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DYNAMICAL COHERENCE AND CENTER BUNCHING

KEITH BURNS AND AMIE WILKINSON

For Yasha Pesin, in friendship and admiration.

Abstract. This paper discusses relationships among the basic notions that have been important in recent investigations of the ergodicity of volume-preserving partially hyperbolic diffeomorphisms. In particular we survey the possible definitions of dynamical coherence and discuss the relationship between dynamical coherence and center bunching.

Introduction. Partial hyperbolicity provides a natural context in which the Hopf argument can be used to prove ergodicity. The most recent result in this direction is our theorem:

Theorem 0.1. [8] *Let f be C^2, volume-preserving, partially hyperbolic and center bunched. If f is essentially accessible, then f is ergodic, and in fact has the Kolmogorov property.*

This result builds on a series of papers [11, 25, 19, 20], initiated by the work of Grayson, Pugh and Shub [11]. In contrast to its predecessors, Theorem 0.1 makes no hypothesis of dynamical coherence. Dynamical coherence and the hypotheses of Theorem 0.1 are defined precisely in the next section.

In this paper, we explore connections between dynamical coherence and the center bunching hypothesis in Theorem 0.1. We also discuss various definitions of dynamical coherence and how they are connected to each other. Along the way, we pose some questions and try to clarify some of the issues involved.

Although dynamical coherence appears to us to be a strong and rather unnatural hypothesis, we know of only one type of non-dynamically coherent diffeomorphism to which Theorem 0.1 applies. This is a family of Anosov diffeomorphisms, described by A. Hammerlindl [12], in which the central bundles are constructed from weak stable bundles. In contrast, there are non-Anosov examples of non-dynamically coherent diffeomorphisms that come arbitrarily close to satisfying the center bunching hypothesis. We describe these and related examples of N. Gourmelon [10] in Section 3.

1. **Partial hyperbolicity, center bunching and accessibility.** We now define the hypotheses in Theorem 0.1, namely, partial hyperbolicity, center bunching, and accessibility. Let $f : M \to M$ be a diffeomorphism of a compact manifold M.

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We say that f is *partially hyperbolic* if the following holds. First, there is a nontrivial splitting of the tangent bundle, $TM = E^s \oplus E^c \oplus E^u$, that is invariant under the derivative map Tf. Further, there is a Riemannian metric for which we can choose continuous positive functions ν, $\hat\nu$, γ and $\hat\gamma$ with

$$\nu, \hat\nu < 1 \quad \text{and} \quad \nu < \gamma < \hat\gamma^{-1} < \hat\nu^{-1} \tag{1}$$

such that, for any unit vector $v \in T_pM$,

$$\|Tfv\| < \nu(p), \qquad \text{if } v \in E^s(p), \tag{2}$$

$$\gamma(p) < \|Tfv\| < \hat\gamma(p)^{-1}, \qquad \text{if } v \in E^c(p), \tag{3}$$

$$\hat\nu(p)^{-1} < \|Tfv\|, \qquad \text{if } v \in E^u(p). \tag{4}$$

It will be convenient to let s, c and u denote the dimensions of E^s, E^c, and E^u, respectively.

We say that f is *center bunched* if the functions $\nu, \hat\nu, \gamma$, and $\hat\gamma$ can be chosen so that:

$$\max\{\nu, \hat\nu\} < \gamma\hat\gamma. \tag{5}$$

Center bunching means that the hyperbolicity of f dominates the nonconformality of Tf on the center. Inequality (5) always holds when $Tf|_{E^c}$ is conformal. For then we have $\|T_pfv\| = \|T_pf|_{E^c(p)}\|$ for any unit vector $v \in E^c(p)$, and hence we can choose $\gamma(p)$ slightly smaller and $\hat\gamma(p)^{-1}$ slightly bigger than

$$\|T_pf|_{E^c(p)}\|.$$

By doing this we may make the ratio $\gamma(p)/\hat\gamma(p)^{-1} = \gamma(p)\hat\gamma(p)$ arbitrarily close to 1, and hence larger than both $\nu(p)$ and $\hat\nu(p)$.

In particular, center bunching holds whenever E^c is one-dimensional.

The center bunching hypothesis considered here is natural and appears in other contexts, e.g. [5, 1, 17, 16]. This hypothesis is much weaker than the center bunching hypothesis in previous ergodicity theorems for partially hyperbolic diffeomorphisms, notably the result of Pugh and Shub in [20]. It appears that a major new idea would be necessary in order to weaken or remove this hypothesis from Theorem 0.1.

The stable and unstable bundles E^s and E^u of a partially hyperbolic diffeomorphism are tangent to foliations, which we denote by $\mathcal{W}^s$ and $\mathcal{W}^u$ respectively [5].

Definition 1.1. A partially hyperbolic diffeomorphism $f : M \to M$ is *accessible* if any point in M can be reached from any other along an *su-path*, which is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s$ or a single leaf of $\mathcal{W}^u$.

The *accessibility class of* $p \in M$ is the set of all $q \in M$ that can be reached from p along an *su*-path. Accessibility means that there is one accessibility class, which contains all points. The following notion is a natural weakening of accessibility.

Definition 1.2. A partially hyperbolic diffeomorphism $f : M \to M$ is *essentially accessible* if every measurable set that is a union of entire accessibility classes has either full or zero volume.

Pugh and Shub have conjectured that essential accessibility implies ergodicity, for a C^2, partially hyperbolic, volume-preserving diffeomorphism [19]. Theorem 0.1 establishes this conjecture under the center bunching hypothesis.

2. **Dynamical coherence.**

2.1. **Integrability.** Before defining dynamical coherence, we make a few general remarks about integrability and unique integrability of distributions.

Let E be a continuous distribution of k-dimensional subspaces of TM.

Definition 2.1. E is *weakly integrable* if through every point in M there exists an injectively-immersed, complete k-dimensional submanifold that is everywhere tangent to E (see [4]).

E is *integrable* if there exists a foliation of M by immersed k-manifolds whose leaves are everywhere tangent to E. Such a foliation is called an *integral foliation* of E.

E is *uniquely integrable* if E is integrable with integral foliation $\mathcal{F}$, and in addition any C^1 path everywhere tangent to E lies in a single leaf of $\mathcal{F}$.

E is *plaquewise uniquely integrable* if E is integrable with integral foliation $\mathcal{F}$, and in addition any k-dimensional immersed disk everywhere tangent to E lies in a single leaf of $\mathcal{F}$.

Note that either form of unique integrability of E implies that E has a unique integral foliation. Having a unique integral foliation is, however, a strictly weaker condition than either form of unique integrability: an example that illustrates this is the 1-dimensional distribution E in $\mathbf{R}^2$ tangent to the foliation $\mathcal{F}$ by the curves $\{(t,(t+c)^3)\colon t \in \mathbf{R}\}_{c\in\mathbf{R}}$. Although $\mathcal{F}$ is the unique foliation tangent to E, the curve $\{(t,0)\colon t \in \mathbf{R}\}$ is everywhere tangent to E and does not lie in a leaf of $\mathcal{F}$. This example is modified in [21] to show that plaquewise unique integrability does not imply unique integrability if $k > 1$. (Note that the two notions do coincide when $k = 1$.)

In [18], Pesin gives a local description of unique integrability:

Definition 2.2. E is *locally uniquely integrable* if each $x \in M$ lies in a k-dimensional smooth submanifold $\mathcal{W}_{loc}(x)$ with the property that any short enough C^1 curve starting at x and tangent to E must lie in $\mathcal{W}_{loc}(x)$.

It is not difficult to see that local unique integrability and unique integrability are equivalent.

It is well-known that a 1-dimensional distribution is uniquely integrable if it is Lipschitz. For higher-dimensional distributions, there is an analogue to this fact:

Proposition 2.3. *Suppose that the distribution E is integrable and has an integral foliation that is transversely Lipschitz (in particular, these hypotheses hold if E is both Lipschitz and integrable). Then E is uniquely integrable.*

See [23] for a proof of this and related results about Lipschitz distributions.

2.2. **Definitions of dynamical coherence.** In the ten years since the notion of dynamical coherence first appeared in [19], it has been redefined nearly ten times. At a minimum, the notion of dynamical coherence should require that each point lie in a c-dimensional submanifold tangent to E^c. Viewed in the proper way, the time-one map of the geodesic flow for a complex (or quaternionic or Cayley) hyperbolic manifold provides an example of a partially hyperbolic diffeomorphism that fails to have this property. The center distribution for these examples combines the flow direction with the planes of weakest curvature. Because the curvature is exactly quarter pinched, these examples lie on the edge of center bunching but are not center bunched. More examples, including center bunched ones, are described in Section 3.

The most common definition of dynamical coherence appears to be that there exist foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ tangent to $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$ respectively, see for example [3, 4, 2].

Proposition 2.4. *If there exist foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$ as above, then there is a foliation $\mathcal{W}^c$ tangent to E^c. Furthermore, $\mathcal{W}^c$ and $\mathcal{W}^u$ subfoliate $\mathcal{W}^{cu}$, while $\mathcal{W}^c$ and $\mathcal{W}^s$ subfoliate $\mathcal{W}^{cs}$.*

The original definition of dynamical coherence in [20] assumed all of the conclusions of this proposition in addition to the existence of the foliations $\mathcal{W}^{cs}$ and $\mathcal{W}^{cu}$.

The main ingredient in the proof of Proposition 2.4 is the following lemma, which applies to any partially hyperbolic diffeomorphism, even one for which the distribution E^{cu} is not everywhere integrable.

Lemma 2.5. *Any disk tangent to E^{cu} is subfoliated by $\mathcal{W}^u$-plaques, and any disk tangent to E^{cs} is subfoliated by $\mathcal{W}^s$-plaques. (A $\mathcal{W}^*$* plaque *is a connected open subset of a $\mathcal{W}^*$-leaf.)*

Proof. Let D be the image of an open disk in $\mathbf{R}^{c+u}$ under an embedding that is everywhere tangent to E^{cu}. Let X be the disjoint union of all of the f-iterates of D, which we call the leaves of X. Each leaf of X is an immersed submanifold of M, tangent to E^{cu}. On the manifold X there is a diffeomorphism F, which acts on each leaf of X as the restriction of f. The diffeomorphism F inherits partial hyperbolicity from f. The stable bundle for F is trivial. The unstable and central bundles for F on a leaf of X are the restrictions of the corresponding bundles for f.

One now constructs the unstable foliation $\mathcal{W}^u_F$ for F in the usual way. While X is not compact, there are uniform estimates on F because of its relationship to f, which does act on a compact manifold. Each leaf of the foliation $\mathcal{W}^u_F$ is a submanifold of M tangent to E^u, and it follows from the unique integrability of E^u that the leaves of $\mathcal{W}^u_F$ are plaques of $\mathcal{W}^u$. But the leaves of $\mathcal{W}^u_F$ subfoliate D, and therefore plaques of $\mathcal{W}^u$ subfoliate D.

Similarly, one shows that plaques of $\mathcal{W}^s$ subfoliate any disk tangent to E^{cs}. $\square$

Proof of Proposition 2.4. Intersecting the leaves of $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$ gives us the leaves of the desired foliation $\mathcal{W}^c$.

It is obvious from its construction that the leaves of $\mathcal{W}^c$ subfoliate the leaves of $\mathcal{W}^{cu}$ and $\mathcal{W}^{cs}$. Lemma 2.5 implies that plaques of $\mathcal{W}^u$ subfoliate plaques of $\mathcal{W}^{cu}$ and that plaques of $\mathcal{W}^s$ subfoliate plaques of $\mathcal{W}^{cs}$. $\square$

The distributions E^u and E^s are always uniquely integrable. Dynamical coherence does not require E^{cs} and E^{cu} or E^c to be uniquely integrable, though no example is known where integrability holds and unique integrability fails for these distributions. Even invariance of the foliations is not required in the definition of dynamical coherence, though Pugh and Shub implicity use f-invariance of these foliations in the proof of their main result in [20]. Note that a unique integral foliation is necessarily invariant. On the other hand, it is conceivable that there might be several integral foliations, some (but not all) of which are invariant.

In defining dynamical coherence, one can impose invariance or any of the unique integrability conditions discussed in the previous subsection. In this way, we obtain 12 definitions of dynamical coherence, summarized in the table below.

	E^{cs} & E^{cu}		E^c
Uniquely integrable	$*$	$\Longrightarrow$	$*$
	$\Downarrow$		$\Downarrow$
Plaquewise uniquely integrable	$*$	$\Longleftrightarrow$	$*$
	$\Downarrow$		$\Downarrow$
Unique integral foliation	$*$		$*$
	$\Downarrow$		$\Downarrow$
Invariant integral foliation	$*$	$\Longrightarrow$	$*$
	$\Downarrow$		$\Downarrow$
Integral foliation	$*$	$\Longrightarrow$	$*$
	$\Downarrow$		$\Downarrow$
Weak integrability	$*$	$\Longrightarrow$	$*$

The implications shown in this table are the ones that are currently known to us. It is possible that the weakest property, existence of an integral foliation for E^c (which appears in the lower right hand corner), is equivalent to the strongest property, unique integrability of E^{cs} and E^{cu} (which appears in the upper left). We have no idea about the relation between the two conditions in the third row.

Both properties in the bottom row of the table hold when E^c is 1-dimensional. The bottom right is a simple consequence of the fact that continuous vector fields have integral curves; the bottom left is Proposition 3.4 in [4].

We now explain the implications in the table. The downward implications in the two columns are immediate consequences of general properties of foliations.

The forward implications in the horizontal rows all use Proposition 2.4: the integral manifolds for E^c are obtained by intersecting the integral manifolds for E^{cs} and E^{cu}. The implications in the two bottom rows are simply this observation. In the fourth row, we also need the observation that we obtain an invariant foliation if this construction is applied to invariant foliations. In the first row, we need to observe that any curve tangent to E^c is also tangent to both E^{cs} and E^{cu}, and therefore must lie in the intersection of a $\mathcal{W}^{cs}$ and a $\mathcal{W}^{cu}$ leaf, if the distributions E^{cs} and E^{cu} are uniquely integrable.

The proof of the equivalence in the second row uses another result. Recall that c, s and u are the respective dimensions of E^c, E^s and E^u.

Proposition 2.6. *Let D be a c-dimensional disk tangent to E^c. Then the set $\bigcup_{x\in D}\mathcal{W}^s_{loc}(x)$ is a C^1 disk tangent to E^{cs} of dimension $c+s$, and $\bigcup_{x\in D}\mathcal{W}^u_{loc}(x)$ is a C^1 disk tangent to E^{cu} of dimension $c+u$.*

This result is the natural generalization to the case of higher-dimensional center of Proposition 3.4 in [4], which assumed that $c = \dim E^c = 1$. The proof given there extends easily to the general case. The result can also be obtained from Theorem 6.1 in [13] by considering the immersion of the disjoint union of all of the iterates of the disk into the manifold, as can Proposition 2.4.

We now establish the equivalence in the second row:

Proposition 2.7. *E^{cs} and E^{cu} are plaquewise uniquely integrable if and only if E^c is plaquewise uniquely integrable.*

Proof. Suppose that E^{cs} and E^{cu} are both plaquewise uniquely integrable. Proposition 2.6 implies that any c-dimensional disk D tangent to E^c can be extended to disks tangent to E^{cs} and E^{cu}. Plaquewise unique integrability of E^{cs} and E^{cu} implies that each of these extended disks lies in a single leaf of the corresponding foliation $\mathcal{W}^{cs}$ or $\mathcal{W}^{cu}$. Thus their intersection, which is D, lies in a single $\mathcal{W}^c$ leaf, implying plaquewise unique integrability of E^c.

We now turn to the converse implication. Suppose that E^c is plaquewise uniquely integrable. Then there is a canonical way to construct center-stable and center-unstable plaques. For any small enough $\delta > 0$ the local center leaf of size δ through x, $\mathcal{W}^c(x, \delta)$, is well-defined, and so we can define

$$\mathcal{W}^{cs}(x,\delta) = \bigcup_{y \in \mathcal{W}^c(x,\delta)} \mathcal{W}^s(y,\delta) \quad \text{and} \quad \mathcal{W}^{cu}(x,\delta) = \bigcup_{y \in \mathcal{W}^c(x,\delta)} \mathcal{W}^u(y,\delta).$$

We will show that the discs $\mathcal{W}^{cs}(x, \delta)$ and $\mathcal{W}^{cu}(x, \delta)$ are local plaques of foliations. Define $\mathcal{W}^*_{loc}(x)$ to be $\mathcal{W}^*(x, \Delta)$ for a suitably small $\Delta > 0$. Then there is a δ_0 such that if $d(p, q) < \delta_0$, then both $\mathcal{W}^{cs}_{loc}(p) \cap \mathcal{W}^u_{loc}(q)$ and $\mathcal{W}^{cu}_{loc}(p) \cap \mathcal{W}^s_{loc}(q)$ are single points.

Proposition 2.4 ensures that center-stable and center-unstable plaques are subfoliated by stable and unstable plaques, respectively. They are also subfoliated by center plaques. Indeed if $x' \in \mathcal{W}^{cs}_{loc}(x)$, then $\mathcal{W}^{cs}_{loc}(x) \cap \mathcal{W}^{cu}_{loc}(x')$ is an immersed c-dimensional manifold tangent to E^c, which must contain $\mathcal{W}^c(x', \epsilon)$ for some small enough $\epsilon > 0$, because E^c is plaquewise uniquely integrable. A similar argument applies to $\mathcal{W}^{cu}$.

We now see that the cs-plaques at different points are compatible: if $x' \in \mathcal{W}^{cs}_{loc}(x)$, then $\mathcal{W}^{cs}_{loc}(x)$ contains $\mathcal{W}^{cs}(x', \delta')$ for some small enough $\delta' > 0$. Similarly, if $x'' \in \mathcal{W}^{cu}_{loc}(x)$, then $\mathcal{W}^{cu}_{loc}(x)$ contains $\mathcal{W}^{cu}(x'', \delta'')$ for some small enough $\delta'' > 0$. Furthermore, if δ is small enough, then either $\mathcal{W}^{cs}(x, \delta) \cap \mathcal{W}^{cs}(x', \delta) = \emptyset$ or $\mathcal{W}^{cs}(x', \delta) \subset \mathcal{W}^{cs}_{loc}(x)$, and similarly, either $\mathcal{W}^{cu}(x, \delta) \cap \mathcal{W}^{cu}(x'', \delta) = \emptyset$ or $\mathcal{W}^{cu}(x'', \delta) \subset \mathcal{W}^{cu}_{loc}(x)$.

This last property ensures that if $\delta > 0$ is small enough and $x \neq x'$, then $\mathcal{W}^{cs}(x, \delta) \cap \mathcal{W}^{cs}(x', \delta) = \emptyset$ if $x' \in \mathcal{W}^u_{loc}(x)$, and similarly $\mathcal{W}^{cu}(x, \delta) \cap \mathcal{W}^{cu}(x', \delta) = \emptyset$ if $x' \in \mathcal{W}^s_{loc}(x)$. For otherwise $\mathcal{W}^{cs}_{loc}(x) \cap \mathcal{W}^u_{loc}(x)$ or $\mathcal{W}^{cu}_{loc}(x) \cap \mathcal{W}^s_{loc}(x)$ would contain both x and x'.

On the other hand, $\bigcup_{x' \in \mathcal{W}^u_{loc}(x)} \mathcal{W}^{cs}(x', \delta)$ is a neighborhood of x for any $\delta > 0$. If y is close enough to x, then $\mathcal{W}^{cs}_{loc}(y)$ intersects $\mathcal{W}^u_{loc}(x)$ in a point x' such that $d(x', y) < \delta$. The compatibility of the cs-plaques means that $y \in \mathcal{W}^{cs}(x', \delta)$.

It is now easy to construct a foliation chart for $\mathcal{W}^{cs}$ that maps the set

$$\bigcup_{x' \in \mathcal{W}^u_{loc}(x)} \mathcal{W}^{cs}(x', \delta)$$

to $\mathbf{R}^n$, sending x to the origin, $\mathcal{W}^u_{loc}(x)$ to $\{0\} \times \mathbf{R}^u$, and each of the $\mathcal{W}^{cs}(x', \delta)$ disks into a hyperplane $\mathbf{R}^{c+s} \times \{v\}$. The compatibility of the cs-plaques ensures that the overlaps for these charts preserve the hyperplanes $\mathbf{R}^{c+s} \times \{\cdot\}$, and so we have foliation charts for $\mathcal{W}^{cs}$. Foliation charts for $\mathcal{W}^{cu}$ are constructed analogously.

Finally, we show that E^{cs} and E^{cu} are plaquewise uniquely integrable. We give the proof for E^{cs}. Let D be a disk with dimension $c+s$ tangent to E^{cs} through the point x. Let D' be the intersection of D with $\mathcal{W}^{cu}(x)$. Then D' is a c-dimensional disk tangent to E^c; plaquewise unique integrability of E^c implies that D' is a plaque of $\mathcal{W}^c$. The union of the $\mathcal{W}^s$-plaques through points of D' is a $\mathcal{W}^{cs}$ plaque, which must contain D. Hence, E^{cs} is plaquewise uniquely integrable. □

All the examples we know of dynamically coherent diffeomorphisms are *robustly* dynamically coherent: any C^1 perturbation is again dynamically coherent. We do not know whether dynamical coherence must always be robust. A result in this direction is the following.

Theorem 2.8. *If E^c is C^1 and integrable, then f is robustly dynamically coherent. In particular, for any g sufficiently C^1 close to f, the distributions E^{cs}_g and E^{cu}_g are integrable, with g-invariant integral foliations.*

This theorem follows from results in Chapter 7 of [13], in particular Theorem 7.6. See also Proposition 3.1 in the survey [6].

The hypothesis that E^c be C^1 in Theorem 2.8 can be weakened to a hypothesis of *plaque expansiveness*, a concept defined in [13]. Another condition that implies plaque expansiveness is the condition that the derivative of f restricted to E^c be isometric. This was stated without proof in [13]; a proof appears in [22] (dynamical coherence under this hypothesis was proved in [3]). In many examples (e.g., the time-1 map of an Anosov flow, skew products, affine transformations of homogeneous spaces), the center distribution is both Lipschitz (in fact, smooth) and integrable; Theorem 2.8 implies that such examples are stably dynamically coherent.

If E^c is Lipschitz and integrable, Proposition 2.3 also implies that E^c is uniquely integrable. It is not known whether perturbations of such examples share this stronger form of dynamical coherence. There are many examples of such perturbations where the center distribution is merely Hölder continuous, and the center foliation fails to be absolutely continuous.

3. **The Borel-Smale examples.** There is an example, attributed by Smale to A. Borel, of a partially hyperbolic diffeomorphism for which the center bundle is C^2 but is not closed under the formation of Lie brackets. The example is also described in [14]. It is presented there and in Smale's original paper [24] as a linear Anosov diffeomorphism on a nilmanifold that is not a torus. Wilkinson [25] observed that if one creates a center subbundle for this diffeomorphism by grouping together invariant weak subbundles, one obtains a partially hyperbolic diffeomorphism whose center subbundle is not integrable. This is described in more detail in [18].

We now give a brief description of these examples. Let H be the Heisenberg group, which is the subgroup of $GL(3, \mathbf{R})$ consisting of matrices of the form:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

Then H is a non-abelian simply-connected nilpotent Lie group, whose Lie algebra $\mathfrak{h}$ is generated by the matrices

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which satisfy the relations:

$$[X, Z] = [Y, Z] = 0; \ [X, Y] = Z.$$

We identify $(\xi, \eta, \zeta) \in \mathbf{R}^3$ with $\xi X + \eta Y + \zeta Z \in \mathfrak{h}$. With this identification, the exponential map $\exp : \mathfrak{h} \to H$ is

$$\exp(\xi, \eta, \zeta) = \begin{pmatrix} 1 & \xi & \zeta + \frac{1}{2}\xi\eta \\ 0 & 1 & \eta \\ 0 & 0 & 1 \end{pmatrix}.$$

It is a diffeomorphism.

There are partially hyperbolic automorphisms of compact quotients of H, which are dynamically coherent. Let $A \in SL(2, \mathbf{Z})$ be a hyperbolic matrix, and let $\Gamma_0 = \exp(\mathbf{Z} \times \mathbf{Z} \times \frac{1}{2}\mathbf{Z})$. It is easily checked that Γ_0 is a subgroup of H; it is discrete and cocompact because the exponential map is a diffeomorphism and the lattice $\mathbf{Z} \times \mathbf{Z} \times \frac{1}{2}\mathbf{Z}$ is cocompact in $\mathfrak{h}$. The map

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

is an automorphism of $\mathfrak{h}$ that preserves the lattice $\mathbf{Z} \times \mathbf{Z} \times \frac{1}{2}\mathbf{Z}$. It induces an automorphism f_A of the compact manifold H/Γ_0 that is partially hyperbolic. The center foliation is a nontrivial fibration of H/Γ_0 by circles that fibers over the torus $\mathbf{T}^2$. The projection of f_A onto this torus is the automorphism induced by A. The diffeomorphism f_A is accessible, because the stable and unstable directions lie in the span of (the left-invariant vector fields corresponding to) X and Y, and $[X, Y] = Z$. Note that there are no Anosov automorphisms of H/Γ_0, because the only Anosov diffeomorphisms of three-dimensional manifolds are on tori (and more generally only tori can support Anosov diffeomorphisms in which one of the distributions E^s or E^u is one-dimensional).

Now consider the group $G = H \times H$. Its Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ is generated by $\{X_1, Y_1, Z_1, X_2, Y_2, Z_2\}$ whose nontrivial bracket relations are:

$$[X_1, Y_1] = Z_1; \quad [X_2, Y_2] = Z_2.$$

We identify $(\zeta_1, \eta_1, \xi_1, \xi_2, \eta_2, \zeta_2) \in \mathbf{R}^3 \times \mathbf{R}^3$ with $\xi_1 X_1 + \eta_1 Y_1 + \zeta_1 Z_1 + \xi_2 X_2 + \eta_2 Y_2 + \zeta_2 Z_2 \in \mathfrak{g}$ (the reason for rearranging the terms in the first $\mathbf{R}^3$ factor will become clear in a minute).

As above, consider a hyperbolic matrix $A \in SL(2, \mathbf{Z})$ and let $\lambda > 1$ and $\lambda^{-1} < 1$ be the eigenvalues of A. Then λ and λ^{-1} are units in the ring of algebraic integers. The field $\mathbf{Q}(\lambda)$ is a quadratic extension of $\mathbf{Q}$; its Galois involution σ interchanges λ and λ^{-1}. Let $\widetilde{\Gamma}$ be the irreducible cocompact lattice in $\mathfrak{g}$ consisting of vectors of the form:[1]

$$(\tfrac{1}{2}w, v, u, \sigma(u), \sigma(v), \sigma(\tfrac{1}{2}w)),$$

with $u, v, w \in \mathbf{Z}[\lambda]$, the ring of algebraic integers in $\mathbf{Q}(\lambda)$. It is easily checked that $\Gamma = \exp(\widetilde{\Gamma})$ is a subgroup of G and is discrete and cocompact. In fact G/Γ is a 2-torus bundle over the 4-torus. The fibers are tangent to $Z_1 \oplus Z_2$.

For any real numbers a and b, the linear map

$$B : (\zeta_1, \eta_1, \xi_1, \xi_2, \eta_2, \zeta_2) \mapsto (\lambda^{a+b}\zeta_1, \lambda^b\eta_1, \lambda^a\xi_1, \lambda^{-a}\xi_2, \lambda^{-b}\eta_2, \lambda^{-a-b}\zeta_2)$$

is an automorphism of $\mathfrak{g}$, and therefore induces a homomorphism $F_B : G \to G$ whose derivative at the identity is B. If a and b are integers, the automorphism B preserves

[1] We have slightly modified the lattice given in [24] to ensure that its image under the exponential map is a subgroup.

$\widetilde{\Gamma}$, the homomorphism F_B preserves Γ, and there is an induced diffeomorphism $f_B : G/\Gamma \to G/\Gamma$. The diffeomorphism f_B is partially hyperbolic if one of $a, b, a+b$ is nonzero and Anosov if all three are nonzero. Smale describes in [24] the cases $a = 1, b = 2$ and $a = 1, b = -3$. More general algebraic constructions of Anosov diffeomorphisms along these lines can be found in [15].

We now assume that $a + b > b \geq a > 0$. In this case f_B is Anosov, with the bundle E^c trivial, E^u spanned by the left invariant vector fields X_1, Y_1, Z_1 and E^u spanned by X_2, Y_2, Z_2. But there is a second way to view f_B as partially hyperbolic, described in [25]: E^u is spanned by Z_1, E^s is spanned by Z_2, and E^c is spanned by X_1, X_2, Y_1, and Y_2. Now f_B is not dynamically coherent, since E^c is not closed under formation of Lie brackets; in fact, $[E^c, E^c]$ is spanned by Z_1 and Z_2.

Nicolas Gourmelon has recently shown that it is possible to deform f_B in the case $a > 0$ to obtain a stably ergodic, volume-preserving, partially hyperbolic diffeomorphism that is neither dynamically coherent nor Anosov [10].

It is perhaps worth discussing why it is possible to obtain dynamically incoherent examples on infranil manifolds but not on tori using this approach. In both cases the exponential map of the group ($\mathbf{R}^n$ or a nilpotent Lie group) conjugates a homomorphism of the group and its derivative at the identity. Being a linear map, the derivative is always dynamically coherent. When the group is $\mathbf{R}^n$, the exponential map is an isometry and it carries a dynamically coherent partially hyperbolic splitting for the derivative to a dynamically coherent partially hyperbolic splitting for the homomorphism. For a nilpotent group, the exponential map still carries an invariant bundle or foliation for the derivative to an invariant bundle or foliation for the homomorphism and vice versa. But the group is noncompact and the exponential map is not an isometry, so the image of a partially hyperbolic splitting may not be partially hyperbolic.

3.1. **Hammerlindl's observation.** Hammerlindl has made the simple and beautiful observation that the diffeomorphim f_B considered above can be viewed as partially hyperbolic in a third way [12]. One chooses the bundle E^u to be spanned by Z_1, Y_1 and X_1, the bundle E^c to be spanned by X_2 and Y_2, and the bundle E^s to be spanned by Z_2. Again E^c is not integrable because $[X_2, Y_2] = Z_2$. But now it is easy to arrange for f_B to be center bunched by choosing the integers a and b suitably. In particular, f_B is center bunched if $a = b$.

It is interesting to note that f_B is essentially accessible when viewed with this partially hyperbolic structure, but is not essentially accessible when viewed with the partially hyperbolic structure introduced above. For Hammerlindl's structure, the unstable leaves are the same as when f_B is viewed as an Anosov diffeomorphism with three dimensional stable and unstable bundles. For the other partially hyperbolic structure, the bundle $E^s \oplus E^u$ is spanned by the left invariant vector fields Z_1 and Z_2 and is thus tangent to the 2-tori that are the fibers of the bundle structure of G/Γ.

4. **Bunching and coherence.** Let us examine the center bunching hypothesis for the diffeomorphism f_B considered in the previous section when $a + b > b \geq a > 0$. We have

$$\nu = \hat{\nu} = \lambda^{-a-b}, \qquad \text{and} \qquad \gamma = \hat{\gamma} = \lambda^{-b}.$$

Consequently,

$$\nu = \hat{\nu} = \lambda^{-a-b} \geq \lambda^{-2b} = \gamma\hat{\gamma},$$

with equality holding if and only if $a = b$. Thus f_B is never center bunched. It falls just short of being center bunched in the best possible case, when $a = b$. In fact, as the next theorem shows, a symmetric construction of this type will never produce a center bunched diffeomorphism that is not dynamically coherent.

Let us say that a partially hyperbolic diffeomorphism is *strongly symmetrically* partially hyperbolic if the inequalities (1)–(5) involving $\nu, \gamma, \hat{\nu}, \hat{\gamma}$ in the Introduction can be satisfied with $\hat{\nu} = \nu$, $\hat{\gamma} = \gamma$ and γ and ν constant. This requires $\nu < \gamma < 1$. A strongly symmetrically partially hyperbolic diffeomorphism is *symmetrically center bunched* if these constants satisfy $\nu < \gamma^2$.

Theorem 4.1. *Suppose that $f : M \to M$ is C^2 and strongly symmetrically partially hyperbolic. If f is symmetrically center bunched and the partially hyperbolic splitting is C^2, then f is dynamically coherent.*

Note that the symmetry condition $\hat{\gamma} = \gamma$ and $\hat{\nu} = \nu$ is the only hypothesis of this theorem not satisfied by Hammerlindl's example, so the asymmetry in his construction is crucial.

Remark 4.2. After Theorem 4.1 appeared in the unpublished preprint [7], several versions of it, also unpublished, have appeared under the weaker hypothesis that E^c is Lipschitz: we are aware of such versions of this theorem due to N. Gourmelon, K. Parwani, R. Saghin and A. Hammerlindl. The generalization follows the argument below, but uses a Lie bracket between Lipschitz vector fields, defined as a distribution. See, e.g. [23] for a description of this Lie bracket.

Before proving this theorem, we make some preliminary observations. We assume that the Riemannian metric on M is chosen so that the inequalities (1)–(5) in the Introduction hold. Such a metric will be called adapted. Note that a rescaling of an adapted metric is still adapted. It will be convenient to assume that the metric is scaled so that the geodesic balls of radius 1 are very small neighborhoods of their centers. Distance with respect to the metric will be denoted by d.

By (if necessary) slightly increasing $\nu = \hat{\nu}$ and slightly decreasing $\gamma = \hat{\gamma}$ and further rescaling the metric to make the local leaves smaller, we may assume that our metric is still adapted and we have the following:

- if $q, q' \in \mathcal{W}^s_{loc}(p)$, then $d(f(q), f(q')) \le \nu d(q, q')$;
- if $q, q' \in \mathcal{W}^u_{loc}(p)$, then $d(f^{-1}(q), f^{-1}(q')) \le \nu d(q, q')$;
- if $q, q' \in \mathcal{W}^{cs}_{loc}(p)$, then $d(f(q), f(q')) \le \gamma^{-1} d(q, q')$; and
- if $q, q' \in \mathcal{W}^{cu}_{loc}(p)$, then $d(f^{-1}(q), f^{-1}(q')) \le \gamma^{-1} d(q, q')$.

Proof of Theorem 4.1. We show that E^{cs} is integrable. The proof that E^{cu} is integrable is very similar. If E^{cs} is not integrable, the Frobenius theorem tells us that E^{cs} is not closed under Lie brackets. It follows that there exist a constant $\eta > 0$, a point $p \in M$, and a sequence of C^1 paths $\kappa_j : [0, 1] \to M$ such that

(1) $\kappa_j(0) = p$ and $\kappa_j(1) \in W^u(p)$,
(2) $\dot{\kappa}_j(t) \in (E^{cs})(\kappa_j(t))$, for all $t \in (0, 1)$,
(3) $\delta_j := \text{length}(\kappa_j) \to 0$ as $j \to \infty$, and
(4) $d(\kappa_j(0), \kappa_j(1)) \ge \eta \delta_j^2$ for all j.

For j sufficiently large, we can choose n_j so that

$$d(f^{n_j+1}(\kappa_j(0)), f^{n_j+1}(\kappa_j(1))) \ge 1,$$

and

$$d(f^i(\kappa_j(0)), f^i(\kappa_j(1))) < 1,$$

for $i \leq n_j$. Since $\delta_j \to 0$ as $j \to \infty$, we have $n_j \to \infty$ as $j \to \infty$. Since $\kappa_j(0)$ and $\kappa_j(1)$ are in the same local unstable manifold,

$$\begin{aligned} 1 &\geq d(f^{n_j}(\kappa_j(0)), f^{n_j}(\kappa_j(1))) \\ &\geq \nu^{-n_j} d(\kappa_j(0), \kappa_j(1)) \\ &\geq \nu^{-n_j} \eta \delta_j^2, \end{aligned}$$

so that

$$\delta_j \leq \sqrt{\frac{\nu^{n_j}}{\eta}}.$$

On the other hand, since κ_j is tangent to E^{cs},

$$\text{length}(f^{n_j+1}(\kappa_j)) \leq \gamma^{-n_j-1}\text{length}(\kappa_j) \leq \gamma^{-n_j-1}\delta_j \leq \sqrt{\frac{\gamma^{-2}}{\eta} \cdot \frac{\nu^{n_j}}{\gamma^{2n_j}}}.$$

Since f is strongly symmetrically partially hyperbolic and center bunched, we have $\nu < \gamma^2$. It follows that

$$\text{length}(f^{n_j+1}(\kappa_j)) \to 0 \quad \text{as} \quad j \to \infty.$$

But this contradicts the fact that

$$d(f^{n_j+1}(\kappa_j(0)), f^{n_j+1}(\kappa_j(1))) \geq 1.$$

□

5. **Further questions.** Two natural questions raised by the above discussion are:

Question: Let $f : M \to M$ be symmetrically partially hyperbolic and center bunched. Is f then dynamically coherent?
and:

Question: Let $f : M \to M$ be a partially hyperbolic affine transformation of a compact homogeneous space G/Γ. If f is accessible, then is f also dynamically coherent?

If f fails to be dynamically coherent, we say that f is dynamically incoherent. The algebraic examples discussed here are stably dynamically incoherent. If f is such an example and f_n is a sequence of dynamically coherent partially hyperbolic diffeomorphisms with $f_n \to f$, then it is easy to see that f must have plaques tangent to E^c. But since E^c is nowhere integrable, this is impossible. We thank the referee for pointing this out.

Since we do not know whether dynamical incoherence in general implies nowhere integrability of E^c, we pose the following:

Question: Is dynamical incoherence a C^1-stable property?

If the answer to this question is "yes" then by the results in [9], in a C^1 neighborhood of any non dynamically coherent diffeomorphism, there is a C^1-dense set of stably accessible and stably dynamically incoherent diffeomorphisms.

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E-mail address: burns@math.northwestern.edu
E-mail address: wilkinso@math.northwestern.edu