p^2\big) + \big((m+p)^2 - m^2\big) \ ,$$

we have that $(m+n+p)^2\alpha \in [3(2r-1), 3(1-2r)] \subset (-r, r)$ and $m+n+p \in B$.

We can iterate Theorem 2.6, leading to the following definitions from [2] and [5]:

Definition 2.7. If S is a nonempty subset of $\mathbb{N}$, define the *difference set* $\Delta(S)$ by

$$\Delta(S) = (S - S) \cap \mathbb{N} = \{b - a \colon a \in S,\ b \in S,\ b > a\} \ .$$

If A is a subset of $\mathbb{N}$, A is a Δ_r^**-set* if $A \cap \Delta(S) \neq \emptyset$ for every subset S of $\mathbb{N}$ with $|S| = r$; A is a Δ^**-set* if $A \cap \Delta(S) \neq \emptyset$ for every infinite subset S of $\mathbb{N}$.

Theorem 2.8. *Every Δ^*-set is a* PW-Bohr_0*-set.*

Every Δ_r^* set is obviously a Δ^*-set and Theorem 2.8 generalizes Theorem II of [2], where it is shown that every Δ_r^* set is a PW-Bohr_0-set. The class of sets of the form $\Delta(S)$ with $|S| = 3$ coincides with the class of sets of the form $\mathrm{S}(E)$ with $|E| = 2$ and thus the classes Δ_3^* and S_2^* are the same. Theorem 2.8 generalizes the case $d = 1$ of Theorem 2.6.

The converse statement of Theorem 2.8 does not hold. However, it is easy to check that every Bohr_0-set is a Δ^*-set (see [2]).

Definition 2.9. Let $d \geq 0$ be an integer and let $P = (p_i)$ be a (finite or infinite) sequence in $\mathbb{N}$. The *set of sums with gaps of length* $< d$ of P is the set $\mathrm{SG}_d(P)$ of all integers of the form

$$\epsilon_1 p_1 + \epsilon_2 p_2 + \cdots + \epsilon_n p_n \ ,$$

where $n \geq 1$ is an integer, $\epsilon_i \in \{0, 1\}$ for $1 \leq i \leq n$, the ϵ_i are not all equal to 0, and the blocks of consecutive 0's between two 1's have length $< d$.

A subset $A \subseteq \mathbb{N}$ is an SG_d^*-set if $A \cap \mathrm{SG}_d(P) \neq \emptyset$ for every infinite sequence P in $\mathbb{N}$.

Note that in this definition, P is a sequence and not a subset of $\mathbb{N}$.

For example, if $P = \{p_1, p_2, \dots\}$, then $\mathrm{SG}_1(P)$ is the set of all sums $p_m + \cdots + p_n$ of consecutive elements of P, and thus it coincides with the set $\Delta(S)$ where $S = \{s, s+p_1, s+p_1+p_2, \dots\}$. Therefore SG_1^*-sets are the same as Δ^*-sets.

For a sequence P, $\mathrm{SG}_2(P)$ consists of all sums of the form

$$\sum_{i=m_0}^{m_1} p_i + \sum_{i=m_1+2}^{m_2} p_i + \cdots + \sum_{i=m_{k-1}+2}^{m_k} p_i + \sum_{i=m_k+2}^{m_{k+1}} p_i \ ,$$

where $k \in \mathbb{N}$ and $m_0, m_1, \dots, m_{k+1}$ are positive integers satisfying $m_{i+1} \geq m_i + 2$ for $i = 0, \dots, k$.

Theorem 2.10. *Every* SG_d^**-set is a* PW-$\mathrm{Nil}_d\,\mathrm{Bohr}_0$*-set.*

Since SG_1^*-sets are the same as Δ^*-sets, Theorem 2.10 generalizes Theorem 2.8. If $|P| = d+1$, then $\mathrm{SG}_d(P) = \mathrm{S}(P)$ and thus Theorem 2.10 generalizes Theorem 2.6.

In general, a $\mathrm{Nil}_d\,\mathrm{Bohr}_0$-set is not a Δ^*-set. To construct an example, take an irrational α and let B be the set of $n \in \mathbb{Z}$ such that $n^2\alpha$ is close to $0 \mod 1$. Then B is a $\mathrm{Nil}_2\,\mathrm{Bohr}_0$-set. On the other hand, by induction we can build an increasing sequence n_j of integers such that $n_j^2\alpha$ is close to $1/3 \mod 1$, while $n_i n_j \alpha \mod 1$ is close to 0 for $i < j$. Taking S to be the set of such n_j, we have that $\Delta(S)$ does not intersect B.

This leads to the following question:

Question 2.11. *Is every* $\mathrm{Nil}_d\,\mathrm{Bohr}_0$*-set an* SG_d^**-set?*

As already noted, the answer to this question is positive for $d = 1$ and we conjecture that it is positive in general.

As our characterizations of the sets SG_d and the class SG_d^* are complicated, we ask the following:

Question 2.12. *Find an alternate description of the sets* SG_d *and of the class* SG_d^*.

2.4. **The method.** The first ingredient in the proof is a modification and extension of the Furstenberg Correspondence Principle. The classical Correspondence Principle gives a relation between sets of integers and measure preserving systems, relating the size of the sets of integers to the measure of some sets of the system. It does not give relations between structures in the set of integers under consideration and ergodic properties of the corresponding system. Some information of this type is provided by our modification (originally introduced in [9]).

We then are left with studying certain properties of the systems that arise from this correspondence. As in several related problems, the properties of the system that we need are linked to certain factors of the system, which are nilsystems. This method and these factors were introduced in the study of convergence of some multiple ergodic averages in [8].

Working within these factor systems, we conclude by making use of techniques for the analysis of nilsystems that have been developed over the last few years. In the abelian setting, a fundamental tool is the Fourier transform, but no analog exists for higher order nilsystems[2]. Another classical tool available in the abelian case is the convolution product, but this too is not defined for general nilsystems. Instead, in Section 5 we build some spaces and measures that take on the role of the convolution. As an example, if G is a compact abelian group we can consider the subgroup

$$\{(g_1, g_2, g_3, g_4) \in G^4 \colon g_1 + g_2 = g_3 + g_4\}$$

of G^4, and we take integrals with respect to its Haar measure. This replaces the role of the convolution product.

These constructions are then used to prove the key convergence result (Proposition 6.4). By studying the limit, Theorem 2.6 is deduced in Section 7. By further iterations, Theorems 2.8 and 2.10 are proved in Section 8.

The strategy used in the proofs of our main results (Theorem 2.6 and 2.10) requires the use of substantial technical machinery. Although Theorem 2.8 is a particular case of Theorem 2.10, to help the reader to understand the main ideas of the paper we include a short proof of this result in Section 4. As well, we use this to point out the differences between this simpler setting and the general context. This proof is almost self-contained, as it only relies on the "modified Correspondence Principle" of Section 3.4.

3. Preliminaries

3.1. **Notation.** We introduce notation that we use throughout the remainder of the article.

If X is a set and $d \geq 1$ is an integer, we write $X^{[d]} = X^{2^d}$ and we index the 2^d copies of X by $\{0,1\}^d$. Elements of $X^{[d]}$ are written as

$$\mathbf{x} = (x_\epsilon \colon \epsilon \in \{0,1\}^d) \ .$$

We write elements of $\{0,1\}^d$ without commas or parentheses.

We also often identify $\{0,1\}^d$ with the family $\mathcal{P}([d])$ of subsets of $[d] = \{1, 2, \dots, d\}$. In this identification, $\epsilon_i = 1$ is the same as $i \in \epsilon$ and $\emptyset = 00\dots0$.

[2]The theory of representations does not help us, as the interesting representations of a nilpotent Lie group are infinite dimensional.

For $\epsilon \in \{0,1\}^d$ and $n \in \mathbb{Z}^d$, we write $|\epsilon| = \epsilon_1 + \ldots + \epsilon_d$ and $\epsilon \cdot n = \epsilon_1 n_1 + \ldots + \epsilon_d n_d$.

If $p\colon X \to Y$ is a map, then we write $p^{[d]}\colon X^{[d]} \to Y^{[d]}$ for the map $(p, p, \ldots, p)$ taken 2^d times. In particular, if T is a transformation on the space X, we define $T^{[d]}\colon X^{[d]} \to X^{[d]}$ as $T \times T \times \ldots \times T$ taken 2^d times and we call $T^{[d]}$ the *diagonal transformation*. We define the *face transformations* $T_i^{[d]}$ for $1 \le i \le d$ by:

$$(T_i^{[d]}\mathbf{x})_\epsilon = \begin{cases} T(x_\epsilon) & \text{if } \epsilon_i = 1 \\ x_\epsilon & \text{otherwise} \end{cases}.$$

Thus for $d = 2$, the diagonal transformation is $T \times T \times T \times T$ and the face transformations are $\mathbf{Id} \times T \times \mathbf{Id} \times T$ and $\mathbf{Id} \times \mathbf{Id} \times T \times T$.

In a slight abuse of notation, we denote all transformations, even in different systems, by the letter T (unless the system is naturally a Cartesian product).

For convenience, we assume that all functions are real valued.

3.2. **Review of nilsystems.**

Definition 3.1. If G is a d-step nilpotent Lie group and $\Gamma \subset G$ is a discrete and cocompact subgroup, the compact manifold $X = G/\Gamma$ is a *d-step nilmanifold*. The *Haar measure* μ of X is the unique probability measure that is invariant under the action $x \mapsto g \cdot x$ of G on X by left translations.

If T denotes left translation on X by a fixed element of G, then (X, μ, T) is a *d-step nilsystem*.

(We generally omit the σ-algebra from the notation, writing (X, μ, T) for a measure preserving system rather than $(X, \mathcal{B}, \mu, T)$, where $\mathcal{B}$ denotes the Borel σ-algebra.)

A d-step nilsystem is an example of a topological distal dynamical system. For a d-step nilsystem, the following properties are equivalent: transitivity, minimality, unique ergodicity, and ergodicity. (Note that the first three of these properties refer to the topological system, while the last refers to the measure preserving system.) Also, the closed orbit of a point in a d-step nilsystem, endowed with the restriction of the original transformation, is a d-step nilsystem, and it follows that this closed orbit is minimal and uniquely ergodic. See [1] for proofs and general references on nilsystems.

We also speak of a nilsystem $(X = G/\Gamma, T_1, \ldots, T_d)$, where $T_1, \ldots, T_d$ are translations by commuting elements of G. All the above properties hold for such systems. In particular, every closed orbit is uniquely ergodic under the induced transformations.

We also make use of inverse limits of systems, both in the topological and measure theoretic senses. All inverse limits are implicitly assumed to be taken along sequences. Inverse limits for a sequence of ergodic nilsystems are the same in both the topological and measure theoretic senses: this follows because a measure theoretic factor map between two ergodic nilsystems is necessarily continuous (see for example Appendix A of [10]).

Many properties of the nilsystems also pass to the inverse limit. In particular, in a topological inverse limit of d-step nilsystems, every closed orbit is minimal and uniquely ergodic.

3.3. **Structure Theorem.** Assume now that (X, μ, T) is an ergodic system.

We recall a construction and definitions from [8], but for consistency we make some small changes in the notation. For an integer $d \geq 0$, a measure $\mu^{[d]}$ on $X^{[d]}$ was built in [8]. Here we denote this measure by $\mu^{(d)}$.

The measure $\mu^{(d)}$ is invariant under $T^{[d]}$ and under all the face transformations $T_i^{[d]}$, $1 \leq i \leq d$. Each of the projections of the measure $\mu^{(d)}$ on X is equal to the measure μ.

If f is a bounded measurable function on X, then

$$\int \prod_{\epsilon \subset [d]} f(x_\epsilon)\, d\mu^{(d)}(\mathbf{x}) \geq 0$$

and we define $|||f|||_d$ to be this expression raised to the power $1/2^d$. Then $|||\cdot|||_d$ is a seminorm on $L^\infty(\mu)$. A main result from [8] is that this is a norm if and only if the system is an inverse limit of $(d-1)$-step nilsystems. More precisely, a summary of the Structure Theorem of [8] is:

Theorem 3.2. *Assume that (X, μ, T) is an ergodic system. Then for each $d \geq 2$, there exist a system (Z_d, μ_d, T) and a factor map $\pi_d \colon X \to Z_d$ satisfying:*

i) *(Z_d, μ_d, T) is the inverse limit of a sequence of $(d-1)$-step nilsystems.*
ii) *For each $f \in L^\infty(\mu)$, $|||f - \mathbb{E}(f \mid Z_d) \circ \pi_d|||_d = 0$.*

For each $d \geq 1$, we call (Z_d, μ_d, T) the *HK-factor of order d* of (X, μ, T). The factor map $\pi_d \colon X \to Z_d$ is measurable, and a priori has no reason to be continuous. For $\ell \leq d$, Z_ℓ is a factor of Z_d, with a continuous factor map.

3.3.1. *The case of an inverse limit of nilsystems.* If (X, μ, T) is an inverse limit of $(d-1)$-step ergodic nilsystems, we define[3] $X^{(d)}$ to be the closed orbit in $X^{[d]}$ of a point $\mathbf{x}_0 = (x_0, \dots, x_0)$ (for some arbitrary $x_0 \in X$) under the transformations $T^{[d]}$ and $T_i^{[d]}$ for $1 \le i \le d$.

When (X, μ, T) is an ergodic $(d-1)$-step nilsystem, then $(X^{(d)}, T^{[d]}, T_1^{[d]}, \dots, T_d^{[d]})$ is an ergodic $(d-1)$ nilsystem and the measure $\mu^{(d)}$ described above is its Haar measure ([8], Section 11).

When (X, μ, T) is an inverse limit of $(d-1)$-step ergodic nilsystems, then system $(X^{(d)}, T^{[d]}, T_1^{[d]}, \dots, T_d^{[d]})$ is an inverse limit of ergodic nilsystems. It is minimal, uniquely ergodic and its unique invariant measure is the measure $\mu^{(d)}$.

3.4. **Furstenberg correspondence principle revisited.** By $\ell^\infty(\mathbb{Z})$, we mean the algebra of bounded real valued sequences indexed by $\mathbb{Z}$.

Let $\mathcal{A}$ be a subalgebra of $\ell^\infty(\mathbb{Z})$, containing the constants, invariant under the shift, closed and separable with respect to the norm $\|\cdot\|_\infty$ of uniform convergence. We refer to this simply as "an algebra." In applications, finitely many subsets of $\mathbb{Z}$ are given and $\mathcal{A}$ is the shift invariant algebra spanned by indicator functions of these subsets.

Given an algebra, we associate various objects to it: a dynamical system, an ergodic measure on this system, a sequence of intervals, etc. We give a summary of these objects without proof, referring to [9] for further details.

3.4.1. *A system associated to $\mathcal{A}$.* By Gelfand's representation, there exist a topological dynamical system (X, T) and a point $x_0 \in X$ such that the map

$$\phi \in \mathcal{C}(X) \mapsto \big(\phi(T^n x_0) \colon n \in \mathbb{Z}\big) \in \ell^\infty(\mathbb{Z})$$

is an isometric isomorphism of algebras from $\mathcal{C}(X)$ onto $\mathcal{A}$. (We use $\mathcal{C}(X)$ to denote the collection of continuous functions on X.)

In particular, if S is a subset of $\mathbb{Z}$ with $\mathbf{1}_S \in \mathcal{A}$, then there exists a subset $\widetilde{S}$ of X that is open and closed in X such that

$$\text{for every } n \in \mathbb{Z}, \quad T^n x_0 \in \widetilde{S} \text{ if and only if } n \in S\ . \tag{1}$$

3.4.2. *Some averages and some measures associated to $\mathcal{A}$.* There also exist a sequence $\mathbf{I} = (I_j \colon j \ge 1)$ of intervals of $\mathbb{Z}$, whose lengths tend

[3]We are forced to use different notation from that in [8], as otherwise the proliferation of indices would be uncontrollable.

to infinity, and an invariant ergodic probability measure μ on X such that

$$\text{for every } \phi \in \mathcal{C}(X), \quad \frac{1}{|I_j|}\sum_{n\in I_j} \phi(T^n x_0) \to \int \phi\, d\mu \text{ as } j \to +\infty\ . \tag{2}$$

Given a subset S of $\mathbb{Z}$, we can chose the intervals I_j such that

$$\frac{|S \cap I_j|}{|I_j|} \to d^*(S) \text{ as } j \to +\infty\ ,$$

where $d^*(S)$ denotes the upper Banach density of S.

In particular, we can assume that the intervals I_j are contained in $\mathbb{N}$.

3.4.3. *Notation.* In the sequel, when $a = (a_n \colon n \in \mathbb{Z})$ is a bounded sequence, we write

$$\lim \mathrm{Av}_{n,\mathbf{I}}\, a_n = \lim_{j\to+\infty} \frac{1}{|I_j|}\sum_{n\in I_j} a_n$$

if this limit exists, and set

$$\limsup \bigl|\mathrm{Av}_{n,\mathbf{I}}\, a_n\bigr| = \limsup_{j\to+\infty} \Bigl|\frac{1}{|I_j|}\sum_{n\in I_j} a_n\Bigr|\ .$$

We omit the subscripts n and/or $\mathbf{I}$ if they are clear from the context.

3.4.4. *Averages and factors of order k.* Recall that Z_k denotes the HK-factor of order k of (X, μ, T) and that $\pi_k \colon X \to Z_k$ denotes the factor map.

The sequence of intervals $\mathbf{I} = (I_j \colon j \geq 1)$ can be chosen such that:

Proposition 3.3. *For every $k \geq 1$, there exists a point $e_k \in Z_k$ such that $\pi_{\ell,k}(e_k) = e_\ell$ for $\ell < k$ and such that for every $\phi \in \mathcal{C}(X)$ and every $f \in \mathcal{C}(Z_k)$,*

$$\lim \mathrm{Av}_{\mathbf{I}}\, \phi(T^n x_0) f(T^n e_k) = \int \phi \cdot f \circ \pi_k\, d\mu = \int \mathbb{E}(\phi \mid Z_k)\, f\, d\mu_k\ .$$

This formula extends (2).

The next corollary is an example of the relation between integrals on the factors Z_k and PW- Nil Bohr-sets. More precise results are proved and used in the sequel.

Corollary 3.4. *Let S be a subset of $\mathbb{Z}$ such that $\mathbf{1}_S$ belongs to the algebra $\mathcal{A}$ and let $\widetilde{S}$ be the corresponding subset of X. Let f be a* *nonnegative continuous function on Z_k with $f(e_k) > 0$, where e_k is as in Proposition 3.3. If*

$$\int \mathbf{1}_{\widetilde{S}}(x) \cdot f \circ \pi_k(x)\, d\mu(x) = 0$$

then $\mathbb{Z} \setminus S$ is a PW- $\mathrm{Nil}_k \mathrm{Bohr}_0$*-set.*

Proof. Let $\Lambda = \{n \in \mathbb{Z} \colon f(T^n e_k) > f(e_k)/2\}$. Then Λ is a $\mathrm{Nil}_k \mathrm{Bohr}_0$-set. Indeed, the function f can be approximated uniformly by a function of the form $f' \circ p$, where f' is a continuous function on a k-step nilsystem Z' and $p \colon Z_k \to Z'$ is a factor map. By Proposition 3.3 and definition (1) of $\widetilde{S}$, the averages on I_j of $1_S(n) f(T^n e_k)$ converge to zero. Thus

$$\lim_{j \to +\infty} \frac{|I_j \cap S \cap \Lambda|}{|I_j|} = 0 \ .$$

Therefore, the subset $E = \bigcup_j I_j \setminus (S \cap \Lambda)$ contains arbitrarily long intervals J_ℓ, $\ell \geq 1$. For every ℓ, $J_\ell \cap (\mathbb{Z} \setminus S) \supset J_\ell \cap \Lambda$. □

3.4.5. It is easy to check that given a sequence of intervals $(J_k \colon k \geq 1)$ whose lengths tend to infinity, we can choose the intervals $(I_j \colon j \geq 1)$ satisfying all of the above properties, and such that each interval I_j is a subinterval of some J_k. To see this, we first reduce to the case that the intervals J_k are disjoint and separated by sufficiently large gaps. We set S to be the union of these intervals. We have $d^*(S) = 1$ and we can choose intervals I'_j with $|S \cap I'_j|/|I'_j| \to 1$. For every $j \in \mathbb{N}$, there exists k_j such that $|I'_j \cap J_{k_j}|/|I'_j| \to 1$ as $j \to +\infty$. We set $I_j = I'_j \cap J_{k_j}$ and the sequence $(I_j \colon j \geq 1)$ satisfies all the requested properties.

3.5. **Definition of the uniformity seminorms.** We recall definitions and results of [9] adapted to the present context. We keep notation as in the previous sections; in particular, Z_k and e_k are as in Proposition 3.3.

Let $\mathbf{I}$ be as in Section 3.4 and let $\mathcal{B}$ be the algebra spanned by $\mathcal{A}$ and sequences of the form $(f(T^n e_k) \colon n \in \mathbb{Z})$, where f is a continuous function on Z_k for some k. By Proposition 3.3, for every sequence $a = (a_n \colon n \in \mathbb{Z})$ belonging to the algebra $\mathcal{B}$, the limit $\lim \mathrm{Av}_{\mathbf{I},n}\, a_n$ exists.

Given a sequence $a \in \mathcal{B}$, for $h = (h_1, \ldots, h_d) \in \mathbb{Z}^d$, let

$$c_{\mathbf{h}} = \lim \mathrm{Av}_{\mathbf{I},n} \prod_{\epsilon \subset [d]} a_{n+\epsilon\cdot\mathbf{h}} \ .$$

Then

$$\lim_{H\to\infty}\frac{1}{H^d}\sum_{h_1,\dots,h_d=0}^{H-1} c_{\mathbf{h}}$$

exists and is nonnegative. We define $\|a\|_{\mathbf{I},d}$ to be this limit raised to the power $1/2^d$.

Proposition 3.5. *Let (Z,T) be an inverse limit of k-step nilsystems and f be a continuous function on Z. Then for every $\delta>0$ there exists $C=C(\delta)>0$ such that for every sequence $a=(a_n\colon n\in\mathbb{Z})$ belonging to the algebra $\mathcal{B}$ and for every $z\in Z$,*

$$\operatorname{limsup}\bigl|\operatorname{Av}_{\mathbf{I}} a_n f(T^n z)\bigr| \leq \delta\|a\|_\infty + C\|a\|_{\mathbf{I},k+1}\ .$$

Proof. By density, we can reduce to the case that (Z,T) is a k-step nilsystem and that the function f is smooth.

In this case, the result is contained in [9] under the hypothesis that the system is ergodic. Indeed, by Proposition 5.6 of this paper, f is a "dual function" on X. By the "Modified Direct Theorem" of Section 5.4 in [9], there exists a constant $\|\!|f|\!\|_k^*\geq 0$ with

$$\operatorname{limsup}\bigl|\operatorname{Av}_{\mathbf{I}} a_n f(T^n z)\bigr| \leq \|\!|f|\!\|_k^*\cdot\|a\|_{\mathbf{I},k+1}\ .$$

This gives the announced inequality with $C=\|\!|f|\!\|_k^*$

In the proofs of [9] we can check that the hypothesis of ergodicity is not used. $\square$

The next proposition was proved in Section 3 of [9] and follows from the Structure Theorem (Theorem 3.2).

Proposition 3.6. *Let ϕ be a continuous function on X with $|\phi|\leq 1$, $k\geq 1$ an integer, and f a continuous function on Z_k with $|f|\leq 1$. Then*

$$\bigl\|\bigl(\phi(T^n x_0)-f(T^n e_k)\colon n\in\mathbb{Z}\bigr)\bigr\|_{\mathbf{I},k+1}\leq 2\bigl\|f-\mathbb{E}(\phi\mid Z_k)\bigr\|_{L^1(\mu_k)}^{1/2^{k+1}}\ .$$

4. The case $d=1$

4.1. **The context.** To help the reader understand the main ideas, and as a warm up, we start with a proof of Theorem 2.8. Throughout, we include comments on the differences between the case $d=1$ and the general settings of Theorems 2.6 and 2.10. The proof does not make use of the machinery developed in the rest of the paper other than the modified correspondence principle of Section 3.4.

Let A be a subset of Z and assume that A is not a PW-Bohr_0-set. We show that A is not a Δ^*-set. Since the families of Δ^*-sets and of SG_1^*-sets are the same, it suffices to show that A is not a SG_1^*-set.

4.2. **A system associated to the set A.** We recall the construction of Section 3.4. Let $\mathcal{A}$ be an algebra, in the sense of Section 3.4, containing $\mathbf{1}_A$. Let (X, T), $x_0 \in X$, $\mathbf{I}$, and μ be associated to this algebra as in Section 3.4 and assume that the intervals I_j are included in $\mathbb{N}$. Let f be the continuous function on X associated to the sequence $1 - \mathbf{1}_A$:

$$f(T^n x_0) = 1 \text{ if } n \notin A \ ; \ f(T^n x_0) = 0 \text{ if } n \in A \ .$$

Since A does not contain arbitrarily long intervals, the density of $\mathbb{Z} \setminus A$ in the intervals I_j does not tend to zero. Thus $\int f \, d\mu > 0$.

Let (Z, ν, T) be the Kronecker factor of (X, μ, T) and let $\pi \colon X \to Z$ be the factor map. We recall that Z is a compact abelian group and that ν is its Haar measure. We use additive notation for Z and the transformation $T \colon Z \to Z$ is given by $Tz = z + \alpha$ for some $\alpha \in Z$. For every bounded measurable function ϕ on X, write $\widetilde{\phi} = \mathbb{E}(\phi \mid Z)$.

The element $e_1 \in Z_1 = Z$ in Proposition 3.3 can be chosen to be the unit element 0 of Z, and this proposition states that:

For every continuous function ϕ on X and every continuous function h on Z,

$$\lim \operatorname{Av}_{\mathbf{I}} \phi(T^n x_0) h(n\alpha) = \int \phi \cdot h \circ \pi \, d\mu = \int \widetilde{\phi} \cdot h \, d\nu \ . \tag{3}$$

4.3. **Two positivity results.** We prove a positivity result (this is a reformulation of Lemma 7.1 in the present context):

Claim 4.1. *Let h be a bounded, nonnegative measurable function on the Kronecker (Z, ν) with $\int h \, d\nu > 0$. Then*

$$\int \widetilde{f}(s) h(t) h(s+t) \, d\nu(s) \, d\nu(t) > 0 \ . \tag{4}$$

Proof of Claim 4.1. For $s \in Z$, define

$$H(s) = \int h(s+t) h(t) \, d\nu(t) \ .$$

Then $H(0) > 0$ and H is a continuous function on Z. The subset Λ of $\mathbb{Z}$ defined by

$$\Lambda = \{n \in \mathbb{Z} \colon H(n\alpha) > H(0)/2\}$$

is a Bohr_0-set.

By definition of the functions f and H, we have that

$$\begin{aligned}\int \widetilde{f}(s)h(t)h(s+t)\,d\nu(s)\,d\nu(t)\\ = \int \widetilde{f}(s)H(s)\,d\nu(s) = \lim \operatorname{Av}_{\mathbf{I}} f(T^n x_0)H(n\alpha)\\ \geq \frac{H(0)}{2}\limsup_{j\to+\infty}\frac{\big|(\mathbb{Z}\setminus A)\cap\Lambda\cap I_j\big|}{|I_j|}\ ,\end{aligned}$$

where the middle equality follows from (3). This limsup is not equal to zero, as otherwise the set $A\cup(\mathbb{Z}\setminus\Lambda)$ would contain arbitrarily long intervals J_i, meaning that A would contain $\Lambda\cap J_i$ and would be a PW- Bohr-set. □

In the general case, the functions $\widetilde{f}$ and h are defined on an inverse limit of nilsystems. Since convolution products are not defined in this context, the corresponding result is more difficult to state. The integral in (4) is replaced by an integral with respect to the Haar measure of some submanifold of a Cartesian power of Z_k defined in Section 5.2. The integral defining $H(s)$ is replaced by the integral with respect to the Haar measure of some other submanifold, depending on s, defined in Section 5.3. In the general case, the positivity of $H(0)$ is shown in Proposition 5.3 and the continuity of the function H in Proposition 5.2.

Claim 4.2. *Let h be a continuous nonnegative function on X with $\int h\,d\mu>0$. There exists an integer n, belonging to some interval I_i, with*

$$h(T^n x_0)>0 \text{ and } \int T^n h\cdot f\,d\mu>0\ .$$

Proof of Claim 4.2. Since $\int\widetilde{h}\,d\nu=\int h\,d\mu>0$, by Claim 4.1,

$$\int \widetilde{h}(t)\cdot\Big(\widetilde{f}(s)\widetilde{h}(s+t)\,d\nu(s)\Big)\,d\nu(t)>0\ . \tag{5}$$

The function defined by the inner integral in this formula is continuous on Z and thus using (3) as above, we have that

$$\lim \operatorname{Av}_{n,\mathbf{I}} h(T^n x_0)\int\widetilde{f}(s)\widetilde{h}(s+n\alpha)\,d\nu(s)>0\ .$$

Since $\widetilde{f}$ and $\widetilde{h}$ are the conditional expectations of the functions f and h, respectively, on the Kronecker factor Z of X, we have that

$$\lim \operatorname{Av}_{n,I}\Big|\int\widetilde{f}(s)\widetilde{h}(s+n\alpha)\,d\nu(s)-\int f(x)h(T^n x)\,d\mu(x)\Big|=0 \tag{6}$$

Thus

$$\lim \mathrm{Av}_{n,\mathbf{I}}\, h(T^n x_0) \int f(x) h(T^n x)\, d\mu(x) > 0$$

and the existence of the integer n with the announced properties follows. □

The convergence result (3) used in this case does not suffice for the general case and is replaced by the deeper Proposition 6.4. The proof of this proposition occupies most of Section 6 and uses the "uniformity seminorms" introduced in [9] and whose properties are recalled in Section 3.5. Proposition 6.1 generalizes (6).

4.4. **End of the proof.** By induction, using Claim 4.2 at each step, we define a sequence of positive integers $(n_j \colon j \geq 1)$, such that the functions $h^{(j)}$ on X, defined inductively by

$$h^{(0)} = f \text{ and } h^{(j)} = T^{n_j} h^{(j-1)} \cdot f \text{ for } j \geq 1$$

satisfy

$$h^{j-1}(T^{n_j} x_0) > 0 \text{ and } \int T^{n_j} h^{(j-1)} \cdot f \, d\mu > 0 \text{ for every } j \geq 1\ .$$

By descending induction on i with j fixed, for $1 \leq i \leq j$ we have that $h^{(i-1)}(T^{n_i+n_{i+1}+\cdots+n_j} x_0) > 0$. Thus $f(T^{n_i+n_{i+1}+\cdots+n_j} x_0) > 0$ and so $n_i + n_{i+1} + \cdots + n_j \notin A$. Setting $E = (n_j \colon j \geq 1)$ we have that $A \cap \mathrm{SG}_1(E) = \emptyset$ and A is not an SG_1^*-set. □

The proof of Theorem 2.10 uses a similar, but more intricate, induction.

5. Some measures associated to inverse limits of nilsystems

5.1. **Standing assumptions.** We assume that every topological system (Z, T) is implicitly endowed with a particular point, called the *base point*. Every topological factor map is implicitly assumed to map base point to base point. For every $k \geq 1$, we take the base point of Z_k to be the point e_k introduced in Section 3.4.4.

If (Z, T) is a nilsystem with $Z = G/\Gamma$, then by changing the group Γ if needed, we can assume that the base point of Z is the image in Z of the unit element of G.

5.2. **The measures $\mu_e^{(m)}$.**

Proposition 5.1. *Let (X,μ,T) be an ergodic inverse limit of ergodic k-step nilsystems, endowed with the base point $e\in X$, and let $m\geq 1$ be an integer.*
a) *The closed orbit of the point $e^{[m]}=(e,e,\dots,e)$ of $X^{(m)}$ under the transformations $T_i^{[m]}$, $1\leq i\leq m$, is*
$$X_e^{(m)}=\{\mathbf{x}\in X^{(m)}\colon x_\emptyset=e\}\ .$$
b) *Let $\mu_e^{(m)}$ be the unique measure on this set invariant under these transformations. Then the image of $\mu_e^{(m)}$ under each of the natural projections $\mathbf{x}\mapsto x_\epsilon\colon X^{[m]}\to X$, $\emptyset\neq\epsilon\subset[d]$, is equal to μ.*
c) *Let (Y,ν,T) be an inverse limit of k-step nilsystems and let $p\colon X\to Y$ be a factor map. Then $\nu_e^{(m)}$ is the image of $\mu_e^{(m)}$ under $p^{[m]}\colon X^{[m]}\to Y^{[m]}$.*
d) *Let (Y,ν,T) be the HK-factor of order $(m-1)$ of X and $p\colon X\to Y$ be the factor map. Then the measure $\mu_e^{(m)}$ is relatively independent with respect to $\nu_e^{(m)}$, meaning that when f_ϵ, $\emptyset\neq\epsilon\subset[d]$, are 2^m-1 bounded measurable functions on X,*
$$\int\prod_{\emptyset\neq\epsilon\subset[d]}f_\epsilon(x_\epsilon)\,d\mu_e^{(m)}(\mathbf{x})=\int\prod_{\emptyset\neq\epsilon\subset[d]}\mathbb{E}(f_\epsilon\mid Y)(y_\epsilon)\,d\nu_e^{(m)}(\mathbf{y})\ .$$
(The existence of these integrals follows from b)*.)*

The uniqueness of the measure $\mu_e^{(m)}$ in b) follows from the fact that $X_e^{(m)}$ is a closed orbit in the system $(X^{(m)},T_1^{[m]},\dots,T_m^{[m]})$, which is an inverse limit of nilsystems (Section 3.3.1). Following our convention, we assume in c) and d) that Y is endowed with a base point and that p maps the base point to the base point.

Proof. We first prove a) and d) assuming that X is a nilsystem. (While the proof is contained in [9], we sketch it here in order to introduce some objects and some notation.)

5.2.1. *The nilmanifold and cubes.* Write $X=G/\Gamma$ and let τ be the element of G defining the transformation T of X. We can assume that the base point e of X is the image in X of the unit element of G. Since (X,μ,T) is ergodic, we can also assume that G is spanned by the connected component G_c of the identity and τ (see for example Section 4 of [3]). This implies that the commutator subgroups G_j, $j\geq 2$, are also connected.

As explained in Section 3.3.1, $X^{(m)}$ is a nilmanifold: $X^{(m)}=G^{(m)}/\Gamma^{(m)}$, where $G^{(m)}$ is a subgroup of $G^{[m]}$ and $\Gamma^{(m)}=\Gamma^{[m]}\cap G^{(m)}$. We recall a convenient presentation of $G^{(m)}$ (see Appendix B of [9] or Appendix E of [7]).

For $g \in G$ and $F \subset \mathcal{P}([m])$, we write g^F for the element of $G^{[m]}$ given by

$$\text{for every } \epsilon \subset [m], \quad \big(g^F\big)_\epsilon = \begin{cases} g & \text{if } \epsilon \in F\ ; \\ 1 & \text{otherwise.} \end{cases}$$

Let $\alpha_1, \dots, \alpha_{2^m}$ be an enumeration of all subsets of $[d]$ such that $|\alpha_i|$ is nondecreasing. In particular, $\alpha_1 = \emptyset$. For $1 \leq i \leq 2^m$, let $F_i = \{\epsilon \colon \alpha_i \subset \epsilon \subset [m]\}$. For every i, F_i is an *upper face* of the cube $\mathcal{P}([m])$, meaning a face containing the vertex $[d]$; its *codimension* is $|\alpha_i|$. Then $F_1, \dots, F_{2^m}$ is an enumeration of all the upper faces, in nonincreasing order with respect to codimension. In particular, F_1 is the whole cube $\mathcal{P}([m])$. We also assume that $\alpha_i = \{i-1\}$ for $2 \leq i \leq m+1$.

Then the group $G^{(m)}$ is the subset of $G^{[m]}$ consisting of elements $\mathbf{h}$ that can be written as

$$\mathbf{h} = g_1^{F_1} g_2^{F_2} \dots g_{2^d}^{F_{2^d}}\ , \text{ where } g_i \in G_{|\alpha_i|} \text{ for every } i \tag{7}$$

(where, by convention, $G_0 = G$) and each element of $G^{(m)}$ has a unique expression of this form.

The diagonal transformation $T^{[m]}$ of $X^{(m)}$ is the translation by the element $\tau^{F_1} = \tau^{[m]}$ of $G^{(m)}$ and, for $1 \leq i \leq m$, the i-th face transformation is the translation by the element $\tau_i^{[m]} := \tau^{F_{i+1}}$. Recall that for $\epsilon \subset [m]$,

$$(\tau_i^{[m]})_\epsilon = \begin{cases} \tau & \text{if } i \in \epsilon\ ; \\ 1 & \text{otherwise.} \end{cases}$$

5.2.2. *Proof of a).* We define

$$G_e^{(m)} = \big\{\mathbf{g} \in G^{(m)} \colon g_\emptyset = 1\big\}\ .$$

This group is closed and normal in $G^{(m)}$ and every element of $G^{(m)}$ can be written in a unique way as $h^{[m]}\mathbf{g}$ with $h \in G$ and $\mathbf{g} \in G_e^{(m)}$. Moreover, $G_e^{(m)}$ is the set of elements of $G^{(m)}$ that are written as in (7) with $g_1 = 1$. From this, it is easy to deduce that the commutator subgroup of this group is equal to $G_e^{(m)} \cap (G_2)^{[m]}$.

Clearly, the subset $X_e^{(m)}$ of $X^{(m)}$ is invariant under $G_e^{(m)}$ and it follows from the preceding description that the action of this group on this set is transitive. Therefore, the subgroup

$$\Gamma_e^{(m)} := \Gamma^{(m)} \cap G_e^{(m)}$$

of $G_e^{(m)}$ is cocompact in $G_e^{(m)}$ and we can identify $X_e^{(m)} = G_e^{(m)}/\Gamma_e^{(m)}$.

Since the groups G_j, $j \geq 2$, are connected, the connected component of the identity of $G_e^{(m)}$ contains all elements of the form (7) where $g_1 = 1$ and g_i lies in the connected component of the identity of G for $2 \leq i \leq m+1$. Since G is spanned by the connected component of its identity and τ, $G_e^{(m)}$ is spanned by the connected component of its identity and the elements $\tau^{F_i} = \tau_i^{[m]}$ for $1 \leq i \leq m$.

Moreover, by using the above description of $G_e^{(m)}$, it is not difficult to check that the action induced by $T_i^{[m]}$, $1 \leq i \leq m$ on the compact abelian group $G_e^{(m)}/(G_e^{(m)})_2\Gamma_e^{(m)}$ is ergodic. By a classical criteria [11], the action of the transformations $T_i^{[m]}$ on $X_e^{(m)}$ is ergodic and thus minimal. In particular, $X_e^{(m)}$ is the closed orbit of the point $e^{[m]}$ under these transformations. This proves a).

5.2.3. *Proof of d).* The HK-factor of order $(m-1)$ (Y, ν, T) of $(X = G/\Gamma, \mu, T)$ is $Y = G/\Gamma G_m$ endowed with its Haar measure.

For every $\epsilon \subset [m]$ with $\epsilon \neq \emptyset$ and every $w \in G_m$, we have $w^{\{\epsilon\}} \in G_e^{(m)}$ and thus the Haar measure $\mu_e^{(m)}$ of $X_e^{(m)}$ is invariant under translation by this element. The result follows.

5.2.4. *Proof of the proposition in the general case.* For a), the generalization to inverse limits is immediate.
b) Let $\epsilon \in [d]$ with $\epsilon \neq \emptyset$. Let $i \in \epsilon$. Then for every $\mathbf{x} \in X^{(m)}$ we have $Tx_\epsilon = (T_i^{[m]}\mathbf{x})_\epsilon$. Since the measure $\mu_e^{(m)}$ is invariant under $T_i^{(m)}$, its image under the projection $\mathbf{x} \mapsto x_\epsilon$ is invariant under T and thus is equal to μ.

Property c) is immediate.
d) Let the functions f_ϵ be as in the statement; without loss we can assume that $|f_\epsilon| \leq 1$ for every ϵ.

Let (X_i, μ_i, T_i), $i \geq 1$, be an increasing sequence of k-step nilsystems with inverse limit (X, μ, T) and let $\pi_i \colon X \to X_i$, $i \geq 1$, be the (pointed) factor maps.

For every ϵ, $\emptyset \neq \epsilon \subset [d]$, we have that

$$\big\| f_\epsilon - \mathbb{E}(f_\epsilon \circ X_i) \circ \pi_i \big\|_{L^1(\mu)} \to 0 \text{ as } i \to +\infty$$

and thus

$$\int \prod_{\emptyset \neq \epsilon \subset [d]} \mathbb{E}(f_\epsilon \mid X_i) \circ \pi_i(x_\epsilon) \, d\mu_e^{(m)}(\mathbf{x}) \to \int \prod_{\emptyset \neq \epsilon \subset [d]} f_\epsilon(x_\epsilon) \, d\mu_e^{(m)}(\mathbf{x}) \tag{8}$$

as $i \to +\infty$.

For every i, let (W_i, σ_i, T) be the HK-factor of order $(m-1)$ of X_i, $q_i \colon X_i \to W_i$ the factor map and $r_i = q_i \circ \pi_i$.

We have shown that for every i, the measure $(\mu_i)_e^{(m)}$ is relatively independent with respect to $(\sigma_i)_e^{(m)}$.

Using c) twice, we have that the second integral in (8) is equal to

$$\int \prod_{\emptyset\neq\epsilon\subset[d]} \mathbb{E}(f_\epsilon \mid W_i)\circ r_i(x_\epsilon)\, d\mu_e^{(m)}(\mathbf{x}) \ .$$

As the systems X_i form an increasing sequence, the systems W_i also form an increasing sequence. Let (W,σ,T) be the inverse limit of this sequence. This system is a factor of X, and writing $r\colon X\to W$ for the factor map, we have that $\mathbb{E}(f_\epsilon \mid W_i)\circ r_i \to \mathbb{E}(f_\epsilon \mid W)\circ r$ in $L^1(\mu)$ for every ϵ. We get

$$\int \prod_{\emptyset\neq\epsilon\subset[d]} f_\epsilon(x_\epsilon)\, d\mu_e^{(m)}(\mathbf{x}) = \int \prod_{\emptyset\neq\epsilon\subset[d]} \mathbb{E}(f_\epsilon \mid W)\circ r(x_\epsilon)\, d\mu_e^{(m)}(\mathbf{x}) \ . \tag{9}$$

This means that the measure $\mu_e^{(m)}$ is relatively independent with respect to $\sigma_e^{(m)}$.

Since W is an inverse limit of $(m-1)$-step nilsystems and is a factor of X, it is a factor of the HK-factor Y of order $(m-1)$ of X. If for some ϵ we have $\mathbb{E}(f_\epsilon \mid Y)=0$, then we have $\mathbb{E}(f_\epsilon \mid W)=0$ and the second integral in (9) is equal to zero. The result follows. $\square$

Passing to inverse limits adds technical issues to each proof. These issues are not difficult and the passage to inverse limits uses only routine techniques, as in the preceding proof. However, it does greatly increase the length of the arguments, and so in general we omit this portion of the argument in the sequel.

5.3. **The measures $\mu_{e,x}^{(m)}$.** In this section, again (X,μ,T) is an ergodic inverse limit of k-step nilsystems, with base point $e\in X$.

For $x\in X$ we write

$$X_{e,x}^{(m)} = \{\mathbf{x}\in X^{(m)}\colon x_\emptyset = e \text{ and } x_{\{m\}} = x\} \ .$$

The set $X_{e,e}^{(m)}$ is the image of the set $X_e^{(m,1)}$ introduced below by a permutation of coordinates.

Proposition 5.2. *For each $x\in X$, there exists a measure $\mu_{e,x}^{(m)}$, concentrated on $X_{e,x}^{(m)}$, such that*

i) *The image of $\mu_{e,x}^{(m)}$ under each projection $\mathbf{x}\mapsto x_\epsilon\colon X^{[m]}\to X$, $\epsilon\neq\emptyset$, $\epsilon\neq\{m\}$, is equal to μ.*

ii) *If f_ϵ, $\epsilon\subset[m]$, $\epsilon\not\subset[1]$, are 2^m-2 bounded measurable functions on X, then the function F on X given by*

$$F(x)=\int\prod_{\substack{\epsilon\subset[d]\\ \epsilon\neq\emptyset,\ \epsilon\neq\{m\}}} f_\epsilon(x_\epsilon)\,d\mu^{(m)}_{e,x}(\mathbf{x})$$

is continuous.

iii) *Moreover, for every bounded measurable function f on X,*

$$\int f(x)F(x)\,d\mu(x)=\int f(x_{\{m\}})\prod_{\substack{\epsilon\subset[d]\\ \epsilon\neq\emptyset,[m]}} f_\epsilon(x_\epsilon)\,d\mu^{(m)}_{e}(\mathbf{x})\ .$$

Proof. It suffices to prove this Proposition in the case that (X,μ,T) is k-step nilsystem, as the general case follows by standard methods.

We write $X=G/\Gamma$ as usual. We can assume that e is the image in X of the unit element 1 of G. We define

$$G^{(m)}_{e,e}=\bigl\{\mathbf{g}\in G^{(m)}\colon g_\emptyset=g_{\{m\}}=1\bigr\}\ .$$

This group is closed and normal in G. It is the set of elements of $G^{(m)}$ that can be written as in (7) with $g_\emptyset=1$ and $g_i=1$ for the value of i such that $\alpha_i=\{m\}$. Recall that $e^{[m]}=(e,e,\dots,e)$.

It is easy to check that $G^{(m)}_{e,e}\cdot e^{[m]}=X^{(m)}_{e,e}$. It follows that

$$\Gamma^{(m)}_{e,e}:=\Gamma^{[m]}\cap G^{(m)}_{e,e}$$

is cocompact in $G^{(m)}_{e,e}$ and that $X^{(m)}_{e,e}$ can be identified with the nilmanifold $G^{(m)}_{e,e}/\Gamma^{(m)}_{e,e}$. We write $\mu^{(m)}_{e,e}$ for the Haar measure of this nilmanifold.

Let $F=\{\epsilon\subset[m]\colon m\in\epsilon\}$. We recall that for $g\in G$, $g^F\in G^{(m)}$ is defined by

$$(g^F)_\epsilon=\begin{cases} g & \text{if } m\in\epsilon\\ 1 & \text{otherwise.}\end{cases}$$

By definition of the sets $X^{(m)}_{e,x}$, the image of $X^{(m)}_{e,e}$ under translation by $g^{[m]}_m$ is equal to $X^{(m)}_{e,g\cdot e}$. Since $G^{(m)}_{e,e}$ is normal in $G^{(m)}$, the image of the measure $\mu^{(m)}_{e,e}$ under g^F is invariant under $G^{(m)}_{e,e}$. Moreover, if $g,h\in G$ satisfy $g\cdot e=h\cdot e$, then we have that $g=h\gamma$ for some $\gamma\in\Gamma$. Since $\gamma^F\cdot e^{[m]}=e^{[m]}$ and by normality of $G^{(m)}_{e,e}$ again, the measure $\mu^{(m)}_{e,e}$ is invariant under γ^F and thus the images of $\mu^{(m)}_{e,e}$ under g^F and h^F are the same.

Therefore, for every $x\in X$ we can define a measure $\mu^{(m)}_{e,x}$ on $X^{(m)}_{e,x}$ by

$$\mu^{(m)}_{e,x}=g^F\cdot\mu^{(m)}_{e,e}\ \text{for every } g\in G \text{ such that } g\cdot e=x\ . \tag{10}$$

In particular, for every $h \in G$ and every $x \in X$,

$$\mu^{(m)}_{e,h\cdot x} = h^F \cdot \mu^{(m)}_{e,x} \ . \tag{11}$$

If T is the translation by $\tau \in G$, then $T^{[m]}_m$ is the translation by τ^F and and so for every integer n,

$$\mu^{(m)}_{e,T^n x} = T^{[m]}_m{}^n \cdot \mu^{(m)}_{e,x} \ . \tag{12}$$

For $1 \leq i < m$, $\tau^{[m]}_i \in G^{(m)}_{e,e}$ and thus, for every $x \in X$, $\mu^{(m)}_{e,x}$ is invariant under $T^{[m]}_i$. As above, it follows that this measure satisfies the first property of the proposition.

To prove the other properties, the first statement of the proposition implies that we can reduce to the case that the functions f_ϵ are continuous. By (10), the map $x \mapsto \mu^{(m)}_{e,x}$ is weakly continuous and the function F is continuous. We are left with showing that

$$\mu^{(m)}_e = \int \mu^{(m)}_{e,x} \, d\mu(x) \ .$$

For $1 \leq i < m$, since for every x the measure $\mu^{(m)}_{e,x}$ is invariant under $T^{[m]}_i$, the measure defined by this integral is invariant under this transformation. By (11), $\mu^{(m)}_{e,Tx} = T^{[m]}_m \cdot \mu^{(m)}_{e,x}$ for every x and it follows that the measure defined by the above integral is invariant under $T^{[m]}_m$. Since it is concentrated on $X^{(m)}_e$, it is equal to the Haar measure $\mu^{(m)}_e$ of this nilmanifold (recall that $(X^{(m)}_e, T^{[m]}_1, \dots, T^{[m]}_m)$ is uniquely ergodic). $\square$

5.4. **A positivity result.** In this section, again (X, μ, T) is an ergodic inverse limit of k-step nilsystems, with base point $e \in X$.

In the next proposition, the notation $\epsilon = \epsilon_1 \dots \epsilon_m \in \{0,1\}^m$ is more convenient that $\epsilon \subset [m]$. We recall that $00\dots0 \in \{0,1\}^m$ corresponds to $\emptyset \subset [m]$ and that $00\dots01 \in \{0,1\}^m$ corresponds to $\{m\} \subset [m]$. For $\epsilon \in \{0,1\}^{m+1}$, $\epsilon_1 \dots \epsilon_m$ corresponds to $\epsilon \cap [m]$.

Proposition 5.3. *Let f_ϵ, $\emptyset \neq \epsilon \in \{0,1\}^m$, be $2^m - 1$ bounded measurable real functions on X. Then*

$$\int \prod_{\substack{\epsilon \in \{0,1\}^{m+1} \\ \epsilon \neq 00\dots0 \\ \epsilon \neq 00\dots01}} f_{\epsilon_1 \dots \epsilon_m}(x_\epsilon) \, d\mu^{(m+1)}_{e,e}(\mathbf{x}) \geq \Big(\int \prod_{\substack{\epsilon \in \{0,1\}^m \\ \epsilon \neq 00\dots0}} f_\epsilon(x_\epsilon) \, d\mu^{(m)}_e(\mathbf{x}) \Big)^2 \ .$$

Proof. We first reduce the general case to that of an ergodic k-step nilsystem. If (X,μ,T) is an inverse limit of an increasing sequence of k-step ergodic nilsystems, then the spaces $X_e^{(m)}$ and $X_{e,e}^{(m+1)}$, as well as the measures $\mu_e^{(m)}$ and $\mu_{e,e}^{(m+1)}$, are the inverse limits of the corresponding objects associated to each of the nilsystems in the sequence of nilsystems converging to X. Thus it suffices to prove the proposition when (X,T,μ) is an ergodic k-step nilsystem. We write $X=G/\Gamma$ as usual.

The groups $G_e^{(m)}$, $\Gamma_e^{(m)}$, $G_{e,e}^{(m+1)}$ and $\Gamma_{e,e}^{(m+1)}$ have been defined and studied above. We recall that $X_e^{(m)}=G_e^{(m)}/\Gamma_e^{(m)}$ and that $\mu_e^{(m)}$ is the Haar measure of this nilmanifold. Also, $X_{e,e}^{(m+1)}=G_{e,e}^{(m+1)}/\Gamma_{e,e}^{(m+1)}$ and $\mu_{e,e}^{(m+1)}$ is the Haar measure of this nilmanifold.

It is convenient to identify $X^{[m+1]}$ with $X^{[m]}\times X^{[m]}$, writing a point $\mathbf{x}\in X^{[m+1]}$ as $\mathbf{x}=(\mathbf{x}',\mathbf{x}'')$, where

$$\mathbf{x}'=(x_{\epsilon_1\dots\epsilon_m 0}\colon \epsilon\in\{0,1\}^m) \text{ and } \mathbf{x}''=(x_{\epsilon_1\dots\epsilon_m 1}\colon \epsilon\in\{0,1\}^m)\ .$$

The *diagonal map* $\Delta_X^{(m)}\colon X^{[m]}\to X^{[m+1]}$ is defined by $\Delta_X^{(m)}(\mathbf{x})=(\mathbf{x},\mathbf{x})$, that is,

$$\text{for } \mathbf{x}\in X^{[m]} \text{ and } \epsilon\in\{0,1\}^{m+1},\quad \bigl(\Delta_X^{(m)}(\mathbf{x})\bigr)_\epsilon=x_{\epsilon_1\dots\epsilon_m}\ .$$

We remark that $\Delta_X^{(m)}(X_e^{(m)})\subset X_{e,e}^{(m+1)}$.

We use similar notation for elements of $G^{[m+1]}$ and define the diagonal map $\Delta_G^{(m)}\colon G^{[m]}\to G^{[m+1]}$. We have that

$$\Delta^{(m)}(G_e^{(m)})\subset G_{e,e}^{(m+1)}$$

and, for every $\mathbf{g}=(\mathbf{g}',\mathbf{g}'')\in G_{e,e}^{(m+1)}$ we have that $\mathbf{g}'$ and $\mathbf{g}''$ belong to $G_e^{(m)}$; in other words, $G_{e,e}^{(m+1)}\subset G_e^{(m)}\times G_e^{(m)}$. We define

$$G_*^{(m)}=\{\mathbf{g}\in G_e^{(m)}\colon (1^{[m]},\mathbf{g})\in G_{e,e}^{(m+1)}\}$$

and we have that $G_*^{(m)}$ is a closed normal subgroup of $G_e^{(m)}$ and that

$$G_{e,e}^{(m+1)}=\bigl\{(\mathbf{g},\mathbf{hg})\colon \mathbf{g}\in G_e^{(m)},\ \mathbf{h}\in G_*^{(m)}\bigr\}\ .$$

It follows that

$$X_{e,e}^{(m+1)}=\bigl\{(\mathbf{x},\mathbf{h}\cdot\mathbf{x})\colon \mathbf{x}\in X_e^{(m)},\ \mathbf{h}\in G_*^{(m+1)}\bigr\}\ .$$

For every $\mathbf{x}\in X^{(m)}$, set

$$V_{\mathbf{x}}=\{\mathbf{y}\in X^{(m)}\colon (\mathbf{x},\mathbf{y})\in X_{e,e}^{(m+1)}\}=\{\mathbf{h}\cdot\mathbf{x}\colon \mathbf{h}\in G_*^{(m+1)}\}\ .$$

Then $V_{\mathbf{x}}$ is a nilmanifold, quotient of the nilpotent Lie group $G_*^{(m+1)}$ by the stabilizer of $\mathbf{x}$. Let $\nu_{\mathbf{x}}$ be the Haar measure of this nilmanifold.

For $\mathbf{x} \in X^{(m)}$ and $\mathbf{g} \in G_e^{(m)}$, we have that

$$\text{the image of } \nu_{\mathbf{x}} \text{ under translation by } \mathbf{g} \text{ is equal to } \nu_{\mathbf{g}\cdot\mathbf{x}} \ . \tag{13}$$

Indeed, this image is supported on $V_{\mathbf{g}\cdot\mathbf{x}}$ and is invariant under $G_*^{(m)}$, since $G_*^{(m)}$ is normal in $G_e^{(m)}$.

We claim that

$$\mu_{e,e}^{(m+1)} = \int \delta_{\mathbf{x}} \times \nu_{\mathbf{x}} \, d\mu_e^{(m)}(\mathbf{x}) \ . \tag{14}$$

The measure on $X_{e,e}^{(m+1)}$ defined by this integral is invariant under translation by elements of the form $(1^{[m]}, \mathbf{h})$ with $\mathbf{h} \in G_*^{(m)}$ (note that each $\delta_{\mathbf{x}} \times \nu_{\mathbf{x}}$ is invariant under such translations). By (13), the measure defined by this integral is also invariant under translation by $(\mathbf{g}, \mathbf{g})$ for $\mathbf{g} \in G_e^{(m)}$. Therefore this measure is invariant under $G_{e,e}^{(m+1)}$. Since it is supported on $X_{e,e}^{(m+1)}$, it is equal to the Haar measure $\mu_{e,e}^{(m+1)}$ of this nilmanifold. The claim is proven.

By (13) again, $\nu_{\mathbf{h}\cdot\mathbf{x}} = \nu_{\mathbf{x}}$ for $\mathbf{h} \in G_*^{(m)}$. Let $\mathcal{F}$ denote the σ-algebra of $G_*^{(m)}$-invariant Borel sets. For every bounded Borel function F on $X_e^{(m)}$,

$$\int F \, d\nu_{\mathbf{x}} = \mathbb{E}(F \mid \mathcal{F})(\mathbf{x}) \quad \mu_e^{(m)}\text{-a.e.} \tag{15}$$

To see this, we note that the function defined by this integral is invariant under translation by $G_*^{(m)}$ and thus is $\mathcal{F}$-measurable. Conversely, if F is $\mathcal{F}$-measurable, then for $\mu_e^{(m)}$ almost every $\mathbf{x}$, it coincides $\nu_{\mathbf{x}}$-almost everywhere with a constant and so the integral is equal almost everywhere to $F(\mathbf{x})$.

Thus for a bounded Borel function F on $X_e^{(m)}$, using (14) and (15), we have that

$$\begin{aligned}
\int F(\mathbf{x}')F(\mathbf{x}'') \, d\mu_{e,e}^{(m+1)}(\mathbf{x}) &= \int \Big(F(\mathbf{x}') \int F(\mathbf{x}'') \, d\nu_{\mathbf{x}'}(\mathbf{x}'') \Big) \, d\mu_e^{(m)}(\mathbf{x}') \\
&= \int F \cdot \mathbb{E}(F \mid \mathcal{F}) \, d\mu_e^{(m)} \\
&= \int \mathbb{E}(F \mid \mathcal{F})^2 \, d\mu_e^{(m)} \geq \Big(\int F \, d\mu_e^{(m)} \Big)^2 \ .
\end{aligned}$$

□

5.5. **The measures $\mu_e^{(m,r)}$.** In this section again, (X, μ, T) is an ergodic inverse limit of k-step nilsystems, with base point $e \in X$. Let m and r be integers with $0 \leq r < m$.

Let $\Delta_{m,r}\colon X^{[m-r]} \to X^{[m]}$ be the map given by

for $\mathbf{x} \in X^{[m-r]}$ and $\epsilon = \epsilon_1 \dots \epsilon_m \in \{0,1\}^m$,

$$(\Delta_{m,r}\mathbf{x})_\epsilon = x_{\epsilon_{r+1}\dots\epsilon_m} \ .$$

We define:

$$X_e^{(m,r)} = \Delta_{m,r}\big(X_e^{(m-r)}\big) \text{ and} \tag{16}$$

$$\mu_e^{(m,r)} \text{ is the image of } \mu_e^{(m-r)} \text{ under } \Delta_{m,r} \ . \tag{17}$$

Recall that $X_e^{(m-r)}$ is the closed orbit of $e^{[m-r]}$ under the transformations $T_i^{[m-r]}$ for $1 \leq i \leq m-r$ and that $\mu_e^{(m-r)}$ is the unique probability measure of this set invariant under these transformations. We have $\Delta_{m,r}e^{[m-r]} = e^{[m]}$, and, for $1 \leq i \leq m-r$, $\Delta_{m,r} \circ T_i^{[m-r]} = T_{r+i}^{[m]} \circ \Delta_{m,r}$. Therefore:

$X_e^{(m,r)}$ is the closed orbit of the point $e^{[m]} \in X^{(m)}$ under the transformations $T_i^{[m]}$ for $r+1 \leq i \leq m$ and $\mu_e^{(m,r)}$ is the unique probability measure on this set invariant under these transformations.

For example, $X_e^{(m,0)} = X_e^{(m)} \subset X^{(m)}$ and $\mu_e^{(m,0)} = \mu_e^{(m)}$.

$X_e^{(r+1,r)} = \{e^{[r]}\} \times \Delta^{[r]} \subset X^{(r+1)}$, where $\Delta^{[r]}$ denotes the diagonal of $X^{[r]}$. $\mu_e^{(r+1,r)}$ is the product of the Dirac mass at $e^{[r]}$ by the diagonal measure of $X^{[r]}$.

Since the image of $\mu_e^{(m-r)}$ under the projections $\mathbf{x} \mapsto x_\epsilon$ with $\epsilon \neq \emptyset$, are equal to μ, we have that:

The images of $\mu_e^{(m,r)}$ under the projections $\mathbf{x} \mapsto x_\epsilon$ for $\epsilon \subset [m]$, $\epsilon \not\subset [r]$, are equal to μ.

Therefore, if h_ϵ, $\epsilon \subset [m]$, $\epsilon \not\subset [r]$, are $2^m - 2^r$ measurable functions on X with $|h_\epsilon| \leq 1$, we have that

$$\Big| \int \prod_{\substack{\epsilon \subset [m] \\ \epsilon \not\subset [r]}} h_\epsilon(x_\epsilon) \, d\mu_e^{(m,r)}(\mathbf{x}) \Big| \leq \min_{\substack{\epsilon \subset [m] \\ \epsilon \not\subset [r]}} \|h_\epsilon\|_{L^1(\mu)} \ . \tag{18}$$

6. A convergence result

In this section, we prove the key convergence result (Proposition 6.4).

6.1. **Context.** We recall our context, as introduced in Sections 3.4 and 3.5.

The system (X,T) is associated to the subalgebra $\mathcal{A}$ of $\ell^\infty(\mathbb{Z})$, μ is an ergodic invariant probability measure on X, associated to the averages on the sequence $\mathbf{I} = (I_j \colon j \geq 1)$ of intervals.

For every $k \geq 1$, let (Z_k, μ_k, T) be the factor of order k of (X, μ, T). We recall that this system is an inverse limit of $(k-1)$-step nilsystems, both in the topological and the ergodic theoretical senses. The system (Z_k, T) is distal, minimal and uniquely ergodic, and Z_k is given with a base point e_k. In a futile attempt to keep the notation only mildly disagreeable, when the base point e_k is used as a subindex, we omit the subscript k.

We write $\pi_k \colon X \to Z_k$ for the factor map. We recall that this map is measurable, and has no reason for being continuous. For $\ell \leq k$, Z_ℓ is a factor of Z_k, with a factor map $\pi_{\ell,k} \colon Z_k \to Z_\ell$ which is continuous and $\pi_{\ell,k}(e_k) = e_\ell$.

We use various different methods of taking limits of averages of sequences indexed by $\mathbb{Z}^r$. For example, in Proposition 6.1, we average over any Følner sequence in $\mathbb{Z}^r$. In the sequel, we use iterated limits: if $\big(a_n \colon n = (n_1, \dots, n_r) \in \mathbb{Z}^r\big)$ is a bounded sequence, we define the iterated limsup of a as

$$\begin{aligned}\operatorname{Iter\,limsup} |\operatorname{Av}_{\mathbf{I},n_1,\dots,n_r} a_{n_1,\dots,n_r}| \\ = \limsup_{j_1\to\infty} \dots \limsup_{j_r\to\infty} \frac{1}{|I_{j_1}|\dots|I_{j_r}|} \Big| \sum_{\substack{n_1\in I_{j_1}\\ \vdots \\ n_r\in I_{j_r}}} a_{n_1,\dots,n_r} \Big| \ .\end{aligned}$$

We define the $\operatorname{Iter\,lim\,Av} a_n$ analogously, assuming that all of the limits exist.

6.2. **An upper bound.**
The next proposition is proved in Section 13 of [8]:

Proposition 6.1. *Let (X, μ, T) be an ergodic system and (Z_d, T, ν) be its factor of order d. Let f_ϵ, $\epsilon \subset [d]$, be 2^d bounded measurable functions on X. For $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$, let*

$$a_n = \int \prod_{\epsilon \subset [d]} f_\epsilon(T^{n\cdot\epsilon} x) \, d\mu(x) \ \text{ and } b_n = \int \prod_{\epsilon \subset [d]} \mathbb{E}(f_\epsilon \mid Z_d)(T^{n\cdot\epsilon} z) \, d\mu_d(z) \ .$$

Then $a_n - b_n$ converges to zero in density, meaning that the averages of $[a_n - b_n|$ on any Følner sequence in $\mathbb{Z}^d$ converge to zero.

Lemma 6.2. *Let $k \geq 1$, $0 \leq r \leq d$ and h_ϵ, $\epsilon \subset [d+1]$, $\epsilon \not\subset [r]$, be $2^{d+1} - 2^r$ continuous functions on Z_k. Then for every $\delta > 0$, there exists $C = C(\delta) > 0$ with the following property:*

Let ψ_ϵ, $\epsilon \subset [r]$, be 2^r sequences belonging to $\mathcal{B}$ (as defined in Section 3.5) with absolute value ≤ 1. Then the iterated limsup *in $n_1, \dots, n_r$*

of the absolute value of the averages on $\mathbf{I}$ of

$$A(n) := \prod_{\epsilon\subset[r]} \psi_\epsilon(n\cdot\epsilon) \int \prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}} h_\epsilon(T^{n\cdot\epsilon}x_\epsilon)\, d\mu_{k\ \ e}^{(d+1,r)}(\mathbf{x})$$

is bounded by

$$\delta + C \prod_{r\in\epsilon\subset[r]} \|\psi_\epsilon\|_{\mathbf{I},k+r}\ .$$

Proof. We write $n = (m_1,\dots,m_{r-1},p)$ and $m = (m_1,\dots,m_{r-1})$. The expression to be averaged can be rewritten as

$$A'(m,p) =$$

$$\prod_{\epsilon\subset[r-1]} \psi_\epsilon(m\cdot\epsilon)\cdot \prod_{r\in\epsilon\subset[r]} \psi_\epsilon(m\cdot\epsilon+p)\cdot \int \prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}} h_\epsilon(T^{m\cdot\epsilon+p\epsilon_r}x_\epsilon)\, d\mu_{k\ \ e}^{(d+1,r)}(\mathbf{x})\ ,$$

where in the term $m\cdot\epsilon$, we only use the first $r-1$ coordinates of ϵ. For $m\in\mathbb{Z}^{r-1}$, we write

$$\Phi_m(p) = \prod_{r\in\epsilon\subset[r]} \psi_\epsilon(m\cdot\epsilon+p) = \prod_{r\in\epsilon\subset[r]} \sigma^{m\cdot\epsilon}\psi_\epsilon(p)$$

where σ is the shift on $\ell^\infty(\mathbb{Z})$. For $\mathbf{x}\in Z_k^{(d+1,r)}$, we also write

$$H(\mathbf{x}) = \prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}} h_\epsilon(x_\epsilon)$$

and for every $\delta>0$, we let $C = C(\delta)$ be associated to this continuous function on $X_k^{(d+1,r)}$ as in Proposition 3.5. We have

$$\prod_{r\in\epsilon\subset[r]} \psi_\epsilon(m\cdot\epsilon+p)\cdot \prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}} h_\epsilon(T^{m\cdot\epsilon+p\epsilon_r}x_\epsilon)$$

$$= \Phi_m(p) H\big((T_r^{[d+1]})^p((T_1^{[d+1]})^{m_1}\dots(T_{r-1}^{[d+1]})^{m_{r-1}}\mathbf{x})\big)$$

and thus

$$\Big|\limsup_j \mathrm{Av}_{p\in I_j} \prod_{r\in\epsilon\subset[r]} \psi_\epsilon(m\cdot\epsilon+p)\cdot \prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}} h_\epsilon(T^{m\cdot\epsilon+p\epsilon_r}x_\epsilon)\Big|$$

$$\le \delta + C\|\Phi_m\|_{\mathbf{I},k+1}$$

for every m and every $\mathbf{x}\in Z_k^{(d+1,r)}$. Taking the integral,

$$\big|\limsup_j \mathrm{Av}_{p\in I_j} A'(m,p)\big| \le \delta + C\|\Phi_m\|_{\mathbf{I},k+1}$$

for every m. Therefore

$$\begin{aligned}
&\operatorname{Iter\,limsup}\bigl|\operatorname{Av}_{\mathbf{I},n_1,\dots,n_r} A(n)\bigr| \\
&\qquad\le \operatorname{Iter\,limsup} \operatorname{Av}_{\mathbf{I},n_1,\dots,n_{r-1}} \bigl|\lim_j \operatorname{Av}_{p\in I_j} A'(m,p)\bigr| \\
&\qquad\le \delta + C \operatorname{Iter\,limsup} \operatorname{Av}_{\mathbf{I},m_1,\dots,m_{r-1}} \|\Phi_m\|_{\mathbf{I},k+1} \\
&\qquad\le \delta + C \operatorname{Iter\,limsup}\Bigl(\operatorname{Av}_{\mathbf{I},m_1,\dots,m_{r-1}} \|\Phi_m\|_{\mathbf{I},k+1}^{2^{r-1}}\Bigr)^{1/2^{r-1}} .
\end{aligned}$$

By Proposition 4.3 in [9], the last limsup is actually a limit and is bounded by

$$\prod_{r\in\epsilon\subset[r]} \|\psi_\epsilon\|_{\mathbf{I},k+r} . \quad \square$$

6.3. **Iteration.**

Proposition 6.3. *Let $k \ge 1$, $0 \le r \le d$ and h_ϵ, $\epsilon \subset [d+1]$, $\epsilon \not\subset [r]$, be $2^{d+1} - 2^r$ bounded measurable functions on Z_k. Let ϕ_ϵ, $\epsilon \subset [r]$, be 2^r continuous functions on X. For $n \in \mathbb{Z}^r$, define*

$$A(n) = \prod_{\epsilon\subset[r]} \phi_\epsilon(T^{n\cdot\epsilon}x_0) \int \prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}} h_\epsilon(T^{n\cdot\epsilon}x_\epsilon)\, d\mu_{k\ e}^{(d+1,r)}(\mathbf{x})$$

and

$$B(n) = \prod_{\epsilon\subset[r-1]} \phi_\epsilon(T^{n\cdot\epsilon}x_0)\cdot$$
$$\int \prod_{r\in\epsilon\subset[r]} \mathbb{E}(\phi_\epsilon \mid Z_{k+r-1})(x_\epsilon) \cdot \prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}} h_\epsilon \circ p_{k+r-1,k}(x_\epsilon)\, d\mu_{k+r-1\ e}^{(d+1,r-1)}(\mathbf{x}) .$$

Then the iterated limit of the averages of $A(n) - B(n)$ is zero.

Proof. We remark that $B(n)$ depends only on $n_1, \dots, n_{r-1}$.

By (18), it suffices to prove the result in the case that the functions h_ϵ are continuous. We can also assume that $|\phi_\epsilon| \le 1$ for every $\epsilon \subset [r]$.

Let $\delta > 0$ be given and let C be as in Lemma 6.2. For each ϵ with $r \in \epsilon \subset [d+1]$, let $\widetilde{\phi}_\epsilon$ be a continuous function on Z_{k+r-1} with $|\widetilde{\phi}_\epsilon| \le 1$, such that $\|\mathbb{E}(\phi_\epsilon \mid Z_{k+r-1}) - \widetilde{\phi}_\epsilon\|$ is sufficiently small. We have that

$$\|(\widetilde{\phi}_\epsilon(T^n e_{k+r-1})\colon n\in\mathbb{Z}) - (\phi_\epsilon(T^n x_0)\colon n\in\mathbb{Z})\|_{\mathbf{I},k+r} \le \delta/2^{r-1}C$$

for every ϵ. This follows from Proposition 3.6.

By Lemma 6.2 the iterated limsup of the absolute value of the averages on $\mathbf{I}$ of

$$A(n)-\prod_{\epsilon\subset[r-1]}\phi_\epsilon(T^{n\cdot\epsilon}x_0)\cdot\prod_{r\in\epsilon\subset[r]}\widetilde{\phi}_\epsilon(T^{n\cdot\epsilon}e_{k+r-1})\cdot\int\prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}}h_\epsilon(T^{n\cdot\epsilon}x_\epsilon)\,d\mu^{(d+1,r)}_{k\ e}(\mathbf{x})$$

is bounded by 2δ. We rewrite the second term in this difference as

$$\prod_{\epsilon\subset[r-1]}\phi_\epsilon(T^{n\cdot\epsilon}x_0)\cdot\prod_{r\in\epsilon\subset[r]}\widetilde{\phi}_\epsilon(T^{n\cdot\epsilon}e_{k+r-1})\cdot \int\prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}}h_\epsilon\circ p_{k+r-1,r}(T^{n\cdot\epsilon}x_\epsilon)\,d\mu^{(d+1,r)}_{k+r-1\ e}(\mathbf{x})$$

and remark that the first product in this last expression depends only on $n_1,\dots,n_{r-1}$.

By definition of the measures and continuity of the functions $\widetilde{\phi}_\epsilon$, the averages in n_r on $\mathbf{I}$ of the above expression converges to

$$\prod_{\epsilon\subset[r-1]}\phi_\epsilon(T^{n\cdot\epsilon}x_0)\cdot\int\prod_{r\in\epsilon\subset[r]}\widetilde{\phi}_\epsilon(x_\epsilon)\cdot\prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}}h_\epsilon\circ p_{k+r-1,k}(x_\epsilon)\,d\mu^{(d+1,r-1)}_{k+r-1}(\mathbf{x})\ .$$

By (18) again, for every $n_1,\dots,n_{r-1}$ the difference between this expression and $B(n)$ is bounded by δ.

The announced result follows. □

Proposition 6.4. *Let $k\geq 1$ and let f_ϵ, $\epsilon\subset[d+1]$, $\epsilon\neq\emptyset$, be $2^{d+1}-1$ continuous functions on X. Then the iterated averages for $n=(n_1,\dots,n_d,n_{d+1})\in\mathbb{Z}^{d+1}$ on $\mathbf{I}$ of*

$$\prod_{\emptyset\neq\epsilon\subset[d+1]}f_\epsilon(T^{n\cdot\epsilon}x_0) \tag{19}$$

converge to

$$\int\prod_{\emptyset\neq\epsilon\subset[d+1]}\mathbb{E}(f_\epsilon\mid Z_d)(x_\epsilon)\,d\mu^{(d+1)}_{d\ e}(\mathbf{x})\ . \tag{20}$$

Proof. For notational convenience we define $f_\emptyset$ to be the constant function 1.

By (2), the averages in n_{d+1} of (19) converge to

$$\prod_{\emptyset\neq\epsilon\subset[d]} f_\epsilon(T^{n\cdot\epsilon}x_0)\cdot\int\prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[d]}} f_\epsilon(T^{n\cdot\epsilon}x)\,d\mu(x) \tag{21}$$

and it remains to show that the iterated averages in $(n_1,\dots,n_d)$ of this expression converge to (20).

By Proposition 6.1, the difference between the quantity (21) and

$$A(n):=\prod_{\epsilon\subset[d]} f_\epsilon(T^{n\cdot\epsilon}x_0)\cdot\int\prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[d]}} \mathbb{E}(f_\epsilon\mid Z_d)(T^{n\cdot\epsilon}x)\,d\mu_d(x)$$

converges to zero in density and we are reduced to study the iterated convergence of the averages of $A(n)$.

We apply Proposition 6.3 with $k=d$ and $r=d$ and left with studying the iterated limit of the averages in $n_1,\dots,n_{d-1}$ of

$$\prod_{\epsilon\subset[d-1]} f_\epsilon(T^{n\cdot\epsilon}x_0)\cdot$$
$$\int\prod_{d\in\epsilon\subset[d]}\mathbb{E}(f_\epsilon\mid Z_{2d-1})(x_\epsilon)\cdot\prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[d]}}\mathbb{E}(f_\epsilon\mid Z_d)\circ p_{2d-1,d}(x_\epsilon)\,d\mu^{(d+1,d-1)}_{2d-1\ e}(\mathbf{x})\ .$$

After $d-r$ steps, we are left with the iterated limit of the averages in $n_1,\dots,n_r$ of an expression of the form

$$\prod_{\epsilon\subset[r]} f_\epsilon(T^{n\cdot\epsilon}x_0)\cdot\int\prod_{\substack{\epsilon\subset[d+1]\\ \epsilon\not\subset[r]}} E(f_\epsilon\mid Z_{\ell(\epsilon)})\circ p_{k,\ell(\epsilon)}(x_\epsilon)\,d\mu^{(d+1,r)}_{k\ e}(\mathbf{x})\ ,$$

where $k=k(r)\geq d$ is an integer and where for every ϵ, $d\leq\ell(\epsilon)\leq k$.

Finally, after d steps, we have that the iterated limit of the expression (21) exists and is equal to

$$\int\prod_{\emptyset\neq\epsilon\subset[d+1]}\mathbb{E}(f_\epsilon\mid Z_{\ell(\epsilon)})\circ p_{k,\ell(\epsilon)}(x_\epsilon)\,d\mu^{(d+1)}_{k\ e}(\mathbf{x})\ ,$$

where k is an integer and $d\leq\ell(\epsilon)\leq k$ for every ϵ.

By Proposition 5.1, the measure $\mu^{(d+1)}_{k\ e}$ is relatively independent with respect to its projection $\mu^{(d+1)}_{d\ e}$ on $Z_d^{(d+1)}$. For every ϵ,

$$\mathbb{E}\big(\mathbb{E}(f_\epsilon\mid Z_{\ell(\epsilon)})\circ p_{k,\ell(\epsilon)}\mid Z_d\big)=\mathbb{E}(f_\epsilon\mid Z_d)$$

and we have that the above limit is equal to (20). □

7. Positivity

In this Section, $\mathcal{A}$, X, μ, $\mathbf{I} = (I_j : j \geq 1)$, ... are as in Sections 3.4 and 3.5. Given a sequence of intervals $(J_k : k \geq 1)$ in $\mathbb{Z}$ whose lengths tend to infinity, we assume that for each $j \geq 1$, there exists some $k = k(j)$ such that the interval I_j is included in J_k.

We simplify the notation: we write Z instead of Z_d, ν instead of μ_d, e instead of e_d. If f is a function on X, $\widetilde{f} = \mathbb{E}(f \mid Z)$.

7.1. **Positivity.**

Lemma 7.1. *Let $B \subset \mathbb{Z}$ be such that $\mathbf{1}_B \in \mathcal{A}$ and let f be the continuous function on X associated to this set:*

$$f(T^n x_0) = \mathbf{1}_B(n) \ .$$

Let $m \geq 1$ be an integer and let h_ϵ, $\emptyset \neq \epsilon \subset [m]$, be $2^m - 1$ nonnegative bounded measurable functions on Z. Assume that

$$\int \prod_{\emptyset \neq \epsilon \subset [m]} h_\epsilon(x_\epsilon)\, d\nu_e^{(m)}(\mathbf{x}) > 0$$

and that

$$\mathbb{Z} \setminus B \text{ is not a } \mathrm{PW}\text{-}\mathrm{Nil}_d\,\mathrm{Bohr}_0 \text{ set.}$$

Then

$$\int \widetilde{f}(x_{\{m+1\}}) \cdot \prod_{\substack{\epsilon \subset [m+1] \\ \epsilon \neq \emptyset, \{m+1\}}} h_{\epsilon \cap [m]}(x_\epsilon)\, d\nu_e^{(m+1)}(\mathbf{x}) > 0 \ .$$

Proof. By Proposition 5.3,

$$\int \prod_{\substack{\epsilon \subset [m+1] \\ \epsilon \neq \emptyset, \{m+1\}}} h_{\epsilon \cap [m]}(x_\epsilon)\, d\nu_{e,e}^{(m+1)}(\mathbf{x}) > 0 \ .$$

For $z \in Z$, define

$$H(z) = \int \prod_{\substack{\epsilon \subset [m+1] \\ \epsilon \neq \emptyset,\ \epsilon \neq \{m+1\}}} h_{\epsilon \cap [m]}(x_\epsilon)\, d\nu_{e,z}^{(m+1)}(\mathbf{x}) \ .$$

We have that $\delta := H(e) > 0$ and, by Proposition 5.2, H is continuous on Z_d. Therefore, the subset

$$\Lambda = \{n \in \mathbb{Z} : H(T^n e) > \delta/2\}$$

is a $\mathrm{Nil}_d\,$Bohr-set.

By the same proposition,

$$\int \widetilde{f}(x_{\{m+1\}}) \cdot \prod_{\substack{\epsilon\subset[m+1]\\ \epsilon\neq\emptyset,\{m+1\}}} h_{\epsilon\cap[m]}(x_\epsilon)\, d\nu_e^{(m+1)} = \int \widetilde{f}(z)H(z)\, d\nu(z)\ .$$

We complete the proof as in the proof of Corollary 3.4. By Proposition 3.3, this last integral is equal to

$$\lim \mathrm{Av}_{\mathbf{I}}\, f(T^n x_0)H(T^n e) \geq \frac{\delta}{2} \limsup_j \frac{1}{|I_j|}|\Lambda\cap B\cap I_j|\ .$$

If this limsup is equal to zero, then there exist arbitrarily long intervals J_ℓ such that $\Lambda\cap B\cap J_\ell = \emptyset$ and thus the set $\mathbb{Z}\setminus B$ contains $\Lambda\cap J_\ell$ for all ℓ. Therefore $\mathbb{Z}\setminus B$ is a PW-Nil_g Bohr-set, hence a contradiction. □

Corollary 7.2. *Let $B\subset\mathbb{Z}$ be such that $\mathbf{1}_B\in\mathcal{A}$ and let f be the continuous function on X associated to this set. Assume that $\mathbb{Z}\setminus B$ is not a* PW-$\mathrm{Nil}_d\,\mathrm{Bohr}_0$*-set. Then, for every m,*

$$\int \prod_{\emptyset\neq\epsilon\subset[m]} \widetilde{f}(x_\epsilon)\, d\nu_e^{(m)}(\mathbf{x}) > 0\ . \tag{22}$$

Proof. We remark first that $\int f\,d\mu>0$. Indeed, if this integral is zero, then the density of the set B in the intervals I_j converges to 0 and $\mathbb{Z}\setminus B$ contains arbitrarily long intervals, a contradiction.

We show (22) by induction. We have that $\nu_e^{(1)} = \delta_e\times\nu$ and thus

$$\int \widetilde{f}(x_1)\, d\nu_{d\ e}^{(1)}(\mathbf{x}) = \int \widetilde{f}\, d\nu = \int f\, d\mu > 0\ .$$

Assume that (22) holds for some $m\geq 1$. Then Lemma 7.1 applied to $h=\widetilde{f}$ shows that it holds for $m+1$. □

7.2. **And now we gather all the pieces of the puzzle.** Recall that if E is a finite subset of $\mathbb{N}$, $\mathrm{S}(E)$ is the set consisting in all sums of distinct elements of E (the empty sum is not considered). A subset A of $\mathbb{Z}$ is a S_m^*-set if $A\cap\mathrm{S}(E)\neq\emptyset$ for every subset E of $\mathbb{N}$ with m elements.

We prove Theorem 2.6:

Theorem. *Let A be a S_{d+1}^* set. Then A is a* PW-Nil_d Bohr*-set.*

Proof. Let $B=\mathbb{Z}\setminus A$, $\mathcal{A}$ a subalgebra of $\ell^\infty(\mathbb{Z})$ containing $\mathbf{1}_B$ and $X,\mu,\mathbf{I},\dots$ are as above. The continuous function f on X is associated to $\mathbf{1}_B$ and we use the same notation as above.

Assume that A is not a PW- Nil_d Bohr-set. By Corollary 7.2,

$$\int \prod_{\emptyset\neq\epsilon\subset[d+1]} \widetilde{f}(x_\epsilon)\, d\nu_e^{(d+1)}(\mathbf{x}) > 0$$

and by Proposition 6.4, this integral is equal to the iterated limit of the averages in $n = (n_1, \dots, n_{d+1})$ of

$$\prod_{\emptyset\neq\epsilon\subset[d+1]} f(T^{n\cdot\epsilon}x_0)\ .$$

This product is nonzero if and only if $\mathrm{S}(\{n_1, \dots, n_{d+1}\}) \subset B$. But the complement A of B in $\mathbb{Z}$ is a S^*_{d+1}-set (recall that the n_i belong to some of the intervals I_j), and so this can not happen. □

8. Proof of Theorem 2.10

We now prove Theorem 2.10 (recall that Theorem 2.8 is a particular case of this theorem):

Theorem. *Every* SG^*_d*-set is a* PW- Nil_d Bohr_0*-set.*

8.1. **The method.** The proof is by contradiction. In this section, $d \geq 1$ is an integer and A is a subset of the integers. We assume that A is not a PW- Nil_d Bohr_0-set and by induction, we build an infinite sequence $P = (p_j : j \geq 1)$ such that $A \cap \mathrm{SG}_d(P) = \emptyset$.

Let $\mathcal{A}$ be a subalgebra of $\ell^\infty(\mathbb{Z})$ containing $\mathbf{1}_A$ and let $X, \mu, \dots$ and the sequence of intervals $\mathbf{I} = (I_j : j \geq 1)$ be as in Sections 3.4 and 3.5. We have the same conventions as in the preceding section for the intervals I_j.

We write $B = \mathbb{Z} \setminus A$ and let f be the continuous function on X associated to $\mathbf{1}_B$ (see Section 3.4):

$$f(T^n x_0) = \begin{cases} 1 & \text{if } n \in B\ ; \\ 0 & \text{otherwise.} \end{cases}$$

As in Section 7, we simplify the notation: we write Z instead of Z_d, ν instead of μ_d, and e instead of e_d. If f is a function on X, $\widetilde{f} = \mathbb{E}(f \mid Z)$.

In this section, it is more convenient to index points of $X^{[d]}$ by $\{0,1\}^d$ instead of by $\mathcal{P}([d])$. Thus a point $\mathbf{x} \in X^{[d]}$ is written $\mathbf{x} = (x_\epsilon : \epsilon \in \{0,1\}^d)$.

By induction, for every $j \geq 0$, we build $2^d - 1$ continuous nonnegative functions $h_\epsilon^{(j)}$, $00\dots0 \neq \epsilon \in \{0,1\}^d$, on X satisfying

$$\int \prod_{\substack{\epsilon\in\{0,1\}^d \\ \epsilon\neq 00\dots0}} \widetilde{h}_\epsilon^{(j)}(x_\epsilon)\, d\nu_e^{(d)}(\mathbf{x}) > 0 \tag{23}$$

and, for every $j \geq 1$, we build an integer p_j (belonging to some interval I_i), satisfying

$$h^{(j-1)}_{100\ldots 0}(T^{p_j}x_0) > 0 \ . \tag{24}$$

Start by setting all of the functions $h^{(0)}_\epsilon$, $00\ldots 0 \neq \epsilon \in \{0,1\}^d$, to be equal to f. By Corollary 7.2 applied with $m = d$ and rewritten in the current notation, we have that property (23) is satisfied for $j = 0$.

8.2. **Iteration.** Assume $j \geq 1$ and that property (23) is satisfied for $j - 1$.

By Proposition 5.3,

$$\int \prod_{\substack{\epsilon\in\{0,1\}^{d+1}\\ \epsilon\neq 00\ldots 0, \epsilon\neq 00\ldots 01}} \widetilde{h}^{(j-1)}_{\epsilon_1\ldots\epsilon_d}(x_\epsilon)\, d\nu^{(d+1)}_{e,e}(\mathbf{x}) > 0 \ .$$

By Lemma 7.1, rewritten in our current notation, we have that

$$\int \widetilde{f}(x_{00\ldots 01}) \cdot \prod_{\substack{\epsilon\in\{0,1\}^{d+1}\\ \epsilon\neq 00\ldots 0, \epsilon\neq 00\ldots 01}} \widetilde{h}^{(j-1)}_{\epsilon_1\ldots\epsilon_d}(x_\epsilon)\, d\nu^{(d+1)}_{e}(\mathbf{x}) > 0 \ .$$

For convenience, we write $h^{(j-1)}_{00\ldots 0} = f$ and rewrite this equation as

$$\int \prod_{\substack{\epsilon\in\{0,1\}^{d+1}\\ \epsilon\neq 00\ldots 0}} \widetilde{h}^{(j-1)}_{\epsilon_1\ldots\epsilon_d}(x_\epsilon)\, d\nu^{(d+1)}_{e}(\mathbf{x}) > 0 \ . \tag{25}$$

By Proposition 6.4, this last integral is the iterated limit of the averages for $n = (n_1, \ldots, n_{d+1})$ of

$$\prod_{\substack{\epsilon\in\{0,1\}^{d+1}\\ \epsilon\neq 00\ldots 0}} h^{(j-1)}_{\epsilon_1\ldots\epsilon_d}(T^{n\cdot\epsilon}x_0) \ .$$

We make a change of indices, writing elements of $\mathbb{Z}^{d+1}$ as $(p, n_1, \ldots, n_d)$ and setting $n = (n_1, \ldots, n_d)$. Elements of $\{0,1\}^{d+1}$ are written as $\eta\epsilon_1 \ldots \epsilon_d$ with $\eta \in \{0,1\}$ and we set $\epsilon = \epsilon_1 \ldots \epsilon_d$. The last product becomes:

$$h^{(j-1)}_{100\ldots 0}(T^p x_0) \prod_{\substack{\epsilon\in\{0,1\}^{d}\\ \epsilon\neq 00\ldots 0}} \big(h^{(j-1)}_{0\epsilon_1\ldots\epsilon_{d-1}} \cdot T^p h^{(j-1)}_{1\epsilon_1\ldots\epsilon_{d-1}}\big)(T^{n\cdot\epsilon}x_0) \ .$$

For $\epsilon \in \{0,1\}^d$, $\epsilon \neq 00\ldots 0$, and for $p \in \mathbb{Z}$, set

$$g_{p,\epsilon} = h^{(j-1)}_{0\epsilon_1\ldots\epsilon_{d-1}} \cdot T^p h^{(j-1)}_{1\epsilon_1\ldots\epsilon_{d-1}}$$

and rewrite the last expression as

$$h^{(j-1)}_{100\dots 0}(T^p x_0)\prod_{\substack{\epsilon\in\{0,1\}^d\\ \epsilon\neq 00\dots 0}} g_{p,\epsilon}(T^{n\cdot\epsilon}x_0)\ .$$

By Proposition 6.4 again, the iterated limit of the averages in $n_1,\dots,n_d$ of this expression converges to

$$h^{(j-1)}_{100\dots 0}(T^p x_0)\int\prod_{\substack{\epsilon\in\{0,1\}^d\\ \epsilon\neq 00\dots 0}} \mathbb{E}\big(g_{p,\epsilon}\mid Z_{d-1}\big)(x_\epsilon)\,d\mu^{(d)}_{d-1\ e}(\mathbf{x})$$
$$= h^{(j-1)}_{100\dots 0}(T^p x_0)\int\prod_{\substack{\epsilon\in\{0,1\}^d\\ \epsilon\neq 00\dots 0}} \widetilde{g_{p,\epsilon}}(x_\epsilon)\,d\mu^{(d)}_e(\mathbf{x})$$

because the measure $\mu^{(d)}_e$ is relatively independent with respect to $\mu^{(m)}_{d-1\ e}$ (see Proposition 5.1, part (d)).

The averages in p over the intervals $\mathbf{I}$ of this expression converge to the limit (25), which is positive. Thus there exists some p (belonging to some I_i) such that this expression is positive. Choosing p_j to be this p, for $00\dots 0\neq\epsilon\in\{0,1\}^d$, we define

$$h^{(j)}_\epsilon = g_{p_j,\epsilon} = h^{(j)}_{0\epsilon_1\dots\epsilon_{d-1}}\cdot T^{p_j}h^{(j-1)}_{1\epsilon_1\dots\epsilon_{d-1}}$$

(recall that $h^{(j-1)}_{00\dots 0}=f$). Since (23) is valid with $h^{(j)}_\epsilon$ substituted for $h^{(j-1)}_\epsilon$, we can iterate. Moreover,

$$h^{(j-1)}_{100\dots 0}(T^{p_j}x_0) > 0\ ,$$

meaning that relation (24) is satisfied.

8.2.1. *Interpreting the iteration.* By induction, it follows that for every $j\geq 0$, the functions $h^{(j)}_\epsilon$, $00\dots 0\neq\epsilon\in\{0,1\}^d$, only depend on the first nonzero digit of ϵ:

$$h^{(j)}_\epsilon = \phi^{(k)}_j \text{ if } \epsilon_1=\dots=\epsilon_{k=1}=0 \text{ and } \epsilon_k=1\ .$$

We have the inductive relations

$$\phi^{(k)}_0 = f \text{ for } 1\leq k\leq d\ ;$$
$$\phi^{(1)}_{j-1}(T^{p_j}x_0)>0\ ;$$
$$\text{for } 1\leq k<d,\quad \phi^{(k)}_j=\phi^{(k+1)}_{j-1}\cdot T^{p_j}\phi^{(1)}_{j-1}\ ;$$
$$\phi^{(d)}_j = f\cdot T^{p_j}\phi_{j-1,1}\ .$$

By induction, $\phi_j^{(1)} \leq \phi_j^{(2)} \leq \cdots \leq \phi_j^{(d)} \leq f$. Moreover, we deduce the following relations between the functions $\phi_j^{(1)}$:

$$\text{for } 1 \leq j < d,\ \phi_j^{(1)} = f \cdot \prod_{k=1}^{j} T^{p_{j-k+1}} \phi_{j-k}^{(1)}$$
$$\text{for } j \geq d,\ \phi_j^{(1)} = f \cdot \prod_{k=1}^{d} T^{p_{j-k+1}} \phi_{j-k}^{(1)} \ .$$

It follows that for every j, there is a finite set E_j of integers with

$$\phi_j^{(1)} = \prod_{q \in E_j} T^q f \ .$$

We have that $E_0 = \{0\}$ and the E_j satisfy the relations

$$\text{for } 1 \leq j < d,\ E_j = \{0\} \cup (E_{j-1} + p_j) \cup (E_{j-2} + p_{j-1}) \cup \ldots \cup (E_0 + p_1) \ ,$$
$$\text{for } j \geq d,\ E_j = \{0\} \cup (E_{j-1} + p_j) \cup (E_{j-2} + p_{j-1}) \cup \ldots \cup (E_{j-d} + p_{j-d+1}) \ .$$

By induction, E_j consists in all sums of the form $\epsilon_1 p_1 + \cdots + \epsilon_j p_j$ where $\epsilon_i \in \{0, 1\}$ for all i, and, after the first occurrence of 1, there can be no block of d consecutive 0's.

By induction, each function $\phi_j^{(1)}$ only takes on the values of 0 and 1 and corresponds to a subset B_j of the integers and we have

$$B_j = \bigcap_{q \in E_j} (B - q) \ .$$

For every j, since $\phi_{j-1}^{(1)}(T^{p_j} x_0) > 0$, we have that $p_j \in B_{j-1}$ and thus that $E_{j-1} + p_j \subset B$.

We conclude that all sums of the form $\epsilon_1 p_1 + \cdots + \epsilon_k p_k$ with $\epsilon_i \in \{0, 1\}$ for all i belong to B, provided the ϵ_i are not all equal to 0 and that the blocks of consecutive 0's between two 1's have length $< d$. In other words, $B \supset \mathrm{SG}_d(\{p_j \colon j \geq 1\})$ and we have a contradiction. $\square$

We note that at each step in the iteration, we have infinitely many choices for the next p. In particular, we can take the p_j tending to infinity as fast as we want. More interesting, in the construction we can choose a different permutation of coordinates at each step. This gives rise to different, but related, structures, which do not seem to have any simple description.

References

[1] L. Auslander, L. Green and F. Hahn. Flows on homogeneous spaces. *Ann. Math. Studies* **53**, Princeton Univ. Press (1963).

[2] V. Bergelson, H. Furstenberg and B. Weiss. Piecewise-Bohr sets of integers and combinatorial number theory. Algorithms Combin. **26**, Springer, Berlin (2006), 13-37.

[3] V. Bergelson, B. Host and B. Kra, with an Appendix by Imre Ruzsa. Multiple recurrence and nilsequences. *Inventiones mathematicae*, 160 (2005), 261–303.

[4] Y. Bilu. Addition of sets of integers of positive density. *J. Numb. Th.* **64** (1997), 233-275.

[5] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory.* Princeton University Press, Princeton, N.J., 1981.

[6] W. T. Gowers. A new proof of Szemerédi's Theorem. *Geom. Funct. Anal.* **11** (2001), 465-588.

[7] B. Green and T. Tao. Linear equations in the primes. To appear, *Ann. Math.*

[8] B. Host and B. Kra. Nonconventional ergodic averages and nilmanifolds. *Ann. Math.* **161** (2005), 397-488.

[9] B. Host and B. Kra. Uniformity norms on ℓ^∞ and applications. To appear, *J. d'Analyse.*

[10] B. Host, B. Kra and A. Maass. Nilsequences and a structure theorem for topological dynamical systems. Preprint, `arXiv:0905.3098`

[11] A. Leibman. Pointwise convergence of ergodic averages for polynomial sequences of translations on a nilmanifold. *Erg. Th. & Dyn. Sys.* **25** (2005), 201–113.

Laboratoire d'analyse et de mathématiques appliquées, Université de Marne la Vallée & CNRS UMR 8050, 5 Bd. Descartes, Champs sur Marne, 77454 Marne la Vallée Cedex 2, France

E-mail address: `bernard.host@univ-mlv.fr`

Department of Mathematics, Northwestern University, 2033 Sheridan Road Evanston, IL 60208-2730, USA

E-mail address: `kra@math.northwestern.edu`