

POLYGONAL \mathbb{Z}^2 -SUBSHIFTS

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ABSTRACT. Let $\mathcal{P} \subset \mathbb{Z}^2$ be a convex polygon with each vertex in it labeled by an element from a finite set and such that the labeling of each vertex $v \in \mathcal{P}$ is uniquely determined by the labeling of all other points in the polygon. We introduce a class of \mathbb{Z}^2 -shift systems, the *polygonal shifts*, determined by such a polygon: these are shift systems such that the restriction of any $x \in X$ to some polygon \mathcal{P} has this property. These polygonal systems are related to various well studied classes of shift systems, including subshifts of finite type and algebraic shifts, but include many other systems. We give necessary conditions for a \mathbb{Z}^2 -system X to be polygonal, in terms of the nonexpansive subspaces of X , and under further conditions can give a complete characterization for such systems.

1. INTRODUCTION

If \mathcal{A} is a finite alphabet, a \mathbb{Z}^2 -shift X is a closed subspace of $\mathcal{A}^{\mathbb{Z}^2}$ that is invariant under the \mathbb{Z}^2 -action by horizontal and vertical shifts. Large classes of shifts have been well studied, including algebraic shifts and shifts of finite type (see for example [26, 15, 19]). We focus on a collection related to these, which we call polygonal shifts.

Roughly speaking, polygonal shifts are a class in which the data in one region determines the data in a larger region. We defer the precise definitions until Sections 2 and 3.1, starting with some examples that motivate the study of these shifts. We refer to an element $x = (x(i, j) : i, j \in \mathbb{Z})$ in a \mathbb{Z}^2 -shift X as *coloring*, and refer to the restriction of $x \in X$ to a set $S \subset \mathbb{Z}^2$ as a *coloring of S* . Perhaps the simplest interesting example is the Ledrappier shift [18]: if $\mathcal{A} = \mathbb{Z}/2\mathbb{Z}$, define X to be the subshift of $\mathcal{A}^{\mathbb{Z}^2}$ such that every $x \in X$ satisfies

$$(1.1) \quad x(i, j) + x(i + 1, j) + x(i, j + 1) = 0 \pmod{2}.$$

The key property is that for the triangle \mathcal{T} with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$, the coloring of a vertex is uniquely determined by the coloring of the other two vertices of the triangle, and this triangle is what motivates the commonly used name *three dot system* for this shift. Shift

Key words and phrases. subshift, automorphism, nonexpansive.

The second author was partially supported by NSF grant DMS-1500670.

invariance of the system implies that the same holds for vertices of any integer translate of \mathcal{T} .

Polygonal systems generalize this idea, and instead of using a triangle, we consider an arbitrary convex polygon $\mathcal{P} \subset \mathbb{R}^2$, and we refer to such a polygon as an (*integer*) *polygon* if all its vertices lie in \mathbb{Z}^2 . The key property of the polygon \mathcal{P} is that the color of each vertex $v \in \mathcal{P}$ is uniquely determined by the coloring of all points of \mathcal{P} other than v (note that these other points may include interior points of the polygon). If X is a \mathbb{Z}^2 -shift and there is a convex polygon with vertices in \mathbb{Z}^2 such that the restrictions of all colorings $x \in X$ to the polygon \mathcal{P} has this property, then we say that the system is *polygonal with respect to \mathcal{P}* and that \mathcal{P} is a *coding polygon* for the shift. We emphasize that by definition coding polygons have their vertices in \mathbb{Z}^2 and hence all edges have rational slopes. It is obvious that if a shift is polygonal with respect to \mathcal{P} , then it is also polygonal with respect to $\mathcal{P} + (i, j)$ for any $i, j \in \mathbb{Z}$, and so the particular choice of polygon is only defined up to translation.

Our primary goal is to characterize the \mathbb{Z}^2 -shifts which are polygonal and for a polygonal shift ascertain to what extent its coding polygon \mathcal{P} is canonical. In general, coding polygons are far from unique. For example, the Ledrappier shift X is polygonal with respect to any of the triangles $\mathcal{T}_n, n > 0$ with vertices $(0, 0), (0, n)$, and $(n, 0)$. It is easy to see that the triangle \mathcal{T} is canonical in the sense that there is no smaller polygon with respect to which X is polygonal.

Key concepts in our investigation are the notions of expansive and nonexpansive. Again, we postpone the formal definitions until Section 2, but we motivate their role. For the Ledrappier system X , it is easy to check that for all but three (up to translation) half spaces in \mathbb{R}^2 , any coloring of its integer lattice points extends uniquely to a coloring of all of \mathbb{Z}^2 ; this is well known, and follows from a more general result given in Proposition 3.2. The only exceptions are the half spaces which are translates of the three half spaces given by the inequalities $x \geq 0$, $y \geq 0$, and $y \leq -x$. To make precise the sense in which data in one region determines data outside this region, we view a half space as being specified by an oriented ray, namely a ray which lies in the boundary of that half space and inherits its orientation from the induced orientation on the boundary. If every coloring of the integer lattice in a half space extends uniquely to the full space, we say that the half space and corresponding oriented ray are *expansive*, and otherwise we say that they are *nonexpansive*. In the Ledrappier system, the only nonexpansive rays are the rays lying in the boundaries of the three specified half spaces and having the appropriate orientation, namely

the rays spanned by the vectors $(1, 0)$, $(-1, 1)$ and $(0, -1)$. Note that these three vectors form the edges (not vertices) of an oriented coding polygon for the Ledrappier shift.

This terminology is consistent with standard notions of expansiveness and nonexpansiveness for one-dimensional subspaces of \mathbb{Z}^2 -systems, as studied, for example, in Boyle and Lind [3]. In particular, they show that the set of nonexpansive subspaces is nonempty when X is infinite. Accordingly, any \mathbb{Z}^2 -shift X we consider is assumed to be infinite. Allowable colorings of a nonexpansive half space do not uniquely determine the coloring of even a single point in the complementary half space, and this behavior again shows up in the Ledrappier system. In our more general setting of polygonal systems, it is exactly the nonexpansive rays that are used to characterize which shifts lie in this class.

Generalizing the Ledrappier example, Kitchens and Schmidt [16, 17] study \mathbb{Z}^d -actions on Markov subgroups. If \mathcal{A} is a finite abelian group, then $\mathcal{A}^{\mathbb{Z}^d}$ is a zero-dimensional compact abelian group when endowed with the operation of component-wise addition. A *Markov subgroup* X is a closed subgroup of this group such that there exists some finite set $S \subset \mathbb{Z}^d$ (called a *shape*) satisfying

$$\sum_{u \in (S+v)} x(u) = 0$$

for each fixed $v \in \mathbb{Z}^d$. For $d = 2$, it is easy to see that any Markov subgroup is polygonal with coding polygon \mathcal{P} given by the convex hull of the finite set S .

The polygonal shifts are a class of zero-dimensional \mathbb{Z}^2 -subshifts that is more general and substantially larger than Markov subgroups or similar systems with a strong algebraic structure. More precisely, a result of Einsiedler [8] shows there are uncountably many \mathbb{Z}^2 -invariant subspaces of the Ledrappier shift X with distinct topological entropies and it follows that there are uncountably many distinct polygonal shifts with the same alphabet and the same polygon \mathcal{T} . In particular, not all polygonal shifts are isomorphic to subshifts of finite type or to \mathbb{Z}^2 -actions on Markov subgroups, as these classes are countable (up to isomorphism).

We limit ourselves to shifts which are polygonal with respect to convex polygons. There is no loss in doing so, as it is easy to see that a shift which is polygonal with respect to a polygon \mathcal{P} is also polygonal with respect to the convex hull $\hat{\mathcal{P}}$ of \mathcal{P} . The advantage of working with

$\hat{\mathcal{P}}$ is that it has strictly fewer edges and vertices than \mathcal{P} , unless \mathcal{P} is already convex.

Another reason to make use of the simplification in the geometry in passing to the convex hull of a shape, rather than more general shapes, is that the edges of a convex coding polygon are closely related to the geometry of nonexpansive subspaces. For example, if X is a Markov subgroup with shape S , then the nonexpansive subspaces are precisely the subspaces parallel to the edges of \mathcal{P} , the convex hull of S .

The fact that all other subspaces are expansive is a special case of a result given in Proposition 3.2. Since for each edge e there are multiple legal colorings of $\hat{\mathcal{P}}$ which differ on e but agree on $\hat{\mathcal{P}} \setminus e$, it follows that the edges are nonexpansive (see Definition 2.8).

In seeking the simplest polygon to represent a shift X we allow ourselves to replace X with a particular kind of isomorphic shift Y which we call a *recoding* of X . The precise definition of recoding is given in Definition 2.3, but again we give an informal motivation. Starting with a finite convex subset $F \subset \mathbb{Z}^2$, we create a new alphabet \mathcal{A}_F consisting of all legal colorings of F . The recoding X_F of X is then the subset of $(\mathcal{A}_F)^{\mathbb{Z}^2}$ with the property that for each $y \in X_F$ there is an $x \in X$ such that for each $i, j \in \mathbb{Z}^2$, the coloring of $y(i, j)$ is the restriction of the coloring x to $F + (i, j)$.

Considering the class $\mathcal{P}(X)$ of all (integer) coding polygons for all recodings of a subshift X , we refer to a polygon $\mathcal{P}_0 \in \mathcal{P}(X)$ as a *minimal recoding polygon* if it has the minimal number of edges of all elements of $\mathcal{P}(X)$ and is minimal under inclusion among coding polygons with that number of edges. Note that a minimal recoding polygon for X is a coding polygon for a recoding of X , not necessarily for X itself.

We show in Proposition 3.13 that if a coding polygon \mathcal{P} for X is equal to $n\mathcal{P}_0$ for some integral polygon \mathcal{P}_0 , then X can be recoded to a polygonal system with a coding polygon \mathcal{P}_0 . Hence a minimal recoding polygon must be primitive in the sense that it is not an integer multiple of a smaller integer polygon. A natural question arises: what are the possible minimal recoding polygons for a polygonal shift?

The geometry of the minimal recoding polygons for a polygonal system X is closely linked to the nonexpansive rays of X . To make this more precise we refine the notion of parallel to distinguish whether parallel objects have orientations which coincide (see Section 2.3 for complete definitions). We view a *ray* in \mathbb{R}^2 as a translate of the set $\ell_v = \{tv : t \geq 0, v \neq 0 \in \mathbb{R}^2\}$, and assume it is oriented in the direction of increasing t . We refer to two rays ℓ_1 and ℓ_2 which are translates

of each other as *positively parallel*, and when the rays ℓ_1 and $-\ell_2$ are positively parallel, we say ℓ_1 and ℓ_2 are *antiparallel*. Thus the standard understanding of rays being parallel means they are either positively parallel or antiparallel. We extend these conventions to oriented line segments, referring to such a segment as positively parallel to a ray ℓ (or to another line segment) if it has the same orientation and otherwise as antiparallel to ℓ (or again to another line segment).

Orientations extend to polygons $\mathcal{P} \subset \mathbb{R}^2$: such a polygon inherits an orientation from \mathbb{R}^2 , and this orientation induces an orientation on the boundary $\partial\mathcal{P}$ and hence an orientation on each edge of \mathcal{P} . In a convex polygon, no two edges can be positively parallel, but pairs of edges may be antiparallel.

Theorem 1.1. *If X is an infinite polygonal shift and \mathcal{P}_0 is a minimal recoding polygon for some recoding of X , then any ray positively parallel to the oriented edges of \mathcal{P}_0 is nonexpansive for X and every other ray is expansive.*

This result is an immediate consequence of Proposition 3.2 and Theorem 5.12 and Theorem 5.13, and it provides a necessary condition for a \mathbb{Z}^2 -system to be polygonal; it must have finitely many nonexpansive rays and they must all have rational slope.

However, an example of Hochman [12] shows that this condition is not sufficient. There is an additional necessary property, called *closing* (see Definition 3.14), that must be satisfied by the nonexpansive rays in polygonal systems. With this additional hypothesis we have both necessity and sufficiency:

Theorem 1.2. *Suppose X is an infinite \mathbb{Z}^2 -subshift with finitely many nonexpansive rays each of which has rational slope and is closing. Then there is a recoding Y of X which is polygonal.*

This result follows from Theorem 5.12, and in Theorem 5.13, we give a version of the converse: if \mathcal{P}_0 is a coding polygon for X , then X can be recoded to a subshift Y with a coding polygon \mathcal{P} having m edges, the minimum possible.

Moreover any two such minimal recoding polygons have parallel edges (and hence have equal corresponding angles). If \mathcal{P}_0 is a triangle, we can say more and in Corollary 5.14, we show that if \mathcal{P}_0 is a minimal recoding triangle for an infinite X , then it is uniquely determined up to translation. We do not know if this generalizes, and in particular do not know if minimal recoding polygons which are not triangles are unique up to translation.

In our setting, we do not know if a system isomorphic to a polygonal system is also polygonal, but in Corollary 5.5 we show that if Y is a recoding of X and X is polygonal then so is Y .

In Section 6, we study various forms of entropy for \mathbb{Z}^2 -systems. For an arbitrary \mathbb{Z}^2 -system X and direction $v \in \mathbb{Z}^2$, there is a seminorm $\|\cdot\|_X$ that captures the directional entropy for X in direction v (see [3] and [21]). In Corollary 6.6 we observe that a result of Milnor implies that for any polygonal system, whose coding polygon has no antiparallel sides, this seminorm $\|\cdot\|_X$ is either identically zero or is a norm. Furthermore, in Proposition 6.10 we show that in this case, if the entropy norms are nontrivial, then the associated seminorms for the family of polygonal systems associated to a given polygon is a quasi-conformal family. Roughly speaking, this means that for any subshift Y in the same family as a subshift X , a sphere in the norm $\|\cdot\|_X$ has bounded eccentricity in $\|\cdot\|_Y$ with a bound that is independent of Y .

More precisely, suppose \mathcal{P} is a rational polygon which has no antiparallel edges and $\mathfrak{F}(\mathcal{P})$ is the family of all \mathbb{Z}^2 -subshifts which are polygonal with respect to \mathcal{P} and which have nontrivial entropy norms. Then we show (Proposition 6.10) that there is a uniform dilatation constant $D > 0$, depending only on \mathcal{P} , which has the property that for all $X \in \mathfrak{F}(\mathcal{P})$ and any $u, v \in S^1$ we have

$$\frac{1}{D} \leq \frac{h_u(X)}{h_v(X)} \leq D.$$

When the polygon is a triangle we obtain a stronger result, showing that they are conformally equivalent.

Corollary 6.9, we show that if X, Y are triangular \mathbb{Z}^2 -subshifts with nontrivial entropy norms with respect to the same rational triangle \mathcal{T} , then there is a constant $C > 0$ such that $\|\cdot\|_X = C\|\cdot\|_Y$ and the constant does not depend on the direction chosen in \mathbb{R}^2 .

Acknowledgment. We thank Van Cyr for many invaluable conversations during the preparation of this paper.

2. BACKGROUND ON SHIFT SYSTEMS

2.1. Shift systems. We assume throughout that \mathcal{A} is a finite set, called the *alphabet*, endowed with the discrete topology. For $d \geq 1$, we endow $\mathcal{A}^{\mathbb{Z}^d}$ with the product topology. We review the standard definitions for $\mathcal{A}^{\mathbb{Z}^d}$ for any $d \geq 1$ when there is no notational difference, but in most of the article we focus on two dimensions.

An element $x: \mathbb{Z}^d \rightarrow \mathcal{A}$ is called a *coloring* and $x(u)$ denotes the color of x at the position $u \in \mathbb{Z}^d$. When we want to make use of both

coordinates in two dimensions, we use $x(i, j)$ to denote the color of x at the position $(i, j) \in \mathbb{Z}^2$.

If $X \subset \mathcal{A}^{\mathbb{Z}^d}$ is closed and invariant under the \mathbb{Z}^d action $(T^u x)(v) = x(u + v)$ for $u \in \mathbb{Z}^d$, then we say that X is \mathbb{Z}^d -*subshift*, and when the context is clear, we shorten this and say that X is a *shift system* or just a *shift*, omitting the transformations from the notation. Thus in two dimensions, such X is implicitly endowed with the horizontal $T^{(1,0)}$ and vertical $T^{(0,1)}$ shifts. When considering more than one shift possibly with different alphabets, we write (X, \mathcal{A}) to emphasize the alphabet. If in addition we need to distinguish the transformations on different shifts, we write (X, T_X) , or (X, \mathcal{A}, T_X) when we need to capture all of the data. When there is no possible ambiguity, we refer to transformations T_X and T_Y on different spaces X and Y as just T .

2.2. Coding and Recoding. Of particular interest is how the coloring information from one region in \mathbb{Z}^d forces the coloring of another region, or perhaps all of \mathbb{Z}^d . We recall a definition from [3] which makes this precise:

Definition 2.1. If X is a \mathbb{Z}^d -subshift and $A, B \subset \mathbb{R}^d$, then A *X-codes* B if for all $x, x' \in X$, whenever x and x' agree on $A \cap \mathbb{Z}^d$, then they also agree on $B \cap \mathbb{Z}^d$. If the shift X is clear from the context, we just say that A *codes* B . In a slight abuse of notation, we say A codes $v \in \mathbb{Z}^d$ to mean that A codes the set $\{v\}$ of a single element.

Note that there is no assumption that the region A is finite, and the definition is stated for A, B as subsets of \mathbb{R}^d . Though the configurations $x, x' \in X$ are only defined on integral coordinates, the more general definition of the subsets gives us necessary flexibility for some of the results.

A trivial example of a region coding another is in a doubly periodic shift, where any set A that contains a full period completely determines an entire configuration and so codes all of $\mathcal{A}^{\mathbb{Z}^2}$. At the opposite extreme is the full shift $\mathcal{A}^{\mathbb{Z}^2}$; no region codes any larger region.

Since by definition a shift system is translation invariant, we have the following immediate fact:

Remark 2.2. Since a shift X is invariant under the \mathbb{Z}^d -action, it follows immediately that for every $v \in \mathbb{Z}^d$, if A codes B then $A + v$ codes $B + v$.

Recall that an *isomorphism* $\Psi: (X, \mathcal{A}, T_X) \rightarrow (Y, \mathcal{A}', T_Y)$ is homeomorphism such that $\Psi \circ T_X = T_Y \circ \Psi$.

Definition 2.3. If (X, \mathcal{A}) is a \mathbb{Z}^2 -shift and F is a finite subset of \mathbb{Z}^2 , we say the \mathbb{Z}^2 -shift (Y, \mathcal{A}') is a *recoding of* (X, \mathcal{A}) *via* F provided there

is an isomorphism of \mathbb{Z}^2 -shifts $\Psi: (X, \mathcal{A}) \rightarrow (Y, \mathcal{A}')$ such that for every $(i, j) \in \mathbb{Z}^2$ and all $x, x' \in X$

$$\Psi(x)(i, j) = \Psi(x')(i, j) \text{ if and only if } x|_{F(i, j)} = x'|_{F(i, j)}$$

where $F(i, j) := F + (i, j)$. Equivalently $\{(i, j)\}$ Ψ^{-1} -codes $F(i, j)$ and $F(i, j)$ Ψ -codes $\{(i, j)\}$.

Note that we slightly overload notation but it should be clear from the context what is meant. We use capital letters such as F or R for subsets of \mathbb{Z}^2 and in this case, for example, $F(i, j)$ denotes the translate of the set $F + (i, j) = \{f + (i, j) : f \in F\}$, while we use lower case letters such as x or y for elements of a shift X and in this case, for example, $x(i, j)$ denotes the color which x assigns to (i, j) .

Given X and any finite subset $F \subset \mathbb{Z}^2$, we define the *canonical recoding* (X_F, \mathcal{A}_F) via F by setting \mathcal{A}_F to be the set of all colorings of F which are the restriction of colorings in X and setting Ψ to be the isomorphism induced by the block map which assigns to a restriction to F of an X -coloring the element of \mathcal{A}_F it represents.

Recall that a map $\Psi: X \rightarrow Y$ is an *r-block code* if for all $x \in X$, the color that $\Psi(x)$ assigns to 0 is determined by the values of $x(i, j)$ with $\|(i, j)\| \leq r$ (when needed, we use the Euclidean norm $\|\cdot\|$ on \mathbb{R}^2).

If $\Psi: (X, \mathcal{A}) \rightarrow (Y, \mathcal{A}')$ is a recoding, then its inverse is an isomorphism induced by a 0-block map $\phi: \mathcal{A}' \rightarrow \mathcal{A}$. It is clear that the relation “ Y is a recoding of X ” is reflexive. It is also transitive, because the composition of two recordings is a recoding. However, this relation is not symmetric, as whenever (Y, \mathcal{A}') is a recoding of (X, \mathcal{A}) , it follows that $\text{card}(\mathcal{A}') \geq \text{card}(\mathcal{A})$ and this inequality is usually strict. Indeed if Y is a recoding of X and X is a recoding of Y , then there is a bijection of their respective alphabets which induces an isomorphism.

We now show if (Y, \mathcal{A}') is a recoding of (X, \mathcal{A}) via a finite set F , then there is an isomorphism of Y with X_F induced by a bijection of \mathcal{A}' and \mathcal{A}_F .

Lemma 2.4. *Suppose X is a \mathbb{Z}^2 -shift, $F \subset \mathbb{Z}^2$ is finite, and X_F is the canonical recoding.*

- (1) *If $v \in \mathbb{Z}^2$ and T_X^v is the shift on X corresponding to v , then $\Psi: (X, \mathcal{A}) \rightarrow (Y, \mathcal{A}')$ is a recoding via F if and only if $T_Y^v \circ \Psi$ is a recoding of X via $T_X^v(F)$.*
- (2) *If (Y, \mathcal{A}') is a recoding of (X, \mathcal{A}) via F , then there is an isomorphism of (Y, \mathcal{A}') and the canonical recoding (X_F, \mathcal{A}_F) induced by a bijection of the alphabet \mathcal{A}' with the alphabet \mathcal{A}_F .*

Proof. The first part follows immediately from the definition of recoding. To prove the second statement, note that if $\alpha \in \mathcal{A}'$, then α

determines a coloring of $\{(0,0)\}$ for the shift Y . Since $\{(0,0)\}$ Ψ^{-1} -codes F , it follows that α determines a unique coloring β of F for X . The assignment $\alpha \mapsto \beta$ determines a bijection from \mathcal{A}' to \mathcal{A}_F which, as a block map, determines an isomorphism $\Psi_F \circ \Psi^{-1}$ from (Y, \mathcal{A}') to (X_F, \mathcal{A}_F) . \square

It is frequently useful to know that a finite set coded by a set A is also coded by a finite subset of A . This follows via an easy compactness argument:

Lemma 2.5. *Assume that X is a \mathbb{Z}^d -subshift. If $A \subset \mathbb{Z}^d$ codes B and B is finite, then there is a finite subset $A_0 \subset A$ such that A_0 codes B .*

Proof. Without loss of generality, it suffices to prove the result when B contains a single point $b \in \mathbb{Z}^d$ which is not an element of A . If the result fails, then for every $m \geq 0$ there exist $x_m, y_m \in X$ such that $x_m(b) \neq y_m(b)$, but $x_m(u) = y_m(u)$ for all $u \in A$ with $\|u\| \leq m$. Since X is compact, by passing to by subsequences if needed, we can assume that $\lim_{m \rightarrow \infty} x_m = x'$ and $\lim_{m \rightarrow \infty} y_m = y'$ for some $x', y' \in X$. Then $x'(b) \neq y'(b)$, but $x'(u) = y'(u)$ for all $u \in A$, a contradiction as A codes $\{b\}$. \square

2.3. Notions of parallel. We summarize the various notions of parallel that we use throughout the sequel.

By a *ray* in \mathbb{R}^2 , we mean a translate of the set $\ell_v = \{tv : t \geq 0, v \neq 0 \in \mathbb{R}^2\}$, and we view a ray as oriented in the direction of increasing t .

Two rays ℓ_1 and ℓ_2 are *positively parallel* if one is a translate of the other, and they are *antiparallel* if ℓ_1 and $-\ell_2$ are positively parallel. We say that two rays are *parallel* if they are either positively parallel or antiparallel.

We extend these conventions to line segments, and we say that an oriented line segment J is *positively parallel* to a ray ℓ if a translate of J lies in ℓ with matching orientations, and we say that the orientated line segment J is *antiparallel* if J is positively parallel to $-\ell$.

Similarly, we say that two oriented line segments are *positively parallel* if a translate of one lies in the other with matching orientations and are *antiparallel* if one is positively parallel with the other with reversed orientation. Since we need to distinguish the various notions our terminology differs a bit from that in [6], where parallel corresponds to our use of positively parallel, while the use of antiparallel is the same.

A polygon $\mathcal{P} \subset \mathbb{R}^2$ inherits an orientation from \mathbb{R}^2 , and this orientation induces an orientation on the boundary $\partial\mathcal{P}$, and this further restricts to an orientation on each edge of \mathcal{P} . For a convex polygon,

no two edges can be positively parallel, but pairs of edges may be antiparallel.

2.4. Expansive and nonexpansive. The fundamental concept related to one region coding another is that of expansivity, defined in Milnor [21] and developed by [3], and we review this in our particular setting of two dimensions. Letting d denote the distance in \mathbb{R}^2 , a subspace L of \mathbb{R}^2 is *expansive* if there exists $r > 0$ such that the r -neighborhood $N_r = \{u \in \mathbb{R}^2 : d(u, L) < r\}$ of L codes \mathbb{R}^2 (the analogous definition can be made in any dimension). Any subspace that is not expansive is called a *nonexpansive* subspace.

Nonexpansive subspaces are common:

Theorem 2.6 (Boyle and Lind [3]). *If X is an infinite compact metric space with a continuous \mathbb{Z}^k -action, then for each $0 \leq j < k$ there exists a j -dimensional subspace of \mathbb{R}^k that is nonexpansive.*

For the two dimensional setting, an immediately corollary is that a system X is finite (and hence doubly periodic) if and only if every subspace of \mathbb{R}^2 is expansive.

For our purposes, the notion of expansiveness can be refined to consider one-sided expansiveness, where the coloring of N_r determines the coloring of one component of the complement of L . We make this more precise (similar notions were considered in [1, 2, 6]):

Lemma 2.7. *Assume X is a \mathbb{Z}^2 -subshift and suppose H is an (open or closed) half space in \mathbb{R}^2 . Then either H codes all of \mathbb{R}^2 or H codes itself but no points of $\mathbb{Z}^2 \setminus H$. In particular, if any subset of H codes any point of $\mathbb{Z}^2 \setminus H$, then H codes all of \mathbb{R}^2 .*

Proof. Suppose there is no $b \in \mathbb{Z}^2 \setminus H$ such that H codes $\{b\}$. In this case, H codes subsets of itself and no other subsets of $\mathbb{R}^2 \cap \mathbb{Z}^2$. Otherwise, there exists $b \in \mathbb{Z}^2 \setminus H$ such that H codes $\{b\}$. We prove this implies H codes \mathbb{R}^2 .

First consider a special case: assume that H is closed and there exists some $z \in \partial H \cap \mathbb{Z}^2$. Let $w = b - z \in \mathbb{Z}^2$ and define $H_1 = w + H$. Then $\partial H_1 = L + w$ contains b . Thus H_1 is a closed half space properly containing H . We claim that if $u \in H_1 \cap \mathbb{Z}^2$, then H codes $\{u\}$. To see this, let $v = u - b$. Since $b \in \partial H_1$ and $u \in H_1 \cap \mathbb{Z}^2$, we have $v + H_1 \subset H_1$ and hence $v + H \subset H$. Since H codes $\{b\}$, we have that $v + H$ codes $v + b = u$. But $v + H \subset H$, proving the claim.

By the claim, it follows that H codes H_1 . Define $H_n = nw + H$. Then since H codes H_1 , it follows from the translation invariance (Remark 2.2) that $H_n = H + nw$ codes $H_{n+1} = H_1 + nw$. Hence H codes

$\bigcup_n H_n = \mathbb{R}^2$, meaning that H codes \mathbb{R}^2 . This completes the proof in the special case that H is closed and $\partial H \cap \mathbb{Z}^2$ is nonempty.

We now turn to the general case, assuming that H is an open or closed half space bounded by L (and no assumption that the line L contains points of \mathbb{Z}^2). The point $b \in \mathbb{Z}^2$ is coded by H , but $b \notin H$. By Lemma 2.5, there is a finite set $A \subset H$ which codes $\{b\}$. For each $a \in A$, let Y_a denote the closed half space contained in H whose boundary is the line L_a which is parallel to L and contains a . If

$$Y = \bigcup_{a \in A} Y_a,$$

then $Y \subset H$ and Y is a closed half space which codes $\{b\}$ and $b \notin Y$. Also ∂Y contains some point of A and hence some point of \mathbb{Z}^2 . It follows that Y satisfies the hypothesis of the first case and so Y codes \mathbb{R}^2 . Since $Y \subset H$, we also have that H codes \mathbb{R}^2 . \square

Note that for any $v \in \mathbb{R}^2$ (not necessarily integral), a half space H is expansive if and only if $v + H$ is expansive. For $v \in \mathbb{Z}^2$, this follows immediately from the translation invariance (Remark 2.2). More generally, for any $v \in \mathbb{R}^2$ there exist $z_1, z_2 \in \mathbb{Z}^2$ such that $z_1 + H \subset v + H \subset z_2 + H$ and so $z_1 + H$ expansive implies $v + H$ is expansive and $v + H$ expansive implies $z_2 + H$ is expansive.

We encapsulate the dichotomy of Lemma 2.7 in the following definition:

Definition 2.8. Assume X is a \mathbb{Z}^2 -subshift. If H is an (open or closed) half space in \mathbb{R}^2 , we say that H is *expansive* if H codes \mathbb{R}^2 and otherwise we say that H is *nonexpansive*. If H is expansive and ℓ is a ray parallel to the boundary of H whose orientation agrees with the orientation ∂H inherits from the standard orientation on H , we say that ℓ is an *expansive ray in X* and otherwise we say that ℓ is a *nonexpansive ray in X* . When it is clear from the context, we shorten this and say that ℓ is *expansive* (or *nonexpansive*).

Remark 2.9. It is easy to see that if (X, \mathcal{A}) and (Y, \mathcal{A}') are isomorphic shifts, then a ray ℓ is expansive for one if and only if it is expansive for the other (see Remark 2.2).

We note that the half space H being nonexpansive is equivalent to the existence of $x_1, x_2 \in X$ with $x_1 \neq x_2$ such that $x_1(i, j) = x_2(i, j)$ for all $(i, j) \in H$. This non-uniqueness in the extension of the half space is often how we make use of this notion.

A one-dimensional subspace L of \mathbb{R}^2 is (two-sided) expansive if for some $r > 0$, the strip $N_r(L) = \{u \in \mathbb{R}^2 : d(u, L) \leq r\}$ codes \mathbb{R}^2 . This

implies that the action on X by a nonzero element $v \in L \cap \mathbb{Z}^2$ is an expansive homeomorphism of X . The following corollary shows that a subspace L is expansive in this sense if and only if both of its complementary half spaces satisfy our definition of one-sided expansiveness (Definition 2.8).

Proposition 2.10. *If H and H' are the two closed half spaces whose common boundary is L and H codes all of \mathbb{R}^2 , then there exists $r > 0$ such that the closed strip $N_r(H) = \{u \in H : d(u, L) \leq r\}$ codes all of H' .*

Proof. Without loss of generality, we can assume that $L = \partial H$ contains some point of \mathbb{Z}^2 : choosing a (not necessarily integral) translate L_0 of L which lies in H and does contain a point of \mathbb{Z}^2 , we can prove the result for $H_0 \subset H$ with $L_0 = \partial H_0$ and obtain the result for H (possibly with a larger value of r).

As in the special case in the proof of Lemma 2.7, H codes some $b \in \text{int}(H') \cap \mathbb{Z}^2$. Choose $z \in L \cap \mathbb{Z}^2$ and set $w = b - z$. There is a finite set $A \subset H$ which codes $\{b\}$. Let $\delta = d(b, L)$. Suppose $u \in H'$ and $d(u, L) \leq \delta$. Setting $v = u - b$, the component of v orthogonal to L has length $\leq \delta$ and so $A + v \subset N_r(H)$ where $r = \delta + \text{diam}(A)$. Also $A + v$ codes $b + v = u$ and so $N_r(H)$ codes the closed strip S whose boundary components are L and $L + w$. The strip S is parallel to L and has width δ . The same argument shows that $S \cup N_r(H)$ codes the strip $S + w$. Inductively, it follows that $N_r(H) \cup (nw + S)$ codes $N_r(H) \cup ((n+1)w + S)$, and so $N_r(H)$ codes H' . \square

The set of expansive rays in \mathbb{R}^2 is open (see [3, 7]). This also follows immediately from Lemma 2.5, which gives the existence of a finite set $A \subset H$ which codes $b \notin H$, and the fact that the set of oriented rays in the plane which span lines separating b from A is an open set.

It thus follows that the set of nonexpansive rays is closed, and it is known to be nonempty if X is infinite (see Theorem 2.6). For the full shift $\mathcal{A}^{\mathbb{Z}^2}$, it is easy to see that all rays are nonexpansive; there are no expansive half spaces. The nonexpansive rays play a significant role in the dynamics of \mathbb{Z}^2 -subshifts because the boundary of a nonexpansive half space forms a barrier to coding. In particular, Lemma 2.7 asserts that if H is nonexpansive, then no subset of H can code a subset of $\mathbb{Z}^2 \setminus H$.

3. DEFINING THE CLASS OF SHIFTS

3.1. Polygonal shifts. We have assembled the tools to define the class we study:

Definition 3.1. Suppose X is an infinite \mathbb{Z}^2 -subshift, \mathcal{P} is a convex integer polygon, and v is a vertex of \mathcal{P} . If $\mathcal{P} \setminus \{v\}$ X -codes $\{v\}$, then we say that \mathcal{P} is a *coding polygon* for the vertex v . If \mathcal{P} is coding for each of its vertices, we say X is *polygonal* with respect to \mathcal{P} or that \mathcal{P} is a *coding polygon* for X .

A polygonal \mathbb{Z}^2 -system is *triangular* if the associated polygon is a triangle.

Note that translation invariance implies that when $\mathcal{P} \setminus \{v\}$ X -codes $\{v\}$, we also have that $(\mathcal{P} + u) \setminus \{v + u\}$ X -codes $\{v + u\}$ for all $u \in \mathbb{Z}^2$. Thus it makes sense to discuss a coding polygon defined only up to translation in \mathbb{Z}^2 . However, coding polygons, even up to this translation, are not unique and it takes work to understand to what extent a coding polygon can be simplified. One notion of simplification is having the fewest number of edges, and this motivates us to restrict our attention to convex polygons. If a non-convex polygon is coding, then its convex hull has fewer sides and is also a coding polygon.

Proposition 3.2. *Suppose X is a \mathbb{Z}^2 -subshift and P is a coding polygon for X . If ℓ is a nonexpansive ray in X , then ℓ is positively parallel to an edge of P whose orientation matches the orientation of ℓ .*

Proof. Let L be the one-dimensional subspace of \mathbb{R}^2 containing ℓ and let H be the open half space bounded by L such that expansiveness of H implies expansiveness of ℓ . Suppose first that L is not parallel to any edge of P . Then there is vertex e of P such that $P \cap (e + L) = \{e\}$ and $P \subset e + H$. Since $P \setminus \{e\} \subset e + H$ codes $e \notin e + H$, Lemma 2.7 implies that $e + H$ is expansive. Hence H is expansive and so is ℓ , a contradiction.

It follows that if ℓ is nonexpansive, then it is parallel to an edge of P . If it is positively parallel to one edge and antiparallel to another, then those edges have opposite orientations and so ℓ is positively parallel to one of them. Finally, if ℓ is antiparallel to a single edge E , then there is a unique vertex $e \in P$ and a unique supporting line L parallel to ℓ such that $L \cap P = \{e\}$. If H is the open half space which is bounded by L and which contains $P \setminus \{e\}$, Lemma 2.7 implies that H is expansive (note that $P \setminus \{e\} \subset H$ codes $e \notin H$). The orientation L inherits from H is the opposite of the orientation E inherits from P . Since the ray ℓ is antiparallel to E and H is expansive, ℓ must be expansive, a contradiction. The only remaining possibility is that ℓ is positively parallel to E . \square

Although coding polygons are not unique, the existence of a coding polygon implies that scaled versions are also coding polygons. To make

this precise, given $P \subset \mathbb{Z}^2$, we write

$$nP = \{nx : x \in P\}.$$

We frequently make use of the following straightforward observation:

Observation 3.3. If X is polygonal with respect to the convex polygon \mathcal{P} , then it is also polygonal with respect to the polygon $n\mathcal{P}$ for every $n \in \mathbb{N}$.

This can be seen by noting that if v is a vertex of \mathcal{P} , there is a translation T such that $T(v) = nv$ and then $T(\mathcal{P})$ is a subpolygon of $n\mathcal{P}$ whose vertex at nv coincides with that of $n\mathcal{P}$.

3.2. Examples of polygonal shifts. We give various examples of polygonal shifts.

Example 3.4. *Ledrappier three-dot system* [18]. Let $\mathcal{A} = \{0, 1\}$ be the field with two elements and take X to be the subshift of $\mathcal{A}^{\mathbb{Z}^2}$ defined by

$$X = \{x \in \mathcal{A}^{\mathbb{Z}^2} : x(i, j) + x(i+1, j) + x(i, j+1) = 0 \pmod{2}\}$$

for $i, j \in \mathbb{Z}$. Note that if $x \in X$ and $R_i(x)$ is the element in the one-dimension shift $\mathcal{A}^{\mathbb{Z}}$ obtained by restricting x to its i^{th} horizontal row, then $R_{i+1}(x) = \phi(R_i(x))$ where ϕ is the endomorphism defined by $\phi(y)_0 = y_0 + y_1 \pmod{2}$. The \mathbb{Z}^2 -subshift X is triangular (with respect to the triangle T with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$). It has three nonexpansive rays, which are the positive x -axis, the negative y -axis, and the ray $(-t, t)$, $t \geq 0$.

Ledrappier's three dot system and related algebraic systems have been studied by Ledrappier [18], Einsiedler [8], and Kitchens and Schmidt [16]. In particular we have the following extension:

Example 3.5. *Einsiedler's examples.* In [8] Einsiedler proves the existence of closed \mathbb{Z}^2 -invariant subsystems of the Ledrappier example realizing any horizontal directional entropy between 0 and $\ln(2)$. Since these are subsystems of the Ledrappier system, they are all triangular with respect to the triangle T . Following [8] and [17], we describe one such example. Taking X to be the Ledrappier system of Example 3.4, consider the subset

$$Y_0 = \{x \in X : x(2v) = 0 \text{ for all } v \in \mathbb{Z}^2\}.$$

Then Y_0 is invariant under the \mathbb{Z}^2 -action obtained by restricting the \mathbb{Z}^2 -action on X to the lattice $(2\mathbb{Z})^2$. While Y_0 is not invariant under the full \mathbb{Z}^2 -action, defining $Y_1 = Y_0 + (1, 0)$, $Y_2 = Y_0 + (0, 1)$, $Y_3 = Y_0 + (1, 1)$, then $Y = Y_0 \cup Y_1 \cup Y_2 \cup Y_3$ is a closed proper \mathbb{Z}^2 -invariant subset of

X . Let R_i denote the restriction of Y_i to the x -axis and R denote the restriction of Y . Then each R_i is a closed subset of the one-dimensional full shift space $\Sigma = \mathcal{A}^{\mathbb{Z}}$ with

$$\begin{aligned} R_0 &= \{y \in \Sigma : y_{2n} = 0 \text{ for all } n \in \mathbb{Z}\} \\ R_1 &= \{y \in \Sigma : y_{2n+1} = 0 \text{ for all } n \in \mathbb{Z}\} \\ R_2 &= \{y \in \Sigma : y_{2n} = y_{2n+1} \text{ for all } n \in \mathbb{Z}\} \\ R_3 &= \{y \in \Sigma : y_{2n} = y_{2n-1} \text{ for all } n \in \mathbb{Z}\}. \end{aligned}$$

Define $\sigma: R \rightarrow R$ to be the left shift and observe that $\sigma^2(R_i) = R_i$. One checks easily that $\sigma^2|_{R_i}: R_i \rightarrow R_i$ is conjugate to the full 2-shift, and so $\sigma^2: R_i \rightarrow R_i$ has topological entropy $\ln(2)$. Since $R_i \cap R_j$ contains at most the two σ -fixed points $\bar{0}$ and $\bar{1}$ for all $i \neq j$, it follows that $\sigma^2: R \rightarrow R$ has topological entropy $\ln(2)$ and hence $h(\sigma) = \ln(2)/2$.

It follows from Einsiedler's results that uncountably many horizontal entropies can be realized in constructing the examples in 3.5. All but countably many of the associated subshifts are not sofic, since there are at most countably many sofic systems with a given alphabet. Thus some of the subshifts realized in Example 3.5 are not sofic and, in particular, are not subshifts of finite type.

Example 3.6. *Low complexity examples.* Polygonal systems arise naturally in studying the Nivat Conjecture, and in this direction, It follows immediately from [6, Corollary 2.6] that (note our terminology differs, and related results appear in [5, 14]):

Proposition 3.7. *Suppose X is a \mathbb{Z}^2 -subshift with alphabet \mathcal{A} , S is a finite convex subset of \mathbb{Z}^2 , and $C(S)$ denotes the number of legal X colorings of S . If*

$$C(S) \leq |S| + |\mathcal{A}| - 2,$$

then X is polygonal.

In particular, it follows from [6] that any counterexample to the Nivat conjecture must be polygonal.

Example 3.8. *Non-abelian groups.* Similar to the construction of the Ledrappier system, one can take a finite (possibly non-abelian) group G as the alphabet and require, for example, that the product of the colors at the vertices of a convex polygon P is the identity (or some other fixed $g \in G$).

Example 3.9. *Enveloping SFTs.* Any polygonal subshift X with coding polygon \mathcal{P} is a closed invariant subset of a polygonal subshift \hat{X} which has the same coding polygon \mathcal{P} and which is a (two-dimensional)

SFT. Namely, let \hat{X} be the SFT whose excluded blocks are precisely the set of colorings of \mathcal{P} which violate the requirements for \mathcal{P} to be a coding polygon. Then \hat{X} is an SFT by definition and is nonempty because it contains the points of X .

Definition 3.10. We say that a \mathbb{Z}^2 -subshift X is a *polygonal subshift of finite type (polygonal SFT)* provided it is a (two-dimensional) SFT and its defining finite set of excluded blocks are all colorings of a single coding polygon \mathcal{P} for X .

We note that a subshift X is a polygonal SFT if and only if it is polygonal and also an SFT. This follows from Observation 3.3 since one can scale up the polygon \mathcal{P} to $n\mathcal{P}$ until it is sufficiently large to hold all excluded colored blocks which define the SFT X .

Example 3.11. Products. Let X and Y be two (one-dimensional) subshifts and extend X to a two dimensional subshift by extending it with period one in the horizontal direction and extend Y to a two dimensional subshift by extending it with period one in the vertical direction. Consider the \mathbb{Z}^2 -subshift obtained by taking the product of the two alphabets, meaning that the entry (i, j) is endowed with the symbol i from the corresponding entry in X and the symbol j from the corresponding entry in Y . This product system is polygonal, and the associated polygon is a 2×2 square.

This last example can be generalized to construct further polygonal shifts:

Proposition 3.12. *Suppose X_1 and X_2 are polygonal \mathbb{Z}^2 -shifts with respect to polygons P_1 and P_2 . If no pairs of edges from the two polygons are parallel, then $X_1 \times X_2$ is also polygonal.*

Proof. Consider the oriented edges of P_1 and P_2 as vectors and order them to form the edges of a convex polygon P . Define an alphabet consisting of ordered pairs of colors from the alphabets of X_1 and X_2 .

To check that this is a polygonal system, note that given a coloring of all but one vertex w of P , we can translate the polygon P_1 associated to X_1 such that P_1 lies in P and a vertex of P_1 coincides with w . This uniquely determines the color of the vertex of P_1 and hence the first component of the pair which is the coloring for $X_1 \times X_2$. The second component of the coloring for $w \in P$ is obtained similarly. \square

Proposition 3.13. *If X is polygonal with respect to the convex integer polygon \mathcal{P} and $\mathcal{P} = n\mathcal{P}_0$ for some integer polygon \mathcal{P}_0 and integer $n > 1$, then there is a recoding Y of X which is polygonal with respect to \mathcal{P}_0 . Hence $m\mathcal{P}_0 = \frac{m}{n}\mathcal{P}$ is also a coding polygon for Y .*

Proof. Let $\{e_i\}_{i=0}^k$ denote the edge vectors of \mathcal{P} taken in some order such that consecutive edges have a vertex in common. Then $v_j = \sum_{i=0}^j e_i$ denote the j^{th} vertex of \mathcal{P} in this ordering. Note that $v_k = \sum_{i=0}^k e_i = 0$. Since $\frac{1}{n}\mathcal{P}$ is an integer polygon, so is $\frac{m}{n}\mathcal{P} = m\mathcal{P}_0$ for $1 \leq m \leq n$.

Define $\mathcal{P}_1 := \frac{n-1}{n}\mathcal{P} = (n-1)\mathcal{P}_0$. Then for each edge e of \mathcal{P} , there are exactly n translates of \mathcal{P}_1 each of which lies in \mathcal{P} and has a translate of the edge $\frac{1}{n}e$ of \mathcal{P}_1 lying in e .

Let $\Psi: X \rightarrow X_{\mathcal{P}_1}$ be the canonical recoding of X (see Definition 2.3) via \mathcal{P}_1 and let $Y = X_{\mathcal{P}_1}$. Then the polygon \mathcal{P} Ψ -codes a translate of $\mathcal{P}_0 = \frac{1}{n}\mathcal{P}$. Likewise \mathcal{P}_0 Ψ^{-1} -codes a translate of \mathcal{P} . It follows that \mathcal{P}_0 is a coding polygon for Y . \square

This result shows that if \mathcal{P} is a coding polygon for X and $\mathcal{P} = n\mathcal{P}_0$, then X can be recoded to Y with a strictly smaller but similar coding polygon. Hence if \mathcal{P} has the minimal number of edges of coding polygons for X , then X can be recoded to a polygonal system with a minimal recoding polygon.

3.3. Refining notions of expansivity. Suppose L is a rational line in \mathbb{R}^2 containing a point of \mathbb{Z}^2 (and hence infinitely many points of \mathbb{Z}^2). Recall that $L + \mathbb{Z}^2$ is a discrete set of lines, meaning there exists $r > 0$ such that any line $z + L$ distinct from L and with $z \in \mathbb{Z}^2$ must have distance from L equal to mr with $m \in \mathbb{N}$. There are two closest integer translates of L which have distance r from L , lying on opposite sides of L .

If $L \subset \mathbb{R}^2$ is a one dimensional subspace, we refer to the intersection of a connected segment of L with \mathbb{Z}^2 as a *block* in $L \cap \mathbb{Z}^2$.

Definition 3.14. Assume $L \subset \mathbb{R}^2$ is a one dimensional subspace with rational slope and suppose L bounds a nonexpansive closed half space H . Let L_0 be the closest line of the form $z + L$ in the complement of H for some $z \in \mathbb{Z}^2$. If there exists $N > 0$ such that every block \mathcal{B} in $L_0 \cap \mathbb{Z}^2$ of length $\geq N$ the set $H \cup \mathcal{B}$ codes $H \cup L_0$ then, we say that H is *closing*. If ℓ is the ray in L whose orientation is inherited from H , we say that ℓ is *closing*.

Note that by definition, a ray that is closing is also nonexpansive and has rational slope.

We example the rationale behind the use of the term closing. In, for example, the Ledrappier system (Example 3.4), the upper half space $H = \{(u, v) : v \geq 0\}$ is nonexpansive. The subspace H is also closing. This latter property is equivalent to the fact that the endomorphism

ϕ defining the system is both right and left closing in the sense of [19, Chapter 8].

We illustrate the property of closing with an example:

Proposition 3.15. *If X is polygonal with coding polygon \mathcal{P} and ℓ is a nonexpansive ray with the same direction as an oriented edge of \mathcal{P} , then ℓ is closing.*

Proof. Without loss of generality, we may assume the ray ℓ is the positive horizontal axis, meaning that the oriented edge lies in the horizontal axis L and \mathcal{P} lies in the closed upper half space H . Then L_0 is the line $L + (0, -1)$. Let N be the number of points in $J := L \cap \mathcal{P}$. Set $\mathfrak{B} = J(0, -1)$ and note that every integer point of the polygon $\mathcal{P} + (1, -1)$ lies in $H \cup \mathfrak{B}$ except one, namely the first point b to the right of \mathfrak{B} in L_0 . Since \mathcal{P} is a coding polygon, the coloring at b is determined by $\mathcal{P} + (1, -1)$ and hence by $H \cup \mathfrak{B}$. Repeating this, it follows that $H \cup \mathfrak{B}$ codes all points to the right of \mathfrak{B} . A similar argument shows it codes all points to the left of \mathfrak{B} . \square

It follows from Remark 2.9 that if a ray is nonexpansive for (X, \mathcal{A}) , then it is also nonexpansive for any isomorphic \mathbb{Z}^2 -shift (Y, \mathcal{A}') . Our next lemma shows that a recoding (and its inverse) preserves closing rays:

Lemma 3.16. *Suppose $\Psi: (X, \mathcal{A}) \rightarrow (Y, \mathcal{A}')$ is a recoding via a finite set F and suppose ℓ is a rational nonexpansive ray in \mathbb{R}^2 . Then ℓ is closing for X if and only if it is closing for Y .*

Proof. Suppose ℓ is closing for one of X or Y . We show it is closing for the other. By a change of coordinates, without loss of generality we can assume that ℓ is the positive x -axis. Let L be the x -axis and let H be the closed upper half space with boundary L . Thus H is closing for X .

By Lemma 2.4, if T is an action on X translating by some element of \mathbb{Z}^2 , then recoding via $T(F)$ is the same as recoding via F and then translating by T . Since translating by T preserves closing half spaces, we can assume that F lies in H and contains $(0, 0) \in L$, but contains no point of $\mathbb{Z}^2 \setminus H$. Let B_0 be the finite set $F \cap L$ and let $L_0 = L + (0, -1)$.

Suppose B is a finite block in L_0 . Since F Ψ -codes $\{(0, 0)\}$, it follows that H Ψ -codes H and $H + (0, -1)$ Ψ -codes $H + (0, -1)$. By the definition of recoding, $\{(0, 0)\}$ Ψ^{-1} -codes F and therefore $\{(i, j)\}$ Ψ^{-1} -codes $F + (i, j)$. Thus it follows that for all i, j , we have that $(i, j) + F$ Ψ -codes $\{(i, j)\}$ and $\{(i, j)\}$ Ψ^{-1} -codes $F + (i, j)$ and hence $\{(i, j)\}$.

Suppose now that H is X -closing and $B \subset L_0$ is a block such that $H \cup B$ X -codes L_0 . We wish to show that $H \cup B$ Y -codes L_0 , and so

H is Y -closing. Since $\{(i, j)\}$ Ψ^{-1} -codes $\{(i, j)\}$, we know that $H \cup B$ Ψ^{-1} -codes $H \cup B$ which X -codes $H \cup L_0 = H + (0, -1)$. This in turn Ψ -codes $H + (0, -1) \supset L_0$ and we have shown that $H \cup B$ Y -codes L_0 . Thus H is Y -closing.

Conversely, suppose H is Y -closing and $B \subset L_0$ is a block such that $H \cup B$ Y -codes L_0 . Since $(i, j) + F$ Ψ -codes $\{(i, j)\}$ we have, in fact, that $H \cup ((i, j) + B_0)$ Ψ -codes $\{(i, j)\}$. It follows that $H \cup ((B + B_0)$ Ψ -codes $H \cup B$. Note that $H \cup ((B + B_0)$ Ψ -codes $H \cup B$ which, in turn Y -codes L_0 . Since $B' := B \cup B_0$ is finite and $H \cup B'$ X -codes L_0 we have shown that H is X -closing. \square

4. CODING CORNERS IN CLOSING LIGHT CONES.

4.1. Spacetimes and light cones. We give way to extend a one dimensional system to a two dimensional version, with a variant of the definition of a spacetime from [7] (there is also a related notion called the complete history in Milnor [21]):

Definition 4.1. If X is a \mathbb{Z}^2 -subshift with oriented basis e_1, e_2 of \mathbb{Z}^2 and if the ray spanned by e_1 is expansive, then $\mathcal{U} = (X, \{e_1, e_2\})$ is called a *spacetime* and $\{e_1, e_2\}$ is called its *distinguished basis*. If $\mathcal{U}_1 = (X_1, \{e_1, e_2\})$ and $\mathcal{U}_2 = (X_2, \{f_1, f_2\})$ are spacetimes, an *isomorphism of spacetimes* $\Psi: \mathcal{U}_1 \rightarrow \mathcal{U}_2$ is a \mathbb{Z}^2 -subshift isomorphism $\Psi: X_1 \rightarrow X_2$ such that $\Psi \circ T_{e_i} = T_{f_i} \circ \Psi$.

Note that this definition of a spacetime \mathcal{U} is more general than that given in [7], where it is required that e_1 be 1-expansive in the sense that the line $L \cap \mathbb{Z}^2$ containing e_1 codes the half space $\{je_1 + me_2 \in \mathbb{Z}^2: m \geq 0\}$. This requirement is equivalent to the existence of an endomorphism ϕ of a \mathbb{Z} -subshift $\sigma: Y \rightarrow Y$ with the same alphabet as \mathcal{U} such that $u \in \mathcal{U}$ if and only if

- (1) for each $j \in \mathbb{Z}$ the sequence $\{y_n = u(n, j): n \in \mathbb{Z}\}$ is an admissible sequence in Y , and
- (2) if $y \in Y$ satisfies $y_n = u(n, j)$ for all $n \in \mathbb{Z}$, then for $\phi^m(y)_n = u(n, j + m)$ for all $n \in \mathbb{Z}$.

When these two conditions are satisfied, we say the spacetime \mathcal{U} is the *spacetime of the endomorphism ϕ* . We show (see Lemma 5.7 below) that if \mathcal{V} is any spacetime with at least one expansive ray, then it can be recoded to be the spacetime of an endomorphism.

If $\phi \in \text{End}(Y, \sigma)$ and $n \geq 0$, following [7] we define $W^+(n, \phi)$ to be the smallest integer such that the ray $[W^+(n, \phi), \infty)$ is ϕ^n -coded by $[0, \infty)$ and define $W^-(n, \phi)$ to be the largest integer such that the ray

$(-\infty, W^-(n, \phi)]$ is ϕ^n -coded by $(-\infty, 0]$. It is straightforward to check that

$$(4.1) \quad W^+(k, \phi\sigma^p) = pk + W^+(k, \phi) \text{ and } W^-(k, \phi\sigma^p) = pk + W^-(k, \phi),$$

for all $p \in \mathbb{Z}$ (see [7] for details).

Definition 4.2. The *future light cone* $\mathcal{C}_f(\phi)$ of $\phi \in \text{End}(X)$ is defined to be

$$\mathcal{C}_f(\phi) = \{(i, j) \in \mathbb{Z}^2 : W^-(j, \phi) \leq i \leq W^+(j, \phi), j \geq 0\}.$$

The *past light cone* $\mathcal{C}_p(\phi)$ of ϕ is defined to be $\mathcal{C}_p(\phi) = -\mathcal{C}_f(\phi)$. The *full light cone* $\mathcal{C}(\phi)$ is defined to be $\mathcal{C}_f(\phi) \cup \mathcal{C}_p(\phi)$.

We emphasize that $\mathcal{C}_p(\phi)$, the past light cone of ϕ , is typically *not* closely related to the light cone of ϕ^{-1} .

The light cone is naturally stratified into levels: define the n^{th} *level* of $\mathcal{C}(\phi)$ to be the set

$$(4.2) \quad \mathcal{I}(n, \phi) := \{i \in \mathbb{Z} : (i, n) \in \mathcal{C}(\phi)\}.$$

Recall that the edges of a light cone have asymptotic slopes defined by

$$\alpha^+ := \lim_{k \rightarrow \infty} \frac{W^+(k, \phi)}{k}$$

and

$$\alpha^- := \lim_{k \rightarrow \infty} \frac{W^-(k, \phi)}{k}.$$

These limits exist by Fekete's Lemma.

The edges of the light cone $\mathcal{C}(\phi)$ are given by the graphs of the functions $i = W^+(k, \phi)$ and $i = W^-(k, \phi)$ and have nice asymptotic properties.

Definition 4.3. The *asymptotic light cone* $A(\phi)$ of ϕ is defined to be the cone in \mathbb{R}^2 bounded by the lines $x = \alpha^+(\phi)y$ and $x = \alpha^-(\phi)y$, meaning that

$$\begin{aligned} A(\phi) = & \{(x, y) \in \mathbb{R}^2 : y \geq 0, \alpha^-(\phi)y \leq x \leq \alpha^+(\phi)y\} \\ & \cup \{(x, y) \in \mathbb{R}^2 : y \leq 0, \alpha^+(\phi)y \leq x \leq \alpha^-(\phi)y\}. \end{aligned}$$

We view $A(\phi)$ as a subset of \mathbb{R}^2 rather than of \mathbb{Z}^2 , as we want to consider lines with irrational slope that may lie in $A(\phi)$ but would intersect $\mathcal{C}_f(\phi)$ only in $\{0\}$.

The rays $t(\alpha^-, 1)$ and $t(-\alpha^+, -1)$ for $t \geq 0$ are nonexpansive rays (see [7, Theorem 4.4]), where the notation $t(\cdot, 1)$ means the set of all positive scalar multiples of the vector $(\cdot, 1)$.

Definition 4.4. We say the asymptotic light cone $A(\phi)$ has *closing edges* if the rays $t(-\alpha^+, -1)$ and $t(\alpha^-, 1)$ are closing (in other words, the two rays forming the left edge of $A(\phi)$ are closing).

For $\alpha \in \mathbb{R}$, define the α *quadrants* in \mathbb{Z}^2 by

$$Q_1(\alpha) = \{(i, j) \in \mathbb{Z}^2 : j \geq 0, i \geq \alpha j\}$$

$$Q_2(\alpha) = \{(i, j) \in \mathbb{Z}^2 : j \geq 0, i \leq \alpha j\}$$

$$Q_3(\alpha) = \{(i, j) \in \mathbb{Z}^2 : j \leq 0, i \leq \alpha j\}$$

$$Q_4(\alpha) = \{(i, j) \in \mathbb{Z}^2 : j \leq 0, i \geq \alpha j\}$$

Even though $Q_i(\alpha)$ is a subset of \mathbb{Z}^2 and has no dependence on any particular spacetime, it frequently is the case that we are interested in $Q_i(\alpha)$ as a subset of the domain of colorings in a space time. This can become confusing when more than one spacetime is involved. Hence for clarity we write $Q_i(\alpha, \mathcal{U})$ to indicate that we are viewing it as a subset of the domain of the colorings in the spacetime \mathcal{U} . We refer to a subset of \mathbb{Z}^2 as a *strip* (respectively, *half strip*) if it is the intersection of \mathbb{Z}^2 with the set of points in \mathbb{R}^2 between two parallel lines (respectively, the intersection of a strip in \mathbb{Z}^2 with a closed half space whose edge is not parallel to the strip).

Lemma 4.5. *Suppose \mathcal{U} is a spacetime, α is rational, and the quadrant $Q_4(\alpha)$ \mathcal{U} -codes the quadrant $Q_1(\alpha)$. Then there exists $N > 0$ such that the half strip*

$$Q_4(\alpha) \cap \left(\bigcup_{j=0}^N L_j \right)$$

\mathcal{U} -codes the quadrant $Q_1(\alpha)$ where $L_j = \{(n, -j) \in \mathbb{Z}^2 : n \in \mathbb{Z}\}$ is the horizontal line in \mathbb{Z}^2 through $(0, -j)$.

Moreover there exists a spacetime \mathcal{V} of a \mathbb{Z} -subshift endomorphism ψ such that \mathcal{V} is a recoding of \mathcal{U} and such that the ray

$$R := ([0, \infty) \times \{0\}) \cap \mathbb{Z}^2 \subset Q_1(\alpha, \mathcal{V})$$

\mathcal{V} -codes the entire quadrant $Q_1(\alpha, \mathcal{V})$.

Proof. First assume that $\alpha \geq 0$. Let (p, q) be the point of $Q_1(\alpha, \mathcal{V})$ closest to $(0, 0)$ such that $p = \alpha q$ with $q > 0$. Hence $\alpha = p/q$ and $p \geq 0$. We claim that there exists $N > 0$ such that the finite set of points

$$T := \{(r, s) \in Q_1(\alpha, \mathcal{V}) : 0 \leq r \leq p, 0 \leq s \leq q\}$$

is \mathcal{U} -coded by the half strip

$$S(N) := \{(i, j) : -N \leq j \leq 0, i \geq \alpha j\} = Q_4(\alpha) \cap ((-\infty, \infty) \times [-N, 0]).$$

Clearly the points with $s = 0$ are coded since they lie in $S(N)$. If not all points of T are coded by $S(N)$, then there exist $x_n, y_n \in \mathcal{U}$ and $(r_0, s_0) \in T$ such that for all $n > 0$ $x_n(r_0, s_0) \neq y_n(r_0, s_0)$, but x_n and y_n agree on the strip $S(n)$. By passing to subsequences if necessary, we can assume that the sequences $\{x_n\}$ and $\{y_n\}$ converge to x_∞ and y_∞ respectively with $x_\infty(r_0, s_0) \neq y_\infty(r_0, s_0)$. Since these two elements of \mathcal{U} agree on $S(n)$ for all n , they agree on the quadrant $Q_4(\alpha)$. This contradicts the hypothesis, proving the claim.

Since $S(N) + (m, 0) \subset S(N)$ for all $m \geq 0$ and the half strip $S(N)$ \mathcal{U} -codes T , it also \mathcal{U} -codes $T + (m, 0)$. Thus it \mathcal{U} -codes the half strip $S(N) + (p, q)$. It then follows by induction on $n \geq 0$ that the strip $S(N) + n(p, q)$ \mathcal{U} -codes $S(N) + (n+1)(p, q)$. Hence the half strip $S(N)$ \mathcal{U} -codes the quadrant $Q_1(\alpha)$. This proves the first assertion of the lemma for $\alpha \geq 0$.

Note that if we define the bi-infinite strip

$$\hat{S}(N) := (-\infty, \infty) \times [-N, 0] = \bigcup_{n \geq 0} (S(N) - (n, 0)),$$

then since $S(N)$ \mathcal{U} -codes $Q_1(\alpha)$, we have that $\hat{S}(N)$ \mathcal{U} -codes the upper half space $j \geq 0$ and $\hat{S}(N) + (i_0, j_0)$ \mathcal{U} -codes the half space $j \geq j_0$.

Next we consider $\Psi: (X, \mathcal{A}) \rightarrow (X_F, \mathcal{A}_F)$, the canonical recoding of X (see Definition 2.3) via the finite set $F \subset \mathbb{Z}^2$ which we define to be the triangle

$$S(N) \cap \{(i, j) : i \leq 0\} = \{(i, j) : -N \leq j \leq 0, \alpha j \leq i \leq 0\}.$$

Then $\mathcal{V} := X_F$ is a spacetime and the horizontal axis $j = 0$ in \mathbb{Z}^2 Ψ^{-1} -codes all horizontal translates of F . But the union of these horizontal translates is $\hat{S}(N)$. Since $\hat{S}(N)$ \mathcal{U} -codes the upper half space $j \geq 0$, it follows that the horizontal axis $j = 0$ \mathcal{V} -codes the half space $j \geq 0$.

The ray $R := [0, \infty) \times \{0\}$ (for \mathcal{V}) Ψ^{-1} -codes the half strip $S(N)$ (for \mathcal{U}). But $S(N)$ \mathcal{U} -codes the quadrant $Q_1(\alpha, \mathcal{U})$. Hence R Ψ^{-1} -codes $Q_1(\alpha, \mathcal{U}) \cup S(N)$. But $Q_1(\alpha, \mathcal{U}) \cup S(N)$ Ψ -codes $Q_1(\alpha, \mathcal{V})$. Thus R \mathcal{V} -codes $Q_1(\alpha, \mathcal{V})$. This completes the second assertion of the lemma for $\alpha \geq 0$.

The proof is analogous for $\alpha < 0$. □

4.2. The role of closing. Recall that if H is a closing half space with boundary L and $L_0 = L + z_0$ where $z_0 \in \mathbb{Z}^2$ is chosen such that L_0 is the closest coset of L in the complement of H , then there is a finite block \mathfrak{B} in $L_0 \cap \mathbb{Z}^2$ such that $H \cup \mathfrak{B}$ codes L_0 . We want to show there

is a constant $\rho > 0$ such that any $(i, j) \in L_0$, close to \mathfrak{B} is coded by the set $\mathfrak{B} \cup (B_\rho(i, j) \cap H \cap \mathbb{Z}^2)$ where $B_\rho(i, j) \subset \mathbb{R}^2$ is the open ball in \mathbb{R}^2 with radius ρ .

Lemma 4.6. *Let ℓ be a rational closing ray contained in the one-dimensional subspace $L \subset \mathbb{R}^2$ which bounds the closing half space H . Suppose $L_0 := z + L$, $z \in \mathbb{Z}^2$, and $\mathfrak{B} \subset L_0$ are as in the definition of closing. Suppose further that $(i, j) \in L_0 \cap \mathbb{Z}^2$ and $\mathfrak{B}(i, j)$ is a translate of \mathfrak{B} in $L_0 \cap \mathbb{Z}^2$ such that $(i, j) \notin \mathfrak{B}$, but $(i, j) \in \mathfrak{B} + e_L$, where e_L is a generator of $L \cap \mathbb{Z}^2$. Then there is a constant $\rho > 0$, independent of i and j such that $\{(i, j)\}$ is coded by the set*

$$\mathfrak{B}(i, j) \cup (B_{\rho/2}(i, j) \cap H \cap \mathbb{Z}^2),$$

where $B_{\rho/2}(i, j)$ is the open ball with radius $\rho/2$ centered at (i, j) .

Proof. Let \mathfrak{B} be the block whose existence is guaranteed by the assumption that ℓ is closing. If the result does not hold, then for any $(i, j) \in L_0$ there exist sequences $\{x_n\}, \{y_n\} \in X$ such that $x_n(i, j) \neq y_n(i, j)$ but x_n and y_n have colorings which agree on $\mathfrak{B}(i, j) \cup (B_n(i, j) \cap H \cap \mathbb{Z}^2)$. Choosing subsequences if necessary we can assume that there exist $x_\infty, y_\infty \in X$ such that

$$\lim_{n \rightarrow \infty} x_n = x_\infty \text{ and } \lim_{n \rightarrow \infty} y_n = y_\infty.$$

Then the restrictions of x_∞ and y_∞ to $\mathfrak{B}(i, j) \cup (H \cap \mathbb{Z}^2)$ are equal but $x_\infty(i, j) \neq y_\infty(i, j)$. This contradicts the fact that ℓ is closing and so there exists some value of ρ with the desired property. Since such a ρ exists for one (i, j) , it follows by translating in L_0 that the same ρ works for any $(i', j') \in L \cap \mathbb{Z}^2$. \square

While in general

$$W^+(k, \phi) = \alpha^+k + o(k),$$

it is not in general true that $W^+(k, \phi) = \lceil \alpha^+k \rceil$. However, with appropriate hypotheses we can recode the spacetime of ϕ to the spacetime of an endomorphism ψ satisfying $W^+(k, \psi) = \lceil \alpha^+k \rceil$. The object of the next three lemmas is to show this holds if ϕ has a closing light cone. We begin with some basic facts about $W^+(k, \phi)$ and its relation to α^+k . Recall that if ψ is a recoding of ϕ then $\alpha^\pm(\phi) = \alpha^\pm(\psi)$ by Proposition 5.3 of [7].

Lemma 4.7. *Suppose \mathcal{U} is the spacetime of an endomorphism ϕ and \mathcal{V} is the spacetime of an endomorphism ψ which is a recoding of \mathcal{U} . Then:*

- (1) $W^+(k, \phi) \geq \lceil \alpha^+ k \rceil$ for all $k \geq 0$ and $W^-(k, \phi) \leq \lfloor \alpha^- k \rfloor$ for all $k \geq 0$.
- (2) $W^+(k, \phi) \geq W^+(k, \psi)$ and $W^-(k, \phi) \leq W^-(k, \psi)$ for all $k \geq 0$.
- (3) If $W^+(k, \phi) = \lceil \alpha^+ k \rceil$ for all $k \geq 0$, then $W^+(k, \psi) = \lceil \alpha^+ k \rceil$ for all $k \geq 0$. Similarly if $W^-(k, \phi) = \lfloor \alpha^- k \rfloor$ for all $k \geq 0$, then $W^-(k, \psi) = \lfloor \alpha^- k \rfloor$, $k \geq 0$.

Proof. By [7, Lemma 4.2], we always have that $W^+(k, \phi) \geq \alpha^+ k$. Since $W^+(k, \phi)$ is an integer, it follows that $W^+(k, \phi) \geq \lceil \alpha^+ k \rceil$ for all $k \geq 0$. Similarly $W^-(k, \phi) \leq \lfloor \alpha^- k \rfloor$ for $k \geq 0$ and so (1) follows.

To prove (2), assume that $F \subset \mathbb{Z}^2$ is finite and $\Psi: \mathcal{U} \rightarrow \mathcal{V}$ is a recoding of \mathcal{U} via F . Let $R(r, s)$ denote the horizontal \mathbb{Z}^2 ray $\{(i, j) : i \geq r, j = s\}$. By the definition of $W^+(n, \phi)$, we have that $R(0, 0)$ \mathcal{U} -codes $R(W^+(n, \phi), n)$ for $n \geq 0$. Hence $R(i, j)$ \mathcal{U} -codes $R(i + W^+(n, \phi), j + n)$ for $n \geq 0$. Therefore $\bigcup_{(i,j) \in F} R(i, j)$ \mathcal{U} -codes

$$\bigcup_{(i,j) \in F} R(i + W^+(n, \phi), j + n) = \bigcup_{(i,j) \in F + (W^+(n, \phi), n)} R(i, j).$$

But the latter Ψ -codes $R(W^+(n, \phi), n)$. It follows that $R(0, 0)$ \mathcal{V} -codes $R(W^+(n, \phi), n)$ for $n \geq 0$. Thus $W^+(n, \phi) \geq W^+(n, \psi)$. The fact that $W^-(k, \phi) \leq W^-(k, \psi)$ is proved similarly.

To prove (3), note that parts (1) and (2) imply that

$$\lceil \alpha^+ k \rceil \leq W^+(k, \psi) \leq W^+(k, \phi) = \lceil \alpha^+ k \rceil.$$

The proof for W^- is similar. □

Lemma 4.8. *Suppose the asymptotic light cone $A(\phi)$ has closing edges and that $\alpha^+(\phi) = p/q$ and $\alpha^-(\phi) = p'/q$ with $p, p' \in \mathbb{Z}^+$ and $q > 0$. Then the spacetime \mathcal{U} of ϕ can be recoded to the spacetime \mathcal{V} of an endomorphism of another shift $\psi \in \text{End}(Y)$, for which*

$$W^+(kq, \psi) = \alpha^+ kq \text{ and } W^-(kq, \psi) = \alpha^- kq$$

when $k > 0$.

Proof. To prove the equality $W^+(kq, \psi) = \alpha^+ kq$, it suffices to consider the special case $\alpha^+ = 0$. Namely, suppose $\alpha^+ = p/q$. It follows from [7, Proposition 3.12]) that $\alpha^+(\phi^m \sigma^k) = k + m\alpha^+(\phi)$. Letting $k = -p$, $m = q$, and $\phi' = \phi^q \sigma^{-p}$ we have that $\alpha^+(\phi') = 0$. Hence if we show that $W^+(k, \phi') = 0$ for all $k \geq 0$, then by Equation 4.1

$$0 = W^+(k, \phi^q \sigma^{-p}) = -pk + W^+(k, \phi^q) = -pk + W^+(kq, \phi),$$

and so $W^+(kq, \phi) = pk = \alpha^+ kq$. Thus it suffices to consider the special case $\alpha^+ = \alpha^+(\phi) = 0$.

Define $\delta^+(n) = W^+(n, \phi) - \alpha^+n$. By [7, Lemma 4.2], the function $\delta^+(n)$ is subadditive, nonnegative, and $\delta^+(n) = o(n)$. Since $\alpha^+ = 0$, it follows that $W^+(n, \phi) = \delta^+(n)$ for all $n \geq 0$ and we are left with showing that $\delta^+(n) = 0$.

Let ρ be the constant given by Lemma 4.6. Without loss of generality, we can assume that ρ is an integer > 1 . Then there exists $C > 0$ and arbitrarily large n_0 with the property that if $r_0 = \delta^+(n_0) + C$, then

$$(4.3) \quad W^+(k, \phi) = \delta^+(k) \leq r_0 \quad \text{for all } 0 \leq k \leq n_0$$

and

$$(4.4) \quad \frac{r_0}{n_0} < \frac{1}{2\rho}.$$

Namely, to prove (4.3), note that if $\delta^+(n)$ is bounded for all $n \geq 1$ we can choose C to be an upper bound, and if $\delta^+(n)$ is unbounded we can choose arbitrarily large n_0 such that for all $0 \leq k \leq n_0$, $\delta^+(k) \leq \delta^+(n_0)$ and let $C = 0$. Then equation (4.4) follows from (4.3) and the fact that $\delta^+(m) = o(m)$.

Observe that if Q is the fourth quadrant $[0, \infty) \times (-\infty, 0]$, then

$$(4.5) \quad Q \text{ } \mathcal{U}\text{-codes } [r_0, \infty) \times [0, n_0],$$

where again $r_0 = \delta^+(n_0) + C$. This holds because $r_0 \geq \delta^+(m) = W^+(m)$ for all m with $0 \leq m \leq n_0$.

But we claim that also $([r_0, \infty) \times [0, \infty]) \cup Q$ codes $[m, \infty) \times [0, \infty]$ for $0 \leq m \leq r_0$. To see this, we first code the vertical line through $(r_0 - 1, 0)$ as follows: use the one-sided expansiveness of the vertical ray $(r_0 - 1, 0) + t(0, 1)$, $t \geq 0$ with a block

$$\mathfrak{B}(r_0 - 1, 0) = \{(m - 1, t) : -N \leq t \leq 0\}.$$

By Lemma 4.6, we have that $\{(r_0 - 1, 1)\}$ is coded by $([r_0, \infty) \times [0, n_0]) \cup Q$ if $n_0 > \rho$. We can repeat this using $\mathfrak{B}(r_0 - 1, 1) = \{(m - 1, t) : -N + 1 \leq t \leq 1\}$ to code $\{(r_0 - 1, 2)\}$ and then $\mathfrak{B}(r_0 - 1, 2) := \{(r_0 - 1, t) : -N + 2 \leq t \leq 2\}$, to code $\{(r_0 - 1, 3)\}$ etc. We can continue coding $\{(r_0 - 1, k)\}$ so long as $k \leq n_0 - \rho$, where $\rho > 1$ is the constant from Lemma 4.6.

If H is the half space to the right of the line $(r_0 - 1, t)$, $t \in \mathbb{Z}$, then $(r_0 - 1, k)$ with $k \leq n_0 - \rho$ satisfies $\mathfrak{B}(r_0 - 1, k - 1) \cup B_{\rho/2}(r_0 - 1, k) \cap H \cap \mathbb{Z}^2 \subset \mathfrak{B}(r_0 - 1, k - 1) \cup ([r_0, \infty) \times [0, \infty]) \cup Q$, which codes $(r_0 - 1, k)$. Thus we have shown that $([r_0, \infty) \times [0, n_0]) \cup Q$ codes $[r_0 - 1, \infty) \times [0, n_0 - \rho]$.

Since $[r_0, \infty) \times [0, n_0] \cup Q$ codes $[r_0 - 1, \infty) \times [0, n_0 - \rho]$, we can repeat this argument to show that $[r_0 - 1, \infty) \times [0, n_0 - \rho] \cup Q$ codes

$[r_0 - 2, \infty) \times [0, n_0 - 2\rho]$, etc. So, as long as m satisfies $n_0 - m\rho > 0$ and $m \leq r_0$ we have that

$$[r_0, \infty) \times [0, n_0] \cup Q \text{ codes } [r_0 - m, \infty) \times [0, n_0 - m\rho].$$

But by Equation 4.4 above

$$\frac{r_0}{n_0} < \frac{1}{2\rho},$$

so $r_0\rho < n_0/2$. Thus if we take $m = r_0$ then $n_0 - m\rho = n_0 - r_0\rho > n_0/2$ so

$$[r_0, \infty) \times [0, n_0] \cup Q \text{ codes } [0, \infty) \times [0, n_0/2].$$

Then by Equation 4.5 we see that Q codes $[0, \infty) \times [0, n_0/2]$. Since n_0 can be arbitrarily large we get that Q codes the full quadrant $[0, \infty) \times [0, \infty)$, and in particular the claim follows.

When \mathcal{U} is the spacetime of the endomorphism ϕ , it follows from Lemma 4.5 that there is a spacetime \mathcal{V}_0 of an endomorphism ψ_0 and a recoding $\Psi: \mathcal{U} \rightarrow \mathcal{V}_0$, with the property that the horizontal ray $[0, \infty) \times \{0\}$ \mathcal{V}_0 -codes the entire first quadrant of \mathbb{Z}^2 . In particular, \mathcal{V}_0 -codes the vertical ray $\{0\} \times [0, \infty)$. In other words $W^+(m, \psi_0) = 0$ for all $m \geq 0$ which is the desired result when $\alpha^+ = 0$. As noted, this suffices to prove the general case that when $\alpha^+(\phi) = p/q$ we have $W^+(kq, \psi_0) = \alpha^+kq$.

By a similar argument we can recode \mathcal{V}_0 to a spacetime \mathcal{V} of the endomorphism ψ with the property that $W^-(kq, \psi) = \alpha^-kq$. Since ψ is a recoding of ψ_0 , using part (3) of Lemma 4.7, it follows that

$$W^+(kq, \psi) = W^+(kq, \psi_0) = \alpha^+kq.$$

Thus the second recoding did not effect the desired equality for W^+ . \square

Proposition 4.9. *Suppose \mathcal{U} is the spacetime of the endomorphism $\phi \in \text{End}(Y_0)$ whose asymptotic light cone $A(\phi)$ has closing edges and asymptotic slopes $\alpha^+ = p/q$ and $\alpha^- = p'/q$. Then \mathcal{U} can be recoded to the spacetime \mathcal{V} of some endomorphism $\psi \in \text{End}(Y_1)$ such that for all $n \geq 0$,*

$$W^+(n, \psi) = \lceil \alpha^+ n \rceil \text{ and } W^-(n, \psi) = \lfloor \alpha^- n \rfloor.$$

Proof. By Lemma 4.8, there exists $m \in \mathbb{Z}^+$ such that

$$W^+(m, \phi) = \alpha^+m \text{ and } W^-(m, \phi) = \alpha^-m.$$

By [7, Lemma 3.10], the function $W^+(n, \phi)$ is subadditive. Hence $W^+(m, \phi) = \alpha^+m$ implies that $W^+(km, \phi) \leq \alpha^+km$ for all $k > 0$. But we always have that $W^+(k, \phi) \geq \alpha^+k$ (see [7, Lemma 4.2]) and so $W^+(km, \phi) = \alpha^+km$.

For fixed i_0, j_0 , define the ray

$$R(i_0, j_0) := \{(i, j) \in \mathbb{Z}^2 : i \geq i_0, j = j_0\}$$

to be the positive horizontal ray emanating from (i_0, j_0) . Then $R(0, 0)$ codes $R(\alpha^+ km, km)$ for all $k \geq 0$. Translating, we obtain that $R(\alpha^+ j, j)$ codes $R(\alpha^+(j + km), j + km)$ for all $j \in \mathbb{Z}$. Hence the half strip $S(m) = \{(i, j) : -m \leq j \leq 0 \text{ and } i \geq \alpha^+ j\}$ codes the quadrant $Q_1(\alpha^+)$.

It follows from the second part of Lemma 4.5 that \mathcal{U} can be recoded to be the spacetime \mathcal{V}_0 of a \mathbb{Z} -subshift endomorphism ψ_0 with the property that the horizontal ray $R(0, 0)$ \mathcal{V}_0 -codes the entire quadrant $Q_1(\alpha, \mathcal{V})$.

Since $(\lceil \alpha^+ j \rceil, j) \in Q_1(\alpha^+)$, the ray $[0, \infty)$ ψ_0^j -codes $(\lceil \alpha^+ j \rceil, \infty)$ and so $W^+(j, \psi_0) \leq \lceil \alpha^+ j \rceil$. But by part (1) of Lemma 4.7, we have that $W^+(j, \psi_0) \geq \lceil \alpha^+ j \rceil$ and so $W^+(j, \psi_0) = \lceil \alpha^+ j \rceil$.

We can apply an analogous argument to the spacetime \mathcal{V}_0 of ψ_0 to obtain a recoding of \mathcal{V}_0 to \mathcal{V} , the spacetime of an endomorphism ψ such that $W^-(n, \psi) = \lfloor \alpha^- n \rfloor$. This recoding still has the property that $W^+(n, \psi) = \lceil \alpha^+ n \rceil$ because part (3) of Lemma 4.7 asserts

$$W^+(n, \psi) = W^+(n, \psi_0) = \lceil \alpha^+ n \rceil.$$

Thus the second recoding did not effect the desired equality for W^+ . \square

Recall that we have defined levels in the light cone of an endomorphism by $\mathcal{I}(n, \phi) := \{i \in \mathbb{Z} : (i, n) \in \mathcal{C}(\phi)\}$. Our next step is the following lemma about light cones:

Lemma 4.10. *Suppose ϕ is an endomorphism of a \mathbb{Z} -subshift (Y, σ) which has a closing asymptotic light cone $A(\phi)$ with $\alpha^+ > \alpha^-$. Then there exist integers m and n_0 with $n_0 > m > 0$ such that $\mathcal{I}(-n, \phi)$ ϕ^m -codes $\mathcal{I}(-n + m, \phi)$ whenever $n \geq n_0$.*

Proof. The endomorphism ϕ is fixed throughout this proof and so we simplify notation by writing $\mathcal{I}(n)$ for $\mathcal{I}(n, \phi)$ and $W^\pm(n)$ for $W^\pm(n, \phi)$. Since the asymptotic slopes α^+ and α^- satisfy $\alpha^+ > \alpha^-$, we have

$$\lim_{n \rightarrow \infty} |\mathcal{I}(-n)| = \infty,$$

where $|\cdot|$ denotes the length of an interval.

By Proposition 4.8, there exists $m > 0$ such that $W^+(jm)/m = j\alpha^+$ and $W^-(jm)/m = j\alpha^-$ for all $j > 0$. Indeed m can be chosen to be the least common multiple of the denominators of α^+ and α^- . Also by Proposition 4.9 we know that $|\mathcal{I}(-n)|$ is monotonically increasing in n .

It follows from [7, Proposition 3.4] that there is a constant C such that the interval $[0, C]$ ϕ^m -codes $\{W^+(m)\}$ and $[-C, 0]$ ϕ^m -codes

$\{W^-(m)\}$. Hence for $t > 0$, we have that the interval $[0, C+t]$ ϕ^m -codes $[W^+(m), W^+(m) + t]$ and the interval $[-C - t, 0]$ ϕ^m -codes $[W^-(m) - t, W^-(m)]$. Translating, it follows that for any $t > 0$, we have that $[W^+(-n), W^+(-n) + C + t]$ ϕ^m -codes $[W^+(-n + m), W^+(-n + m) + t]$. Therefore a left-aligned subinterval of $\mathcal{I}(-n)$ with length $C + t$ ϕ^m -codes a left-aligned subinterval of $\mathcal{I}(-n + m)$ with length t whenever $t \leq |\mathcal{I}(-n + m)|$ and otherwise ϕ^m -codes all of $\mathcal{I}(-n + m)$.

Let $t = t(n) := |\mathcal{I}(-n)| - C$. Then by monotonicity of $|\mathcal{I}(-n)|$, we have that

$$t(n) > |\mathcal{I}(-n + m)| - C.$$

Since $|c\mathcal{I}(-n + m)|$ tends to infinity with n , there exists $n_0 > 0$ such that $n \geq n_0$ implies

$$t = t(n) \geq |\mathcal{I}(-n + m)| - C > \frac{|\mathcal{I}(-n + m)|}{2}.$$

Thus $\mathcal{I}(-n)$ codes a left-aligned subinterval of $\mathcal{I}(-n + m)$ with length t which is greater than half the length of $\mathcal{I}(-n + m)$. An analogous argument shows that $\mathcal{I}(-n)$ codes a right-aligned subinterval of $\mathcal{I}(-n + m)$ with length greater than half the length of $\mathcal{I}(-n + m)$. We conclude that $\mathcal{I}(-n)$ codes $\mathcal{I}(-n + m)$ when $n > n_0$. \square

5. CORNER CODING SECTORS

5.1. Corner coding.

Definition 5.1. Suppose ℓ_1 and ℓ_2 are nonparallel rays in \mathbb{R}^2 emanating from the origin labeled such that for $e_i \neq 0 \in \ell_i$, the basis $\{e_1, e_2\}$ is positively oriented and the angle γ between e_1 and e_2 satisfies $0 < \gamma < \pi$.

Define the *sector* \mathcal{S} determined by ℓ_1 and ℓ_2 to be

$$\mathcal{S} = \mathbb{Z}^2 \cap (\ell_1 \cup \ell_2 \cup \{v \in \mathbb{R}^2 : \{e_1, v, e_2\} \text{ is positively cyclically ordered}\})$$

for any nonzero $e_1 \in \ell_1, e_2 \in \ell_2$. The *supplementary sector* to \mathcal{S} is defined to be the sector determined by ℓ_2 and $-\ell_1$ and is denoted by \mathcal{S}_s . The sector \mathcal{S} is *rational* if the two rays determining it are rational.

Note that the sector determined by ℓ_1 and ℓ_2 is the set of points of \mathbb{Z}^2 lying either between these rays or on them.

Definition 5.2. A rational sector \mathcal{S} for a \mathbb{Z}^2 -shift X is *corner coding* if for any finite set $\mathcal{F} \subset \mathcal{S}$, the set $\mathcal{S} \setminus \mathcal{F}$ X -codes all of \mathcal{S} . A rational sector \mathcal{S} for a \mathbb{Z}^2 -shift X is *weakly corner coding* if there is a finite set \mathcal{F}_0 such that for any finite set $\mathcal{F} \subset \mathcal{S}$ the set $\mathcal{S} \setminus \mathcal{F}$ X -codes all of $\mathcal{S} \setminus \mathcal{F}_0$.

Equivalently, the rational sector \mathcal{S} is corner coding if the set $\mathcal{S} \setminus \{(0,0)\}$ X -codes $\{(0,0)\}$ (and hence all of \mathcal{S}). This is easily checked by induction on the cardinality of F .

We sometimes make use of sectors whose vertex v is not at the origin, for example $\mathcal{S} = \mathcal{S}_0 + v$ for some \mathcal{S}_0 a sector as defined in Definition 5.1 with its vertex at the origin. Extending the definition, we say that such an \mathcal{S} is corner coding if \mathcal{S}_0 is corner coding. In particular, if v is a vertex of a polygon \mathcal{P} , then $\mathcal{S}(v)$, the *sector based at the vertex* v is defined to be the sector with vertex v and rays emanating from v containing the edges of \mathcal{P} which meet at v . Thus a sector based at a vertex other than the origin is a translate of a sector based at the origin.

Proposition 5.3. *Suppose X is a \mathbb{Z}^2 -subshift and \mathcal{P} is a convex integer polygon. If for each vertex $v \in \mathcal{P}$ the sector $\mathcal{S}(v)$ based at v is corner coding, then for sufficiently large $n > 0$, $n\mathcal{P}$ is a coding polygon for X . Conversely, if \mathcal{P} is a coding polygon for X , then for each vertex $v \in \mathcal{P}$ the sector $\mathcal{S}(v)$ based at v is corner coding.*

Proof. Assume that for each $v \in \mathcal{P}$, the sector $\mathcal{S}(v)$ is corner coding. Then there is a finite set $G(v) \subset \mathcal{S}(v)$ with $v \notin G(v)$ that codes v (see Lemma 2.5). The sector $n\mathcal{S}(v)$ with vertex nv has the property that $nv + G(v)$ codes its vertex nv . It follows that for n_0 sufficiently large, any $n \geq n_0$ satisfies $nv + G(v) \subset n\mathcal{P}$ and hence $n\mathcal{P}$ codes nv . Repeating this for each vertex of \mathcal{P} , we obtain $n > 0$ such that for each vertex w of $n\mathcal{P}$, the set $n\mathcal{P} \setminus \{w\}$ codes w . Hence $n\mathcal{P}$ is a coding polygon.

The converse follows immediately from the definition of corner coding. \square

Lemma 5.4. *If a rational sector \mathcal{S} is corner coding for X and $\Psi: X \rightarrow Y$ is a recoding, then \mathcal{S} is corner coding for Y .*

Proof. Suppose $\Psi: X \rightarrow Y$ is a recoding via the finite set F . By the equivalent formulation of Definition 5.2, it suffices to show that the set $\check{\mathcal{S}} := \mathcal{S} \setminus \{(0,0)\}$ Y -codes $\{(0,0)\}$. But the set $\{(0,0)\}$ Ψ^{-1} codes F , and so $\{(i,j)\}$ Ψ^{-1} codes $F(i,j) := F + (i,j)$. Hence $\check{\mathcal{S}}$ Ψ^{-1} codes

$$\bigcup_{(i,j) \in \check{\mathcal{S}}} F(i,j).$$

But

$$\bigcup_{(i,j) \in \check{\mathcal{S}}} F(i,j) = \bigcup_{(r,s) \in F} ((r,s) + \check{\mathcal{S}})$$

and since \mathcal{S} is corner coding for X , each translate $((r, s) + \check{\mathcal{S}})$ X -codes (r, s) . Since this holds for each $(r, s) \in F$, it follows that $\check{\mathcal{S}}$ Ψ^{-1} -codes F . Thus $\check{\mathcal{S}}$ Y -codes $(0, 0)$ and hence \mathcal{S} is corner coding for Y . \square

Corollary 5.5. *If X is a polygonal subshift with coding polygon \mathcal{P} and $\Psi: X \rightarrow Y$ is a recoding, then Y is a polygonal subshift with coding polygon $n\mathcal{P}$ for some $n > 0$.*

Proof. Since \mathcal{P} is a coding polygon for X , each sector based at a vertex of the polygon \mathcal{P} is corner coding for X . By Lemma 5.4, each of these sectors is corner coding for Y . Then by Proposition 5.3, for sufficiently large n , the polygon $n\mathcal{P}$ is a coding polygon for Y . \square

However, an example of Salo [25] shows that a system isomorphic to a polygonal system need not itself be polygonal: he constructs a system isomorphic to the Ledrappier system with an isomorphism that does not preserve the polygonal property.

Proposition 5.6. *If a rational sector \mathcal{S} is weakly corner coding for X , then there is a recoding Y of X for which \mathcal{S} is corner coding.*

Proof. By the definition of weakly corner coding, there is a finite set F_0 such that for any finite set $F \subset \mathcal{S}$, the set $\mathcal{S} \setminus F$ X -codes all of $\mathcal{S} \setminus F_0$. Without loss of generality we can assume that $0 \in F_0$ and F_0 is convex.

Choose \mathcal{K} to be a strip in \mathbb{R}^2 with several properties we now describe (see Figure 1). Assume that \mathcal{K} crosses both sides of \mathcal{S} transversely such that each of its edges intersects the edges of \mathcal{S} in points of \mathbb{Z}^2 . We choose \mathcal{K} so that $\mathcal{S} \setminus \mathcal{K}$ has two parts separated by \mathcal{K} : the first \mathcal{B}_0 is finite and the second \mathcal{B}_∞ is unbounded. Assume further that \mathcal{K} is chosen so that \mathcal{B}_0 contains the finite set F_0 . Let $\mathcal{D} = \mathcal{K} \cap \mathcal{S} \cap \mathbb{Z}^2$. Then \mathcal{D} is a finite subset whose convex hull is a trapezoid $\hat{\mathcal{D}}$. Two edges of $\hat{\mathcal{D}}$ are antiparallel and lie in the two edges of \mathcal{K} , and the other two sides of $\hat{\mathcal{D}}$ lie in the two edges of \mathcal{S} . Note that $\mathcal{D} + (i, j) \subset \mathcal{S}$ for any $(i, j) \in \mathcal{S}$. We also assume that \mathcal{K} has been chosen to be sufficiently wide such that

$$\mathcal{D} \cup \mathcal{B}_\infty = \bigcup_{(i,j) \in \mathcal{S}} (\mathcal{D} + (i, j)).$$

Finally, let (m, n) be the closest point of \mathcal{D} to $(0, 0)$ and note that without loss of generality we can assume that $(m, n) \in \ell_1$.

Note that it suffices to show that the translate $\mathcal{S}(m, n)$ of \mathcal{S} is strongly corner coding.

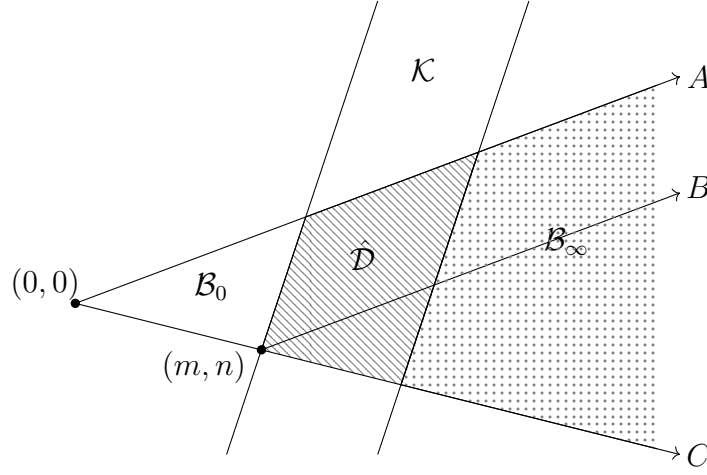


Figure 1: The rays from the origin $(0,0)$ through A and through C are the edges of the sector \mathcal{S} , and the rays from (m,n) through B and through C are the edges of the sector $\mathcal{S}(m,n)$.

Set $\mathcal{D}' = \mathcal{D} - (m,n)$ and let $\Psi: X \rightarrow X_{\mathcal{D}'}$ be the canonical recoding of X via \mathcal{D}' to $X_{\mathcal{D}'}$. (We use \mathcal{D}' instead of \mathcal{D} because we want \mathcal{D} to Ψ -code (m,n) , but using the canonical recoding $X_{\mathcal{D}}$, the set \mathcal{D} codes $(0,0)$ not (m,n) .) Thus with the recoding Ψ via \mathcal{D}' , we have that \mathcal{D}' codes $(0,0)$ and so $\mathcal{D} = \mathcal{D}' + (m,n)$ Ψ -codes (m,n) .

Thus if $y \in X_{\mathcal{D}}$, then the singleton $\{(m,n)\}$ Ψ^{-1} -codes \mathcal{D} for the shift X . It follows that the sector $\mathcal{S}(m,n)$ Ψ^{-1} -codes

$$\bigcup_{(i,j) \in \mathcal{S}} (\mathcal{D} + (i,j)) = \mathcal{D} \cup B_{\infty}$$

for the shift X . Furthermore, $\mathcal{D} \cup B_{\infty}$ Ψ -codes the sector $\mathcal{S}(m,n)$ for $X_{\mathcal{D}}$. Since for any finite set F the set $(\mathcal{D} \cup B_{\infty}) \setminus F$ X -codes $\mathcal{D} \cup B_{\infty}$, it follows that $\mathcal{S}(m,n) \setminus F$ $X_{\mathcal{D}}$ -codes $\mathcal{S}(m,n)$ for $X_{\mathcal{D}}$. Thus $\mathcal{S}(m,n)$ is strongly corner coding for $X_{\mathcal{D}}$. Since $\mathcal{S}(m,n)$ is a translate of \mathcal{S} , it follows that \mathcal{S} is strongly corner coding for $X_{\mathcal{D}}$. \square

Proposition 5.7. *Suppose X is a \mathbb{Z}^2 -subshift and \mathcal{C} is a component of the open set of expansive rays for X . Then there is a basis $\{u_1, u_2\}$ of \mathbb{Z}^2 and a \mathbb{Z}^2 -subshift Y such that:*

- (1) *The rays ρ_1 containing u_1 and ρ_2 containing u_2 lie in \mathcal{C} .*
- (2) *There is a recoding $\Psi: X \rightarrow Y$.*
- (3) *The one-dimensional subspace L_1 containing u_1 has the property that $L_1 \cap \mathbb{Z}^2$ codes all of Y and hence it Ψ^{-1} -codes all of X .*

In particular, Y endowed with the basis $\{-u_1, -u_2\}$ is a spacetime \mathcal{U} of an endomorphism of a \mathbb{Z} -subshift.

Proof. Let ℓ_1 and ℓ_2 be rays bounding the component \mathcal{C} . Choose a ray ρ in the interior of \mathcal{C} with irrational slope λ . Let p_1/q_1 and p_2/q_2 be successive convergents for the continued fraction expansion of λ which are chosen such that the subspaces L_n with slopes p_n/q_n have slopes sufficiently close to λ that the vectors $u_1 = (p_n, q_n)$ and $u_2 = (p_{n+1}, q_{n+1})$ determine rays ρ_1, ρ_2 which lie in the interior of \mathcal{C} . Since p_n/q_n and p_{n+1}/q_{n+1} are successive convergents in the continued fraction expansion of λ , it follows that $\{u_1, u_2\}$ is a basis of \mathbb{Z}^2 (see Olds [24, Section 3.4]). Switching the roles of u_1 and u_2 we may assume it is a positively oriented basis.

Let L be the line containing ρ_1 . Since ρ_1 is expansive, one of the complementary components of L (call this one H) codes the other H' . By a change of basis, we can assume that L is the horizontal axis and $u_1 = (-1, 0)$ and so $u_2 = (0, -1) \in H$. See Figure 2.

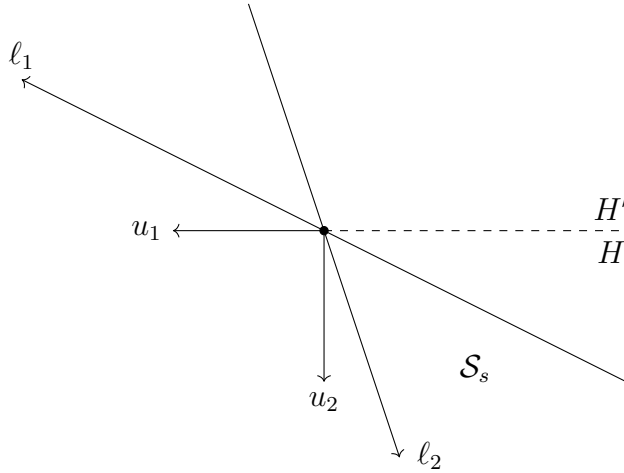


Figure 2: The rays ℓ_1 and ℓ_2 and the sector \mathcal{S}_s .

Since the negative horizontal axis is an expansive ray, there exists $r > 0$ such that the strip S consisting of points of $L \cup H$ with distance at most r from L X -codes the half space H' .

Let F be the ball in \mathbb{Z}^2 of radius r around 0 and let Y be the shift X_F obtained by the canonical recoding $\Psi: X \rightarrow X_F$. Then Y together with the basis $\{-u_1, -u_2\}$ is the spacetime of an automorphism of the projective subdynamics obtained by restricting Y to L . \square

Lemma 5.8. *Suppose \mathcal{S} is the sector for the \mathbb{Z}^2 -subshift X determined by the rays ℓ_1 and ℓ_2 . If the supplementary sector \mathcal{S}_s (bounded by ℓ_2 and $-\ell_1$) is weakly corner coding, then any ray ℓ in the interior of the sector \mathcal{S} is expansive.*

Proof. If L is the line containing ℓ , then $\mathcal{S}_s \setminus \{(0,0)\}$ lies in the complementary half space H of L whose orientation determines an orientation of L matching that of ℓ . Let H' be the other complementary component of L . Translate \mathcal{S}_s by an element of \mathbb{Z}^2 to obtain \mathcal{S}'_s such that $B := \mathcal{S}'_s \cap H'$ is finite, nonempty, and such that some element $b \in B$ is coded by $\mathcal{S}'_s \cap H$. Then by Lemma 2.7, H , and hence ℓ , is expansive. See Figure 2. \square

Proposition 5.9. *Suppose ℓ_1 and ℓ_2 are nonparallel closing rays for a \mathbb{Z}^2 -subshift X_0 and let \mathcal{S}_0 be the sector they determine. Assume that every ray interior to \mathcal{S}_0 is expansive. Let \mathcal{S}_s be the supplementary sector to \mathcal{S}_0 (the sector determined by ℓ_2 and $-\ell_1$). Then X_0 can be recoded to a \mathbb{Z}^2 -subshift X_1 such that the sector \mathcal{S}_s is strongly corner coding for X_1 .*

Note that in this lemma it is not the sector \mathcal{S}_0 which has the corner coding property, but its supplement \mathcal{S}_s .

Proof. By hypothesis, the set of all rays in the interior of \mathcal{S}_0 is a component of the space of expansive rays for X . By Lemma 5.7, after recoding we can assume that X is the spacetime $(\mathcal{V}, \{u_1, u_2\})$ of an endomorphism ψ with $u_1, u_2 \in \mathcal{S}_0$. The lines containing the edges of the asymptotic light cone $A(\psi)$ of ψ must be the lines containing ℓ_1 and ℓ_2 , since these edges are nonexpansive and there are no other nonexpansive rays in \mathcal{S}_0 . Then \mathcal{S}_s is the lower half of the asymptotic light cone of this endomorphism. (See Figure 2.) It follows from Lemma 4.10 that \mathcal{S}_s is weakly corner coding. By Proposition 5.6, \mathcal{S}_s is strongly corner coding. \square

5.2. Recoding to obtain polygonal shifts. The primary aim of this section is to prove that if a \mathbb{Z}^2 -shift X has finitely many nonexpansive rays, all of which are rational and closing, then X recodes to a polygonal shift. Hence, given a finite set $\mathcal{E} := \{\ell_i\}$ of rational closing rays, which includes all nonexpansive rays, we want to construct a coding polygon (for a recoding) whose oriented edges are positively parallel to the elements of \mathcal{E} . Recall (see Section 2.3) that an oriented edge E is positively parallel to a ray ℓ if a translate of E lies in ℓ with matching orientations, meaning that they are parallel with matching orientations.

Abstractly, given a set of rays, a necessary condition for the existence of a convex polygon with one oriented edge positively parallel to each ray is that we can find a nonzero vector in each ray such that the sum of the vectors is 0. These vectors are just the edge vectors of the polygon. Thus, given the finite set \mathcal{E} of nonexpansive rays, we want to find a nonzero integer vector $e_i \in \ell_i$ for each $\ell_i \in \mathcal{E}$ such that $\sum e_i = 0$ and

show these vectors form the edges of a polygon. We first consider a degenerate case where the polygon turns out to be a line segment.

Lemma 5.10. *Suppose X has two closing and nonexpansive antiparallel rays, ℓ and $-\ell$, which lie in the rational line L . If one of the two components of the complement of L does not intersect any nonexpansive rays, then ℓ and $-\ell$ are the only nonexpansive rays and the line L determines a periodic direction for X . In particular, X recodes to a polygonal shift with a degenerate coding polygon which is a line segment parallel to L .*

Proof. Without loss of generality, we can assume that L is vertical, taking L to be the y -axis, and further assume that the left half space $H := \{(x, y) : x < 0\}$ is disjoint from nonexpansive rays. As we can recode without affecting nonexpansive directions or periodic directions, we can do so and further assume that X is the spacetime of an endomorphism ϕ (see Proposition 5.7). Recall the left upwardly oriented edge of the top half of the asymptotic light cone is a nonexpansive ray and there are no other nonexpansive rays between it and the negative x -axis (see [7, Theorem 4.4]). Hence this ray must be either ℓ or $-\ell$; we assume without loss that it is ℓ . The same argument shows that the left downwardly oriented edge of the bottom half of the asymptotic light cone is positively parallel to $-\ell$. It follows that both edges of the asymptotic light cone of ϕ must be the line L ; in other words, the light cone is degenerate.

Since ℓ and $-\ell$ are closing, it follows from Proposition 4.9 that $[0, \infty) \times \{0\}$ codes the first quadrant and hence $[0, \infty) \times \{1\}$. Likewise $(-\infty, 0] \times \{0\}$ codes the second quadrant and hence $(-\infty, 0] \times \{1\}$. Therefore there exists $b > 0$ such that for all sufficiently large c , the set $[0, c] \times \{0\}$ codes $[0, c - b] \times \{1\}$ and similarly $[-c, 0] \times \{0\}$ codes $[-c + b, 0] \times \{1\}$. Translating the second by c , we have that $[0, c] \times \{0\}$ codes $[b, c] \times \{1\}$. If $c > 2b$ this implies $[0, c] \times \{0\}$ codes $[0, c] \times \{1\}$, and so the strip $[0, c] \times [0, \infty)$ is periodic. It is easy to check that this implies $[0, c] \times (-\infty, \infty)$ is periodic. Hence L is periodic, which implies any non-vertical ray is expansive. \square

Lemma 5.11. *Suppose X is an infinite \mathbb{Z}^2 -subshift with finitely many nonexpansive subspaces, all of which are rational and closing.*

- (1) *If $E \subset \mathbb{R}^2$ is a one-dimensional rational expansive subspace of X , there exist nonexpansive rays ρ_1 and ρ_2 and $u_i \neq 0 \in \rho_i$, $i = 1, 2$, such that u_1 and u_2 lie in different components of the complement of E . In particular, there are at least 2 distinct nonexpansive rays.*

- (2) If X has only two nonexpansive rays, then they must be antiparallel.
- (3) Suppose $\{\ell_i\}_{i=1}^n$ is the complete set of nonexpansive rays for X and $n \geq 3$. Then there exist nonzero vectors $e_i \in \ell_i \cap \mathbb{Z}^2$ such that $\{e_i\}_{i=0}^n$ (cyclically ordered by angle with an axis) are the edges of a convex polygon \mathcal{P} .

Proof. The number of nonexpansive rays is nonzero by [3, Theorem 3.7], since X is an infinite, compact metric space. Part (1) essentially follows from [7, Theorem 4.4]. More precisely, if we make a change of basis such that E is horizontal and recode (per Lemma 5.7), then X is the spacetime of an endomorphism ϕ and the asymptotic light cone $A(\phi)$ of ϕ is not empty. By the same result of [7], the ray ρ_1 which forms the the left edge of the part of the asymptotic light cone $A(\phi)$ which is above the horizontal axis is a nonexpansive ray. Similarly the ray ρ_2 which is the left edge of the part of $A(\phi)$ which is below the horizontal axis is a nonexpansive ray. This proves (1).

To prove (2), note that if the two rays are not antiparallel there is an expansive subspace L with the nonzero vectors in both rays lying on the same side. This contradicts (1).

To prove (3), we first claim that 0 lies in the interior of the convex hull of $\{u_i\}_{i=1}^n$ for any choice of $u_i \neq 0 \in \ell_i \cap \mathbb{Z}^2$, and we proceed by contradiction. Recall that $n \geq 3$. If 0 does not lie in the interior of the convex hull of $\{u_i\}_{i=1}^n$ and $u_i \neq 0 \in \ell_i \cap \mathbb{Z}^2$, then there is a one-dimensional subspace L bounding a closed half space H such that u_i lies in H for all i .

If each u_i lies in the interior of H , then L is expansive, a contradiction of (1). If one u_i lies in L and all the remaining ones lie in the interior of H , then there is a subspace L' lying arbitrarily close to L such that all of $\{u_i\}$ lie in the interior of one component of its complement, again contradicting (1).

Finally, suppose two of the u_i lie in L , and any others lie in the interior of H . Then the nonexpansive rays containing these two are antiparallel and by Lemma 5.10, there are no others. Hence we have contradicted the assumption that there are $n \geq 3$ nonexpansive rays.

This completes the proof of the claim that 0 lies in the interior of the convex hull of $\{u_i\}_{i=1}^n$ for any choice of $u_i \neq 0 \in \ell_i \cap \mathbb{Z}^2$.

We next proceed to the proof of the existence of \mathcal{P} . Observe that the claim implies that given $\{u_i\}_{i=1}^n$ as above, there are $\{t_i\}_{i=1}^n \subset (0, \infty)$ such that

$$\sum t_i = 1 \text{ and } \sum t_i u_i = 0.$$

Since $u_i \in \mathbb{Z}^2$, all of the t_i are rational. Let $e_i = mt_i u_i$ where $m > 0$ is chosen such that the vectors $e_i \in \mathbb{Z}^2$. Then $\sum e_i = 0$ and $e_i \in \ell_i \cap \mathbb{Z}^2$. We label the e_i such they are cyclically ordered by the angle they make with the x -axis, and form a polygonal curve \mathcal{P} by concatenating translates of the e_i end-to-end in order. Since $\sum e_i = 0$, it follows that this defines a closed polygonal curve. The vertex where the end of the translate of e_i meets the start of the translate of e_{i+1} has an exterior angle equal to the angle between ℓ_i and ℓ_{i+1} . Since these exterior angles are all positive and sum to 2π , it follows that \mathcal{P} is a simple closed curve which is convex. \square

Theorem 5.12. *Assume that X is a \mathbb{Z}^2 -subshift with a finite nonempty set of nonexpansive rays, each of which is rational and closing. Then X can be recoded to be a polygonal shift with a polygon \mathcal{P} having each oriented edge positively parallel to one of the nonexpansive rays and each nonexpansive ray positively parallel to an edge of \mathcal{P} .*

Proof. By part (1) of Lemma 5.11, there are at least two nonexpansive rays. If there are exactly two, Lemma 5.10 implies that X can be recoded to a (periodic) polygonal system with a degenerate polygon with oriented edges positively parallel to the two expansive rays.

Hence we may assume there are at least three nonexpansive rays. By part (3) of Lemma 5.11, there are vectors $\{e_i\}$ forming the edges of a polygon \mathcal{T} with $e_i \in \ell_i$, where ℓ_i denotes the i^{th} nonexpansive ray in X and the rays $\{\ell_i\}_{i=1}^n$ are cyclically ordered by the angle made with the positive horizontal axis. Let \mathcal{S}^i be the sector determined by ℓ_i and ℓ_{i+1} . Note that the angle determined by \mathcal{S}^i is an exterior angle of the polygon \mathcal{T} , and there are no nonexpansive rays in the interior of \mathcal{S}^i . By Proposition 5.9, we can recode X such that the supplementary sector \mathcal{S}_s^i is corner coding. The sector \mathcal{S}_s^i is the sector determined by e_{i+1} and $-e_i$, meaning it is a translate of the i^{th} vertex of \mathcal{T} , and we denote this vertex by w_i . By repeated recoding, we can guarantee that each \mathcal{S}_s^i is corner coding, and it follows from Lemma 5.4 that each additional recoding does not affect the corner coding properties of previous corners. Denote the final recoding by Y .

By Lemma 2.5, there is a finite set $G_i \subset (\mathcal{S}_s^i \setminus \{(0,0)\})$ such that G_i Y -codes $\{(0,0)\}$ for each i . Setting $G'_i := G_i + w_i$, we have that G'_i Y -codes w_i . Choosing n sufficiently large, we can guarantee that the polygon $\mathcal{P} := n\mathcal{T}$ contains G'_i for all i . It follows that for each i , the set $\mathcal{P} \setminus \{w_i\}$ Y -codes $\{w_i\}$, meaning that the polygon \mathcal{P} is a coding polygon for Y . \square

In the spirit of a converse to Theorem 5.12, we have:

Theorem 5.13. *Let X be an infinite polygonal \mathbb{Z}^2 subshift. Assume that \mathcal{P} is a coding polygon for a recoding Y of X such that \mathcal{P} has the minimal number of sides among all coding polygons for recodings of X . Then each of the oriented edges of \mathcal{P} determines a ray which is closing for X , and these rays are the only nonexpansive rays for X .*

Proof. By Lemma 5.8, every subspace not parallel to an edge of \mathcal{P} is expansive. By Proposition 3.15), oriented edges that are nonexpansive determine rays which are closing. It then follows that every oriented edge determines a nonexpansive ray since otherwise by Theorem 5.12 we could produce a recoding with a coding polygon having fewer sides. \square

Corollary 5.14. *Let X be a polygonal \mathbb{Z}^2 subshift. Then any two minimal recoding polygons for X which are similar differ by a translation. In particular, if X is triangular, any two minimal recoding polygons for X differ by a translation.*

Proof. By Theorem 5.13, any two similar minimal coding polygons \mathcal{P} and \mathcal{P}' have oriented edges which make the same angles with respect to the axes of \mathbb{R}^2 . In other words, two coding polygons differ at most by a homothety and a translation. By translating, we can assume that they differ only by a homothety and there are corresponding edges e and e' which emanate from the origin. Suppose the homothety carrying \mathcal{P} to \mathcal{P}' is multiplication by the rational $r > 0$. If $r > 1$, then \mathcal{P} is a proper subset of \mathcal{P}' and if $r < 1$, then \mathcal{P}' is a proper subset of \mathcal{P} . Hence $r = 1$ and $\mathcal{P} = \mathcal{P}'$.

If \mathcal{P} and \mathcal{P}' are triangles, they must have the same angles and so are similar. \square

Example 5.15. We contrast Corollary 5.14 with a polygonal system X whose coding polygon \mathcal{P} is an $m \times n$ rectangle whose edges are horizontal and vertical. If we let \mathcal{P}' be an $(m + 1) \times (n + 1)$ rectangle containing \mathcal{P} , then X is polygonal with respect to \mathcal{P}' . It is easy to check that if X_F is the canonical recoding of X via $F = \mathcal{P}$, then a 2×2 square is a coding polygon for X_F and is the minimal recoding polygon for X .

6. DIRECTIONAL ENTROPIES OF POLYGONAL SYSTEMS

6.1. Linear polygonal entropy. We turn to the study of entropy for two dimensional shifts. If X is a \mathbb{Z}^2 -shift with at least one expansive ray, then any finite region in the shift is coded by an interval in the expansive direction. In particular, this implies that the two dimensional entropy of any \mathbb{Z}^2 -shift with at least one expansive ray is zero,

and so we restrict ourselves to linear entropy. This leads us to define a generalization of directional entropy that depends on a polygon, rather than a line. One of the goals of this section is to show that for a polygonal \mathbb{Z}^2 -subshift with polygon \mathcal{P} , there are strong relations between $\mathcal{H}(X, \mathcal{P})$ and the directional entropies.

For a polygon \mathcal{P} in \mathbb{R}^2 and $r > 0$, we denote the r -neighborhood of the polygon by \mathcal{P}_r , meaning that

$$\mathcal{P}_r = \{u \in \mathbb{Z}^2 : d(u, \mathcal{P}) < r\}.$$

If X is a \mathbb{Z}^2 -subshift and $\mathcal{S} \subset \mathbb{R}^2$, we denote the complexity of \mathcal{S} in X by $P(X, \mathcal{S})$, meaning that $P(X, \mathcal{S})$ is the number of X -colorings of $\mathcal{S} \cap \mathbb{Z}^2$.

Definition 6.1. If X is a \mathbb{Z}^2 -subshift and \mathcal{P} is a polygon in \mathbb{R}^2 , define the *linear polygonal entropy* of \mathcal{P} by

$$\mathcal{H}(X, \mathcal{P}) = \lim_{r \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\ln P(X, (n\mathcal{P})_r)}{n}.$$

Note that we allow the \mathcal{P} to be a degenerate polygon, meaning that we allow \mathcal{P} to be a line segment. If $v \in \mathbb{R}^2$ and $I_v = \{tv : t \in [0, 1]\}$, then $\mathcal{H}(X, I_v)$ is the *directional entropy* $h_v(X)$ in the direction v as defined by Milnor[21]. Abusing notation slightly, for $v \in \mathbb{R}^2$ we write $\mathcal{H}(X, v)$ for $\mathcal{H}(X, I_v)$, where I_v denotes the interval $\{tv : 0 \leq t \leq 1\}$ (considered as a degenerate polygon in \mathbb{R}^2).

If $X = \mathcal{A}^{\mathbb{Z}^2}$, then $P(X, n\mathcal{P})$ is exponential in the area of $n\mathcal{P}$, and thus is an exponential function of something quadratic that is in n . In particular, this means that in this setting $\mathcal{H}(X, \mathcal{P}) = \infty$. On the other hand, this quantity is finite for any \mathbb{Z}^2 -system with at least one expansive ray (see the remark following Lemma 6.2).

We record the following elementary properties of polygonal entropy:

Lemma 6.2. *For a \mathbb{Z}^2 -subshift X , the polygonal entropy $\mathcal{H}(X, \mathcal{P})$ satisfies the following properties:*

- (1) *If $v \in \mathbb{Z}^2$, the directional entropy $h_v(X)$ corresponding to v is equal to $\mathcal{H}(X, v)$.*
- (2) *For $v \in \mathbb{R}^2$, $\mathcal{H}(X, \mathcal{P} + v) = \mathcal{H}(X, \mathcal{P})$.*
- (3) *For $r > 0$, $\mathcal{H}(X, r\mathcal{P}) = r\mathcal{H}(X, \mathcal{P})$. In particular for $v \in \mathbb{R}^2$, $\mathcal{H}(X, rv) = r\mathcal{H}(X, v)$.*
- (4) *If \mathcal{P}_1 and \mathcal{P}_2 are polygons in \mathbb{R}^2 and there are $v_1, v_2 \in \mathbb{R}^2$ such that $\mathcal{P}_1 + v_1 \subset \mathcal{P}_2 \subset r\mathcal{P}_1 + v_2$ for some $r \in \mathbb{Q}$, then*

$$\mathcal{H}(X, \mathcal{P}_1) \leq \mathcal{H}(X, \mathcal{P}_2) \leq r\mathcal{H}(X, \mathcal{P}_1)$$

Remark 6.3. If X is a \mathbb{Z}^2 -subshift with at least one expansive ray, then $P(X, n\mathcal{P})$ is bounded above by a linear function of n and so $\mathcal{H}(X, \mathcal{P})$

is finite. To see this, note that a long interval J parallel to a one-sided expansive ray codes a triangle \mathcal{T} with J on one side. Taking J to be sufficiently long, then \mathcal{T} is large enough to contain a translate of \mathcal{P} . It follows from the properties in Lemma 6.2 that

$$\mathcal{H}(X, \mathcal{P}) \leq \mathcal{H}(X, \mathcal{T}) = \mathcal{H}(X, J) < \infty.$$

We recall the following definition (see [20] for example).

Definition 6.4. If X is a \mathbb{Z}^2 -subshift, the entropy seminorm for X on \mathbb{R}^2 is defined by

$$\|v\|_X = h_v(X).$$

In general $\|v\|_X$ defines a seminorm (see [3]), but when X is polygonal with respect to \mathcal{P} and no two sides of \mathcal{P} are antiparallel, then $\|\cdot\|_X$ is either identically 0 or a norm. To prove this, we make use of a small variation of a result of Milnor [21] (see also Boyle and Lind [3, Theorem 6.3, part 4]):

Lemma 6.5 (Milnor [21]). *Suppose X is a \mathbb{Z}^2 -subshift with finitely many nonexpansive rays and assume that for each ray $\ell \subset \mathbb{R}^2$, at least one of ℓ or $-\ell$ is an expansive ray. Then the directional entropy $h_v(X)$ is either 0 for all $v \in \mathbb{R}^2$ or is nonzero for all $v \neq 0$. Thus the entropy seminorm $\|\cdot\|_X$ is either trivial or a norm.*

Proof. A special case of [3, part 4, Theorem 6.9] implies that the directional entropy function $h_v(X)$ is continuous in v . Thus the set

$$Z = \{v \in \mathbb{R}^2: \|v\| = 1 \text{ and } h_v(X) = 0\}$$

is a closed subset of the unit circle $S^1 \subset \mathbb{R}^2$. We show that the set Z is also open, and hence is either empty or is all of S^1 .

Let $v \neq 0$ and let J be an interval in \mathbb{R}^2 that is parallel to v and contains 0. Then for some $n, r > 0$, the set (nJ) codes a rectangle R on one side of nJ with two of its edges parallel to J . If $h_v(X) = 0$, then $\mathcal{H}(X, J) = 0$ implies $\mathcal{H}(X, R) = 0$. This implies that $\mathcal{H}(X, I) = 0$ for any interval I with endpoints on the ends of R which are perpendicular to nJ . But the unit vectors v_I parallel to such I (with orientation determined by the orientation nJ inherits from v) form a neighborhood of v in S^1 . Since for each such v_I we have $h_{v_I} = 0$, it follows that the set Z is open. \square

Corollary 6.6. *Suppose X is a polygonal \mathbb{Z}^2 -shift with a coding polygon having no pairs of antiparallel sides. Then the entropy seminorm $\|\cdot\|_X$ is either trivial or a norm.*

Proof. By Proposition 3.2, X satisfies the hypotheses of Lemma 6.5 and the statement follows. \square

6.2. Triangular \mathbb{Z}^2 -systems. Define the *girth* $\mathcal{G}(X, \mathcal{P})$ of a polygon \mathcal{P} in the direction v for some nonzero $v \in \mathbb{Z}^2$ to be the maximal length of a line segment that is the intersection of \mathcal{P} with a line parallel to v . Equivalently, the girth is the smallest distance between two parallel lines which enclose \mathcal{P} and are orthogonal to v .

For triangles, we are able to say more:

Proposition 6.7. *Suppose X is a triangular \mathbb{Z}^2 -subshift with coding polygon \mathcal{T} . If $v \in \mathbb{R}^2$, then the directional entropy of X corresponding to a vector v is*

$$h_v(X) = \frac{\mathcal{H}(X, \mathcal{T})}{\mathcal{G}(\mathcal{T}, v)} \|v\|.$$

This means that for a triangular polygonal system X , depending on whether or not $\mathcal{H}(X, \mathcal{T}) = 0$, the directional entropy $h_v(X)$ is either identically 0 or is nonzero for all $v \neq 0$.

Proof. Assume first that the vector v is not parallel to one of the sides of \mathcal{T} . The girth $\mathcal{G}(\mathcal{T}, v)$ is the length of a line segment J parallel to v with one end on a vertex of \mathcal{T} (which we assume without loss to be $(0, 0)$) and the other end on the side of \mathcal{T} opposite to this vertex. The vector from one end of J to the other is

$$\mathcal{G}(\mathcal{T}, v) \frac{v}{\|v\|}.$$

Hence

$$\mathcal{H}(X, J) = \frac{\mathcal{G}(\mathcal{T}, v)}{\|v\|} h_v(X),$$

and so

$$h_v(X) = \frac{\mathcal{H}(X, J)}{\mathcal{G}(\mathcal{T}, v)} \|v\|.$$

We complete the proof by showing that $\mathcal{H}(X, J) = \mathcal{H}(X, \mathcal{T})$. The interval J divides \mathcal{T} into two smaller triangles U and W which share the common side J . The triangle U shares a vertex $u \neq 0$ with \mathcal{T} and W shares a vertex $w \neq 0$ with \mathcal{T} . The sectors $\mathcal{S}(u)$ and $\mathcal{S}(w)$ whose edges are positively parallel to the edges emanating from u and w , respectively, are corner coding sectors for X . Let L be the subspace containing J and choose $r > \text{diam}(\mathcal{P})$ so that all the translates $P(e) := \mathcal{P} + e$, with $e \in L$, lie in L_r . Note that for sufficiently large n , the union of the translates $P(e)$ which lie in nP codes all of nU .

Therefore there exists $n_0 > 0$ and $s \in (0, 1]$ such that $(nJ)_r$ X -codes $(nU)_{sr}$ for all $n > n_0$. Similarly, we can assume that $(nJ)_r$ X -codes $(nW)_{sr}$ for all $n > n_0$ and hence $(nJ)_r$ X -codes $(n\mathcal{T})_{sr}$. By the definition of $\mathcal{H}(X, \cdot)$, we conclude $\mathcal{H}(X, nJ) \geq \mathcal{H}(X, n\mathcal{T})$ for all

$n > n_0$. Clearly $(nT)_r$ X -codes $(nJ)_r$ and so by the definition of $\mathcal{H}(X, \cdot)$ and part (4) of Lemma 6.2, we have $\mathcal{H}(X, nJ) \leq \mathcal{H}(X, nT)$. Thus $\mathcal{H}(X, J) = \mathcal{H}(X, \mathcal{T})$.

If v is parallel to one of the sides J of \mathcal{T} , then that side has length $\mathcal{G}(\mathcal{T}, v)$. A similar argument shows that $\mathcal{H}(X, J) = \mathcal{H}(X, \mathcal{T})$, and again the result follows. \square

Corollary 6.8. *Suppose X is a rational triangular \mathbb{Z}^2 -subshift and suppose $E(\mathcal{T})$ is the set of oriented edges of \mathcal{T} . If $\mathcal{H}(X, \mathcal{T}) \neq 0$, then the unit sphere in the entropy norm $\|\cdot\|_X$ is*

$$\frac{1}{\mathcal{H}(X, \mathcal{T})} S_X$$

where S_X is the convex hexagon whose oriented edges are $\{\pm e : e \in E(\mathcal{T})\}$.

Proof. If e is an oriented edge of \mathcal{T} , then the girth $\mathcal{G}(\mathcal{T}, e) = \|e\|$. Thus by Proposition 6.7, it follows that $h_e(X) = \mathcal{H}(X, \mathcal{T})$. Hence if e is a positively or negatively oriented edge of \mathcal{T} , then $h_e(X)$ lies on the sphere of radius $\mathcal{H}(X, \mathcal{T})$ in the norm $\|\cdot\|_X$. Suppose w_0 is a vertex of \mathcal{T} and e_1, e_2 are the edges emanating from w_0 . If v is a vector from w_0 to a point on the opposite side of \mathcal{T} , then the girth $\mathcal{G}(\mathcal{T}, v) = \|v\|$. So by Proposition 6.7 we have $h_v(X) = \mathcal{H}(X, \mathcal{T})$ and hence v lies on the sphere of radius $\mathcal{H}(X, \mathcal{T})$ in the norm $\|\cdot\|_X$. \square

Corollary 6.9. *Suppose X, Y are triangular \mathbb{Z}^2 -subshifts with non-trivial entropy norms and assume both are polygonal with respect to the same rational triangle \mathcal{T} . Then there is a constant $c > 0$ such that the entropy norms of X and Y satisfy $\|v\|_X = c\|v\|_Y$ for all $v \in \mathbb{R}^2$.*

Proof. Let

$$c = \frac{\mathcal{H}(X, \mathcal{T})}{\mathcal{H}(Y, \mathcal{T})}.$$

The result then follows from Corollary 6.8. \square

Note this implies that if X, Y are polygonal \mathbb{Z}^2 -subshifts with respect to the same triangle \mathcal{T} , then the ratio of their directional entropies of X in the direction v is independent of the choice of v . The \mathbb{Z}^2 -subshifts X and Y can be different and even have different alphabets, but the shape of the unit ball in the entropy norm depends only on the triangle \mathcal{T} not the shift X .

Next we turn to the relationship between entropy norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ when X and Y with respect to the same polygon \mathcal{P} , but with no restriction on the number of edges in the polygon.

Proposition 6.10. *Suppose X is a polygonal \mathbb{Z}^2 -subshift with coding polygon \mathcal{P} and assume that \mathcal{P} has no antiparallel sides. If $\mathfrak{F}(\mathcal{P})$ is the family of all \mathbb{Z}^2 -subshifts which are polygonal with respect to \mathcal{P} and which have nontrivial entropy norms, then there is a uniform dilatation constant $D > 0$, depending only on \mathcal{P} , which has the property that for all $u, v \in S^1$ we have*

$$\frac{1}{D} \leq \frac{h_u(X)}{h_v(X)} \leq D$$

for all $X \in \mathfrak{F}(\mathcal{P})$.

Thus the conclusion means that $\mathfrak{F}(\mathcal{P})$ is a *quasi-conformal family*.

Proof. Suppose v is a unit vector and let L_1 and L_2 be the supporting lines for \mathcal{P} which are parallel to v . Since \mathcal{P} has no antiparallel sides, at least one of these lines contains a vertex w of \mathcal{P} . Without loss, assume that this line is L_1 . Let T_v be the unique triangle such that

- (1) The vertex w of \mathcal{P} is also a vertex of T_v .
- (2) The two edges of T_v which meet at w contain the two edges of \mathcal{P} which meet at w .
- (3) The other two vertices of T_v lie in L_1 .

Let $W := W(v)$ be the side of T_v which lies in L_1 and let $|W| = |(W(v))|$ denote the length of W .

Because the vertex w of \mathcal{P} is corner coding, if we replace \mathcal{P} by $n\mathcal{P}$ for some large $n > 0$ (still calling it \mathcal{P}), then for a given $r > 0$ there is $s \in (0, 1]$ such that W_r X -codes T_{sr} . Clearly T_r X -codes W_r . Thus it follows from Lemma 6.2 that $\mathcal{H}(X, W) = \mathcal{H}(X, T_v)$. Choose $K := K_v > 0$ sufficiently large such that a translate of $K\mathcal{P}$ contains T_v . Then K depends only on \mathcal{P} and v . Since $\mathcal{P} \subset T_v$, we have that

$$(6.1) \quad \mathcal{H}(X, \mathcal{P}) \leq \mathcal{H}(X, T_v) \leq K(v)\mathcal{H}(X, \mathcal{P}).$$

Note that $h_v(X) = C_v\mathcal{H}(X, T_v)$, where $C_v = |(W(v))|$ which depends only on \mathcal{P} and v . Since $h_v(X)$ is Lipschitz in v with a Lipschitz constant independent of X (see [3, part 4, Theorem 6.9]) there is a neighborhood N_v of v in S^1 which is independent of X and such that for all $u \in N_v$,

$$\frac{C_v}{2}\mathcal{H}(X, T_v) \leq h_u(X) \leq 2C_v\mathcal{H}(X, T_v).$$

Since S^1 is compact, there is a finite subcovering $\{N_{v_i}\}_{i=1}^m$ of $\{N_v\}$. Let $K = \max K(v_i)$. Then Equation 6.1 implies that

$$\mathcal{H}(X, \mathcal{P}) \leq \mathcal{H}(X, T_{v_i}) \leq K\mathcal{H}(X, \mathcal{P})$$

for all $1 \leq i \leq m$. Setting $C = \max C_{v_i}$ and $c = \min C_{v_i}$, then for each i and $u \in N_{v_i}$

$$\frac{c}{2}\mathcal{H}(X, \mathcal{P}) \leq \frac{c}{2}\mathcal{H}(X, T_{v_i}) \leq h_u(X) \leq 2C\mathcal{H}(X, T_{v_i}) \leq 2CK\mathcal{H}(X, \mathcal{P}).$$

Therefore

$$\frac{c}{2}\mathcal{H}(X, \mathcal{P}) \leq h_u(X) \leq 2CK\mathcal{H}(X, \mathcal{P})$$

for all $u \in S^1$.

It follows that for all u_1, u_2 in S^1

$$\frac{h_{u_1}(X)}{h_{u_2}(X)} \leq \frac{2CK\mathcal{H}(X, \mathcal{P})}{\frac{c}{2}\mathcal{H}(X, \mathcal{P})} = \frac{4CK}{c}.$$

Setting $D := \frac{4CK}{c}$, since C, K and c are independent of X , the result follows. \square

7. FURTHER DIRECTIONS

We have several questions we are unable to answer, and we collect some of these in this section. The first is if there is a canonical way to represent a polygonal system:

Question 7.1. *If X is an infinite polygonal system, are minimal recoding polygons for X unique up to translation?*

Corollary 5.14 shows this holds when X has a triangular coding polygon and Example 5.15 shows that this holds when X has a rectangular coding polygon with sides parallel to the axes. However, even adding an assumption that the polygonal system has no antiparallel sides, we can not answer this question.

One of our results has the hypothesis of a coding polygon with no two antiparallel sides, or equivalently no two nonexpansive rays with opposite directions. We ask:

Question 7.2. *Does the conclusion of Corollary 6.6 remain valid without the assumption of no antiparallel sides? In other words, is the entropy seminorm for a polygonal system always either a norm or trivial?*

There is an example of Hochman [13] which has exactly two nonexpansive rays, each of which is the negative of the other and neither of which is closing. Both this example and its Cartesian product with a polygonal system are not polygonal (see Proposition 3.12). It seems likely that there exists an example with finitely many nonexpansive rays, no two of which are antiparallel, with at least one of them not closing, but we do not know how to construct such an example.

In Corollary 6.8, we showed that for a triangular system whose coding triangle \mathcal{T} has $\mathcal{H}(X, \mathcal{T}) \neq 0$, the unit sphere in the entropy norm is determined by the triangle \mathcal{T} . In fact this sphere is

$$\frac{1}{\mathcal{H}(X, \mathcal{T})} S_X,$$

where S_X denotes the convex hexagon whose oriented edges are $\{\pm e : e \in E(\mathcal{T})\}$.

Question 7.3. *Does the analogous result hold for systems that are not necessarily triangular polygonal systems? More precisely, if X has a minimal recoding polygon \mathcal{P} with n sides and $\mathcal{H}(X, \mathcal{P}) \neq 0$, must the unit sphere in the entropy norm of X be the $2n$ -gon*

$$\frac{1}{\mathcal{H}(X, \mathcal{P})} S_X,$$

where S_X denotes the convex polygon whose oriented edges are $\{\pm e : e \in E(\mathcal{P})\}$ and $E(\mathcal{P})$ denotes the set of oriented edges of \mathcal{P} ?

This question may be easier to answer under the additional hypothesis that \mathcal{P} has no antiparallel sides. A positive answer to this question would imply a positive answer to Question 7.1 for systems with $\mathcal{H}(X, \mathcal{P}) \neq 0$, meaning that for such systems, the minimal recoding polygon for such an X is unique.

In Example 3.9 we observed that any polygonal shift X with coding polygon \mathcal{P} is a closed invariant subshift of \hat{X} , an SFT which is polygonal and has the same coding polygon \mathcal{P} . However, at least *a priori* the SFT \hat{X} depends on \mathcal{P} . This leads to the following:

Question 7.4. *If \mathcal{P} and \mathcal{P}' are two coding polygons for X is the construction of \hat{X} in Example 3.9 independent of the polygon? More generally how canonical is an enveloping SFT \hat{X} , and in particular is it the smallest SFT containing X with the same alphabet as X ?*

Results of Einsiedler [8] show that for a large class of algebraic systems defined over a compact abelian group, including the Ledrappier example, there are uncountably many invariant subspaces realizing distinct directional entropies. We ask if this is true in greater generality.

Question 7.5. *If X is a nontrivial polygonal shift (not necessarily algebraic), is there a rational direction for which there uncountably many closed \mathbb{Z}^2 -invariant subspaces of X realizing distinct directional topological entropies? In particular does this hold if X is a polygonal SFT?*

Is it possible that all values in an open interval could be realized as the directional entropies in a fixed direction of closed subsystems of X .

Question 7.6. *Are there higher dimensional analogues for our results. What is the appropriate definition of a polyhedral \mathbb{Z}^d -shift?*

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