A SIMPLE SPECIAL CASE OF SHARKOVSKII'S THEOREM

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In this note $I \subset \mathbb{R}$ is a bounded closed interval and $f : I \to I$ will be a continuous map; f^n denotes the n-fold composition of f with itself. A point $x \in I$ is a periodic point for f with period p if $f^p(x) = x$ and has *least period* p if in addition $f^k(x) \neq x$ for $1 \leq k \leq p-1$. The note presents a short proof of the following result.

Proposition. *If f has a periodic point that is not fixed, then f has a periodic point of least period* 2*.*

This is a special case of Sharkovskii's famous theorem, which states that if f has a periodic point with least period p and q comes after p in the ordering

$$3,\ 5,\ 7,\ \ldots,\ 2\cdot 3,\ 2\cdot 5,\ 2\cdot 7,\ \ldots,\ 2^2\cdot 3,\ 2^2\cdot 5,\ 2^2\cdot 7,\ \ldots,\ 2^3,\ 2^2,\ 2,\ 1,$$

then f has a periodic point with least period q.

The proposition corresponds to the fact that 2 is the penultimate number in this ordering. The other simple special case of Sharkovskii's theorem, namely that an orbit of least period 3 forces the existence of an orbit of least period q for any q, corresponds to the fact that 3 is the first number in the ordering. It was presented in this journal by Li and Yorke [6], who rediscovered it independently of Sharkovskii's work.

According to the history given in [1], the proposition was Sharkovskii's first step towards his theorem. Coppel [3] proved in the 1950's that $f^n(x)$ converges as $n \to \infty$ for all $x \in I$ if and only if f has no periodic points with least period 2. In 1960 Sharkovskii [8] reproved Coppel's result and observed that it implies the proposition — because $f^n(x)$ does not converge if x is a periodic point whose least period is 2 or more. Sharkovskii completed the proof of his theorem in two subsequent papers [9, 10].

Proof of the Proposition. We prove that if f has a periodic point with least period $p > 2$, then f has a periodic point that is not fixed and has least period less than p. A descending induction then shows that there is a periodic point with least period 2.

Let $x_1 < x_2 < \cdots < x_p$ be the points on an orbit of least period p. We consider the directed graph with vertices $1, \ldots, p-1$ in which vertex i is joined to vertex j if and only if $f([x_i, x_{i+1}]) \supset [x_j, x_{j+1}]$. Each vertex i must be joined to at least one vertex $j \neq i$, because otherwise f would have to permute the endpoints of $[x_i, x_{i+1}]$, which is impossible since these points lie on a periodic orbit for f with least period $p \geq 3$.

Starting at the vertex 1, choose an edge that joins 1 to a different vertex. Then we join this vertex to a different vertex, and so on. The path can be extended indefinitely with

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each edge joining two different vertices. After at most $p-1$ vertices it must return to a previously visited vertex. This gives us a loop $i_1, \dots, i_q$ that passes through at least 2 and at most $p-1$ vertices.

Set $I_k = [x_{i_k}, x_{i_k+1}]$ for $k = 1, \dots, q$. Then $f(I_k) \supset I_{k+1}$ for $k = 1, \dots, q-1$ and $f(I_q) \supset I_1$. There is a closed subinterval $I_1' \subset I_1$ such that $f(I_1') = I_2$. In order to see this, look at the intersection of the graph of f with the rectangle $I_1 \times I_2$. At least one component of this intersection must join the top and bottom edges of $I_1 \times I_2$. The interval I_1' is the projection to I_1 of such a component.

Since $f^q(I_1') = f^{q-1}(f(I_1')) = f^{q-1}(I_2) \supset I_1 \supset I_1'$, it follows from the intermediate value theorem that f^q has a fixed point z in I_1'. Clearly z is a periodic point for f whose least period is a factor of q and therefore less than p.

We now show that z is not a fixed point for f. Since $z \in I_1' \subset I_1$ and $f(z) \in f(I_1') = I_2$, we can have $z = f(z)$ only if $z \in I_1 \cap I_2$. But I_1 and I_2 have disjoint interiors and their endpoints belong to an orbit with least period $p > 1$. □

The modern proof of the full Sharkovskii theorem is more intricate than the above argument, but does not require any more sophisticated tools. It was first given in [2] and can be found in many texts on dynamical systems, for example [1, 4, 5, 7]. One of the steps is a special case of our proposition, namely that f has a periodic point with least period 2 if f has a periodic point with even least period. The argument presented here is simpler than the standard proof of this step.

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