A note on stable holonomy between centers

Preliminary version

Keith Burns and Amie Wilkinson

June 11, 2005

There is a slight disparity in smooth ergodic theory, between Pesin theory and the Pugh-Shub partially hyperbolic theory. Both theories assume a weakened form of hyperbolicity and conclude ergodicity. While the two theories have considerable overlap, Pesin theory assumes a $C^{1+\delta}$ hypothesis, whereas the Pugh-Shub stable ergodicity theorem assumes a C^2 hypothesis. The purpose of this paper is to close the gap between $C^{1+\delta}$ and C^2.

Let $f : M \to M$ be a partially hyperbolic diffeomorphism of a compact manifold M. Partially hyperbolic means the following. There is a nontrivial splitting of the tangent bundle, $TM = E^u \oplus E^c \oplus E^s$, that is invariant under the derivative map Tf. Further, there is a Riemannian metric for which we can choose continuous positive functions ν, $\hat{\nu}$, and $\hat{\gamma}$ with

$$\nu, \hat{\nu} < 1 \quad \text{and} \quad \nu < \gamma \leq \hat{\gamma}^{-1} < \hat{\nu}^{-1} \tag{1}$$

such that, for a unit vector $v \in T_pM$,

$$\|Tfv\| \leq \nu(p), \qquad \text{if } v \in E^s(p), \tag{2}$$

$$\gamma(p) \leq \|Tfv\| \leq \hat{\gamma}(p)^{-1}, \qquad \text{if } v \in E^c(p), \tag{3}$$

$$\hat{\nu}(p)^{-1} \leq \|Tfv\|, \qquad \text{if } v \in E^u(p). \tag{4}$$

A partially hyperbolic diffeomorphism is *dynamically coherent* if there are foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to $E^c \oplus E^s$ and $E^c \oplus E^u$ respectively. In this case there is also a foliation $\mathcal{W}^c$ tangent to E^c whose leaves are the intersections of the leaves of $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$. Each leaf of $\mathcal{W}^{cs}$ is foliated by leaves of $\mathcal{W}^c$ and $\mathcal{W}^s$; leaves of $\mathcal{W}^{cu}$ have the analogous property.

Theorem 0.1 *([PSW], Theorem B) Let $f : M \to M$ be a C^2, partially hyperbolic, dynamically coherent diffeomorphism of a compact manifold M satisfying*

$$\nu < \gamma\hat{\gamma}.$$

Then the stable distribution is a uniformly C^1 bundle when restricted to any leaf of the center-stable foliation. Consequently, the stable holonomy map between center transversals is C^1.

If the assumption that f is C^2 is relaxed, then the first conclusion of Theorem 0.1 does not hold; that is, the stable bundle may no longer be a C^1 bundle over every leaf of the center-stable foliation. This is easy to see in the case where the unstable bundle is trivial. Let $F : \mathbf{T}^2 \to \mathbf{T}^2$ be a linear Anosov diffeomorphism of the 2-torus, and let $f = \varphi \circ F \circ \varphi^{-1}$, where $\varphi : \mathbf{T}^2 \to \mathbf{T}^2$ is a $C^{1+\delta}$ diffeomorphism that is not C^2. Then f is an Anosov diffeomorphism, with $E^s_f = \varphi_*(E^s_F)$ and $E^u_f = \varphi_*(E^u_F)$. We regard f as a partially hyperbolic diffeomorphism, with stable bundle E^s_f and center bundle E^u_f. Then the center-stable foliation has one leaf, the whole 2-torus, and the stable bundle $E^s_f = \varphi_*(E^s_F)$ is merely Hölder continuous, since $T\varphi$ is Hölder-continuous, but not C^1. Note that the hypothesis $\nu < \gamma\hat{\gamma}$ is satisfied by this example, since the center bundle of f is 1-dimensional.

While the first conclusion of Theorem 0.1 does not hold for this example, the second one does; the stable holonomy for f between center transversals is C^1, and in fact, $C^{1+\delta}$. This is because the stable holonomy for f is conjugate via φ to the stable holonomy for F.

Note also that, even under the hypothesis that f is C^2, the techniques behind the proof of Theorem 0.1 do not give a $C^{1+\beta}$ condition on the stable holonomy; for this, a $C^{2+\delta}$ hypothesis on f is required. Our main result shows that we may relax the C^2 assumption to $C^{1+\delta}$ and still obtain that the stable holonomy for f between center transversals is C^1. The price of relaxing the differentiability of f is that we require a more stringent bunching condition, a condition that is nonetheless satisfied by many examples. As an added benefit, we obtain that the holonomy is $C^{1+\beta}$, for some $\beta > 0$.

Theorem 0.2 *Let $f : M \to M$ be a $C^{1+\delta}$, partially hyperbolic, dynamically coherent diffeomorphism of a compact manifold M satisfying*

$$\nu^\theta < \gamma\hat{\gamma},$$

where $\theta \in (0, \delta)$ is a Hölder exponent for E^c. Then there exists $\beta > 0$ such that the stable holonomy map between center transversals is $C^{1+\beta}$.

As far as we know, this result is new, although there are related results for codimension-1 hyperbolic systems [PR], in the nonuniform setting when E^c is a hyperbolic Lyapunov subspace in an Oseledec decomposition [BPS], and for Lie group cocycles [PW] .

For any partially hyperbolic diffeomorphism, the stable and unstable subbundles E^s and E^u are tangent to foliations, which we denote by $\mathcal{W}^s$ and $\mathcal{W}^u$ respectively [BP]. These foliations induce an equivalence relation on M: we say that $p \sim_{us} q$ if there is a sequence of points $p = p_0, \ldots, p_k = q$ such that any two consecutive points in the sequence lie in the same $\mathcal{W}^s$-leaf or the same $\mathcal{W}^u$-leaf. A partially hyperbolic diffeomorphism has the *accessibility property* if there is only one $\sim_{us}$-class, i.e. if $p \sim_{us} q$ for any $p, q \in M$. It has the *essential accessibility property* if a set that is measurable (with respect to the volume) and is a union of $\sim_{us}$-classes must have 0 or full measure. In [BW], we use Theorem 0.2 to prove the following:

Theorem 0.3 *Let f be $C^{1+\delta}$, volume preserving, and partially hyperbolic. Suppose that f satisfies the* strong center bunching condition*:*

$$\max\{\nu, \hat{\nu}\}^\theta < \gamma\hat{\gamma}, \tag{5}$$

where $\theta \in (0, \delta)$ satisfies the inequalities:

$$\nu\gamma^{-1} < \mu^\theta, \qquad \hat{\nu}\hat{\gamma}^{-1} < \hat{\mu}^\theta. \tag{6}$$

If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.

This has the corollary:

Corollary 0.4 *Let $f : M \to M$ be a $C^{1+\delta}$, volume-preserving, partially hyperbolic, dynamically coherent diffeomorphism with 1-dimensional center bundle. If f is accessible, then f is ergodic.*

Note that Theorem 0.2 can equally be applied to the stable foliation of $C^{1+\delta}$ Anosov diffeomorphism by regarding the unstable bundle as a center bundle. By doing so, we obtain, for example, the following:

Corollary 0.5 *Let f be a $C^{1+\delta}$ conformal Anosov diffeomorphism. Then for every $\beta < \delta$, there exists a neighborhood $\mathcal{U}$ of f in $\mathit{Diff}^{1+\delta}(M)$, such that, for every $g \in \mathcal{U}$, the stable and unstable holonomies for g are $C^{1+\beta}$.*

1 Proof of Theorem 0.2

The proof follows similar lines to the proof of absolute continuity of the stable foliation for partially hyperbolic systems. The main difference is that, unlike the Jacobian map $x \mapsto \mathrm{Jac}_x f$, the multiplicative cocycle $x \mapsto T_x f$ over f is not abelian. We exploit instead the fact that the map $v \mapsto \|Tfv\|$ is an abelian multiplicative cocycle over the projectivized tangent map $f_* : P(TM) \to P(TM)$. This map is projective in the fibers, with Lipschitz norm $(\gamma\hat{\gamma})^{-1}$. The inequality $\nu^\theta < \gamma\hat{\gamma}$ allows us to prove that the stable holonomy h^s has a projectivized derivative h^s_*. Once this is established, a distortion estimate using the $v \mapsto \|Tfv\|$ cocycle gives us that h^s is differentiable.

We denote by h^s the stable holonomy between two center leaves, omitting reference to domain and codomain. We denote by π^s a fixed, uniformly $C^{1+\theta}$, local approximation to h^s. By this we mean that there is a constant $C > 0$ such that for any $p, q \in M$ with $q \in W^s(p, 1)$ we have a $C^{1+\theta}$ diffeomorphism

$$\pi^s_{p,q} : W^c_{loc}(p) \to W^c(q)$$

such that:

1. $d(\pi^s_{p,q}(p), q) \leq Cd(p, q)$;
2. $d(T\pi^s_{p,q}v, v) \leq Cd(p, q)^\theta$, for all $v \in P(E^c(p))$;
3. if $\mathcal{W}^c_{loc}(p) \cap \mathcal{W}^c_{loc}(p') \neq \emptyset$, and the codomains of $\pi^s_{p,q}$ and $\pi^s_{p',q'}$ also intersect nontrivially, then $\pi^s_{p,q} = \pi^s_{p',q'}$ on $\mathcal{W}^c_{loc}(p) \cap \mathcal{W}^c_{loc}(p')$.

This can be accomplished by fixing a $C^{1+\theta}$ normal bundle to each leaf of the center foliation $\mathcal{W}^c$. Since the leaves of $\mathcal{W}^c$ vary continuously in the $C^{1+\theta}$ topology, this choice of normal bundle can be made to vary continuously from leaf to leaf. The fibers of this bundle foliate a uniform neighborhood of each leaf. The holonomy of this local foliation between a local center leaf $\mathcal{W}^c_{loc}(p)$ and a neighboring center leaf $\mathcal{W}^c_{loc}(q)$ gives the map $\pi^s_{p,q}$. By rescaling the metric if necessary, we may assume that $\pi^s_{p,q}$ is well-defined on the domain $\mathcal{W}^c_{loc}(p)$, for all $q \in W^s(p, 1)$. We will sometimes drop the reference to p and q and write π^s for short.

Fix $p, q \in M$ with $q \in W^s(p, 1)$, and let

$$h^s_n = f^{-n} \circ \pi^s_{p_n,q_n} \circ f^n.$$

Note that $h_n^s : \mathcal{W}^c_{loc}(p) \to \mathcal{W}^c(q)$ is well-defined, for all $n \geq 0$, since $\pi^s_{p_n,q_n}$ extends uniquely to the domain $f^n(\mathcal{W}^c_{loc}(p))$.

The proof now proceeds in four steps alluded to above. In the first step, we show that h_n^s converges in the C^0 topology to h^s. This first step uses only partial hyperbolicity and dynamical coherence. In the second step, we show that the projective linear bundle maps $h^s_{n*} : P(E^c) \to P(E^c)$ converge in the C^0 topology to a projective linear bundle map $h^s_* : P(E^c) \to P(E^c)$. In Step 3, we show that the derivatives $Th_n^s : E^c \to E^c$ converge in the C^0 topology to a bundle isomorphism $Th^s : E^c \to E^c$. It follows that h^s is C^1, with derivative Th^s. Finally, we show that Th^s is Hölder continuous. The last three steps use the bunching hypothesis $\nu^\theta < \gamma\hat{\gamma}$.

Step 1: the sequence h_n^s converges to h^s in the space of all continuous maps from $\mathcal{W}^c_{loc}(p)$ to $\mathcal{W}^c(q)$.

Invariance of $\mathcal{W}^s$ under f implies that $f^{-n}h^s_{p_n,q_n}f^n = h^s$, where the stable holonomy $h^s_{p_n,q_n}$ maps from $f^n(\mathcal{W}^c_{loc}(p))$ to $\mathcal{W}^c(q_n)$, sending p_n to q_n.

Note that since f^n contracts distances in $\mathcal{W}^s(x,1)$ by a factor of $\nu_n(x)$, we have that $d(x_n, h^s_{p_n,q_n}(x_n)) \leq \nu_n(x)$, for all $x \in \mathcal{W}^c_{loc}(p)$ and $n \geq 1$. By property 1. of π^s, we then obtain that

$$d(\pi^s(x_n), h^s_{p_n,q_n}(x_n)) \leq C\nu_n(x), \tag{7}$$

for all $x \in \mathcal{W}^c_{loc}(p)$ and $n \geq 1$.

Now fix $x \in \mathcal{W}^c_{loc}(p)$ and $n \geq 1$. We have:

$$\begin{aligned} d(h_n^s(x), h^s(x)) &= d(f^{-n}\pi^s(x_n), f^{-n}h^s_{p_n,q_n}(x_n)) \\ &\leq \gamma_{-n}(x)\, d(\pi^s(x_n), h^s_{p_n,q_n}(x_n)) \\ &\leq C\gamma_{-n}(x)\nu_{-n}(x). \end{aligned}$$

Partial hyperbolicity implies that there exists $\kappa < 1$ such that the function $\gamma_{-n}\nu_n$ is uniformly bounded by κ^n. Hence h_n^s converges uniformly to h^s.$\diamond$

Step 2: the sequence $h^s_{n*} : P(T\mathcal{W}^c_{loc}(p)) \to P(T\mathcal{W}^c(q))$ is Cauchy.

We fix some notation. For $w \in P(TM)$ and $k \in \mathbf{Z}$, let $w_k = f^k_*(w) \in P(TM)$. We first prove some lemmas.

Lemma 1.1 *There exists $\delta > 0$ such that, for every $v \in P(E^c(x))$ and $w \in P(E^c(y))$:*

1. *if $d(v_k, w_k) \leq \delta$, for $k = 0, \ldots, n-1$, then*

$$d(v_n, w_n) \leq (\gamma\hat{\gamma})_{-n}(x_n) d(v, w)^\theta;$$

2. *if $d(v_{-k}, w_{-k}) \leq \delta$, for $k = 0, \ldots, n-1$, then*

$$d(v_{-n}, w_{-n}) \leq (\gamma\hat{\gamma})_{-n}(x) d(v, w)^\theta.$$

In particular, if $d(v, w) \leq \delta(\gamma\hat{\gamma})_n(x)$, then the conclusions in 1. hold, and if $d(v, w) \leq \delta(\gamma\hat{\gamma})_n(x_n)$, then the conclusions in 2. hold.

Proof. It suffices to prove the lemma in the case $n = 1$; the other cases are proved inductively. In writing $f_*(v)$ for $v \in S(T_x M)$, we deliberately suppress the basepoint x. For the following calculation, the basepoint is relevant, so we shall write $v = (x, \zeta)$ and $w = (y, \xi)$. Recall that the Lipschitz norm of f_* restricted to the fiber $P(E^c(x))$ is $\gamma\hat{\gamma}(x)^{-1} = (\gamma\hat{\gamma})_{-1}(x_1)$, and the Lipschitz norm of f_*^{-1} restricted to the fiber $P(E^c(x))$ is $\gamma\hat{\gamma}(x_{-1})^{-1} = (\gamma\hat{\gamma})_{-1}(x)$. We estimate:

$$\begin{aligned}
d(v_1, w_1) &= d(f_*(x, \zeta), f_*(y, \xi)) \\
&\leq d(f_*(x, \zeta), f_*(x, \xi)) + d(f_*(x, \xi), f_*(y, \xi)) \\
&\leq (\gamma\hat{\gamma})_{-1}(x_1) d(\zeta, \xi) + H d(x, y)^\delta \\
&\leq (\gamma\hat{\gamma})_{-1}(x_1) \left(d(\zeta, \xi)^{1-\theta} + H d(x, y)^{\delta - \theta} \right) \sup\{d(x, y), d(\zeta, \xi)\}^\theta.
\end{aligned}$$

If $d(v, w) = \sup\{d(\zeta, \xi), d(x, y)\}$ is sufficiently small, then

$$d(\zeta, \xi)^{1-\theta} + H d(x, y)^{\delta-\theta} < 1,$$

and we have

$$d(v_1, w_1) \leq (\gamma\hat{\gamma})_{-1}(x_1) d(v, w)$$

This completes the proof of 1, in the case $n = 1$. The case 2. is proved similarly.$\diamond$

The next Lemma will also be useful in Step 3.

Lemma 1.2 *For $n \geq 1$, and $k \leq n$, let*

$$v'_k = f_*^{-n+k} \pi_*(v_n),$$

where $v \in P(T_x \mathcal{W}^c_{loc}(p))$. Then, for n sufficiently large, we have:

$$d(v_k, v'_k) \leq C \nu_k(x)^\theta,$$

for all $k \leq n$.

Proof. Property 2. of π^s and uniform contraction of $\mathcal{W}^s(p, 1)$ by ν together imply that

$$\begin{aligned} d(v_n, v'_n) &= d(v_n, \pi^s_*(v_n)) \\ &\leq C d(x_n, h^s_{p_n, q_n}(x_n))^\theta \\ &\leq C \nu_n(x)^\theta. \end{aligned}$$

Let $\delta > 0$ be given by Lemma 1.1. The bunching hypothesis $\nu^\theta < \gamma\hat{\gamma}$ implies that if n is sufficiently large, we have:

$$d(v_n, v'_n) \leq \delta (\gamma\hat{\gamma})_n(x).$$

Assume n is this large.

We prove Lemma 1.2 by backward induction on k. We have shown that

$$d(v_n, v'_n) \leq \min\{C\nu_n(x)^\theta, \delta(\gamma\hat{\gamma})_n(x)\}.$$

Lemma 1.1 now implies that for all $1 \leq k \leq n$, we have

$$\begin{aligned} d(v_k, v'_k) &= d(f_*^{-n+k}(v_n), f_*^{-n+k}(v'_n)) \\ &\leq (\gamma\hat{\gamma})_{k-n}(x) d(v_n, v'_n) \\ &\leq \delta(\gamma\hat{\gamma})_k(x) \end{aligned}$$

Suppose now that for some k we have

$$d(v_k, v'_k) \leq C\nu_k(x)^\theta.$$

Then, since $d(v_k, v'_k) \leq \delta$, we have

$$\begin{aligned} d(v_{k-1}, v'_{k-1}) &= d(f_*^{-1}(v_k), f_*^{-1}(v'_k)) \\ &\leq (\gamma\hat{\gamma})_{-1}(x_k) d(v_k, v'_k) \\ &\leq (\gamma\hat{\gamma})_{-1}(x_k) C\nu_k(x)^\theta \\ &\leq C\nu_{k-1}(x)^\theta, \end{aligned}$$

since $\nu^\theta < \gamma\hat{\gamma}$. $\diamond$

Returning to the proof of Step 2, we fix $v \in P(T\mathcal{W}^c_{loc}(p))$, and $n \geq 1$. Then

$$\begin{aligned} d(h_{n*}(v), h_{n+1*}(v)) &= d(f_*^{-n}\pi^s_*(v_n), f_*^{-n-1}\pi^s_* f_*(v_n)) \\ &\leq (\gamma\hat{\gamma})_{-n}(x)\, d(\pi^s_*(v_n), f_*^{-1}\pi^s_* f_*(v_n)) \\ &\leq (\gamma\hat{\gamma})_{-n}(x)\left[d(\pi^s_*(v_n),(v_n)) + d(f_*^{-1}f_*(v_n), f_*^{-1}\pi^s_* f_*(v_n)\right] \\ &\leq (\gamma\hat{\gamma})_{-n}(x) C\nu_n(x)^\theta + (\gamma\hat{\gamma})_{-n-1}(x) C\nu_{n+1}(x)^\theta \end{aligned}$$

Since $\nu^\theta < \gamma\hat{\gamma}$, the sequence is Cauchy.

Let $h^s_* = \lim_{n\to\infty} h^s_{n*}$.

Step 3: the sequence $Th^s_n : T\mathcal{W}^c_{loc}(p) \to T\mathcal{W}^c(q)$ is Cauchy.

Since we have already found h_*, it suffices to show that the sequence of functions $v \mapsto \|Th_n v\|$, for $v \in P(T_x\mathcal{W}^c_{loc}(p))$, is Cauchy. This is uses a standard $C^{1+\theta}$ distortion estimate.

Lemma 1.3 *There exists a $K \geq 1$, such that, for all $v \in P(T_x\mathcal{W}^c_{loc}(p))$ and $n \geq 1$, we have:*

$$K^{-1} \leq \frac{\Pi_{k=0}^{n-1}\|Tfv_k\|}{\Pi_{k=0}^{n-1}\|Tfv'_k\|} \leq K.$$

Proof. For $w \in P(TM)$, let $\varphi(w) = \log\|Tfw\|$. Note that φ is a δ-Hölder continuous function. Fix $v \in P(T_x\mathcal{W}^c_{loc}(p))$ and $n \geq 1$. Then

$$\begin{aligned} \left|\log\frac{\Pi_{k=0}^{n-1}\|Tfv_k\|}{\Pi_{k=0}^{n-1}\|Tfv'_k\|}\right| &= \left|\sum_{k=0}^{n-1}\varphi(v_k) - \varphi(v'_k)\right| \\ &\leq \sum_{k=0}^{n-1} H d(v_k, v'_k)^\delta \\ &\leq H\sum_{k=0}^{n-1} C^\delta \nu_k(x)^{\theta\delta} \\ &\leq HC^\delta(1-\overline{\nu}^{\theta\delta})^{-1}, \end{aligned}$$

where $\overline{\nu} = \sup_{z\in M}\nu(z) < 1$. Setting $K = \exp(HC^\delta(1-\overline{\nu}^{\theta\delta})^{-1})$ completes the proof. $\diamond$

Using this lemma, we fix $v \in P(T_x \mathcal{W}^c_{loc}(p))$ and $n \geq 1$ and estimate:

$$\begin{aligned}
|\|Th^s_n v\| - \|Th^s_{n+1} v\|| &= \frac{\Pi_{k=0}^{n-1}\|Tfv_k\|}{\Pi_{k=0}^{n-1}\|Tfv'_k\|}|\|T\pi v_n\| - \|T(f\pi f^{-1})v_n\|| \\
&\leq C\|T\pi v_n - T(f\pi f^{-1})v_n\| \\
&\leq C\|T\pi v_n - v_n\| + \|T(ff^{-1})v_n - T(f\pi f^{-1})v_n\| \\
&\leq C'\nu_n(x)^\theta
\end{aligned}$$

Hence the sequence $\|Th^s_n\| : P(T\mathcal{W}^c_{loc}(p)) \to \mathbf{R}$ is Cauchy. Let $Th = \lim_{n\to\infty} Th^s_n$.

Step 4: the function $Th^s : T\mathcal{W}^c_{loc}(p) \to T\mathcal{W}^c(q)$ is Hölder continuous.

We first show that $h^s_* : P(T\mathcal{W}^c_{loc}(p)) \to P(T\mathcal{W}^c(q))$ is Hölder continuous, uniformly in $p \in M$ and $q \in W^c(p,1)$. Choose $\varepsilon > 0$ so that $\nu^\theta < (\gamma\hat{\gamma})^{1+\varepsilon}$, and let $\rho = (\gamma\hat{\gamma})^{(2+\varepsilon)/\theta}$. Suppose that $p' \in W^c_{loc}(p)$. Then there exists an $n \geq 1$ such that $\min\{d(p,p'),1\} \in [\rho_n(p), \rho_{n-1}(p))$.

Our previous calculations show that there exists a $K > 0$ such that

$$d_{C^0}(h^s_*, h^s_{n*}) \leq K\nu^\theta_n(\gamma\hat{\gamma})_{-n}.$$

Let $\beta = \frac{\theta\varepsilon}{2+\varepsilon}$, and note that

$$\nu^\theta(\gamma\hat{\gamma})_{-1} < \rho^\beta,$$

for all n. Thus it suffices to show that

$$d(h_{n*}(p,\xi), h_{n*}(p',\xi)) \leq O(\rho_n(p)^\beta),$$

for all $n \geq 1$, $p \in M$ and $p' \in W^c(p, \rho_n(p))$.

Lemma 1.1 implies that

$$d(f^n_*(p,\xi), f^n_*(p',\xi)) \leq (\gamma\hat{\gamma})_{-n}(p_n)\rho_n(p)^\theta < (\gamma\hat{\gamma})_n(p)^{1+\varepsilon}.$$

Now property 2. of π^s implies that

$$\begin{aligned}
d(\pi^s f^n_*(p,\xi), \pi^s f^n_*(p',\xi)) &\leq d(\pi^s f^n_*(p,\xi), f^n_*(p,\xi)) + d(f^n_*(p,\xi), f^n_*(p',\xi)) \\
&\quad + d(f^n_*(p',\xi), \pi^s f^n_*(p',\xi)) \\
&\leq Cd(p_n, \pi^s(p_n))^\theta + d(f^n_*(p,\xi), f^n_*(p',\xi)) + Cd(p'_n, \pi^s(p'_n))^\theta \\
&\leq C\nu_n(p)^\theta + (\gamma\hat{\gamma})_n(p)^{1+\varepsilon} + C\nu_n(p')^\theta \\
&\leq O((\gamma\hat{\gamma})_n(p)^{1+\varepsilon})
\end{aligned}$$

Again applying Lemma 1.1, we obtain that

$$\begin{aligned} d(h_{n*}(p,\xi), h_{n*}(p',\xi)) &\leq (\gamma\hat{\gamma})_{-n}(p'_n)O((\gamma\hat{\gamma})_n(p)^{1+\varepsilon}) \\ &\leq O((\gamma\hat{\gamma})_n(p)^{\varepsilon}) \\ &= O(\rho_n(p)^{\beta}). \end{aligned}$$

We now show that $\log \|Th^s\| : P(T\mathcal{W}^c_{loc}(p)) \to \mathbf{R}_+$ is Hölder continuous. Let $v = (p,\xi)$ and $w = (p',\xi)$, and suppose as above that $\min\{d(p,p'),1\} \in [\rho_n(p), \rho_{n-1}(p))$. We showed in Step 3 that

$$\log \|Th^s(v)\| = \sum_{i=0}^{\infty} \varphi(v_i) - \varphi(h^s_*(v_i)),$$

where $\varphi = \log \|Tf\|$. From this it follows that

$$\begin{aligned} |\log \|Th^s(p,\xi)\| - \log \|Th^s(p',\xi)\|| &= \left| \sum_{i=0}^{\infty} (\varphi(v_i) - \varphi(h^s_* v_i)) - \sum_{i=0}^{\infty} (\varphi(w_i) - \varphi(h^s_* w_i)) \right| \\ &\leq \sum_{i=0}^{n-1} |\varphi(v_i) - \varphi(w_i)| + \sum_{i=0}^{n-1} |\varphi(h^s_* v_i) - \varphi(h^s_* w_i)| \\ &\quad + \sum_{i=n}^{\infty} |\varphi(v_i) - \varphi(h^s_* v_i)| + \sum_{i=n}^{\infty} |\varphi(w_i) - \varphi(h^s_* w_i)| \\ &\leq O(\sum_{i=0}^{n-1} d(v_i, w_i)^{\delta} + \sum_{i=0}^{n-1} d(h^s_* v_i, h^s_* w_i)^{\delta} \\ &\quad + \sum_{i=n}^{\infty} d(v_i, h^s_* v_i)^{\delta} + \sum_{i=n}^{\infty} d(w_i, h^s_* w_i)^{\delta}) \\ &\leq O(\sum_{i=0}^{n-1} ((\gamma\hat{\gamma})_{-i}\rho_n^{\theta})^{\delta} + \sum_{i=0}^{n-1} ((\gamma\hat{\gamma})_{-i}\rho_n^{\theta})^{\beta\delta} \\ &\quad + \sum_{i=n}^{\infty} \nu_i^{\theta\delta}) \\ &\leq O\left(\left(((\gamma\hat{\gamma})_{-n}\rho_n^{\theta}) \right)^{\beta\delta} + \nu_n^{\theta\delta} \right) \end{aligned}$$

Recall that we have chosen n so that

$$d(v_n, w_n) \leq (\gamma\hat{\gamma})_{-n}(p_n)\rho_n(p)^{\theta} < (\gamma\hat{\gamma})_n(p)^{1+\varepsilon} = \rho_n(p)^{\theta(1+\varepsilon)/(2+\varepsilon)}.$$

We also have that

$$\nu^{\theta} < (\gamma\hat{\gamma})^{1+\varepsilon}$$

From these facts we conclude that

$$|\log\|Th^s(p,\xi) - \log\|Th^s(p',\xi)\||| \quad \leq \quad O(\rho_n(p)^{\beta'}),$$

where

$$\beta' = \delta\beta\frac{\theta(1+\varepsilon)}{2+\varepsilon} = \frac{\delta\theta\varepsilon(1+\varepsilon)}{(2+\varepsilon)^2}.$$

References

[BPS] Barreira, L., Pesin, Ya., and J. Schmeling, *Dimension and product structure of hyperbolic measures*, Ann. of Math. (2) **149** (1999), no. 3, 755–783.

[BP] Brin, M, and Ja. Pesin, *Partially hyperbolic dynamical systems*, Math. USSR Izvestija **8** (1974), 177–218.

[BW] Burns, K. and A. Wilkinson, *On the ergodicity of partially hyperbolic systems*, preprint.

[PR] Pinto, A. A. and D. A. Rand, *Smoothness of holonomies for codimension 1 hyperbolic dynamics.* Bull. London Math. Soc. **34** (2002), no. 3, 341–352.

[PW] Pollicott, M., and P. Walkden, *Livšic theorems for connected Lie groups*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2879–2895.

[PSW] Pugh, C., Shub, M., and A. Wilkinson, *Hölder foliations*, Duke Math. J., **86** (1997), no. 3, 517–546