ANOSOV MAGNETIC FLOWS, CRITICAL VALUES AND TOPOLOGICAL ENTROPY

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ABSTRACT. We study the magnetic flow determined by a smooth Riemannian metric g and a closed 2-form Ω on a closed manifold M. If the lift of Ω to the universal cover $\widetilde{M}$ is exact, we can define a critical value $c(g,\Omega)$ in the sense of Mañé [29] for the lift of the flow to $\widetilde{M}$. We have $c(g,\Omega)<\infty$ if the lift of Ω has a bounded primitive. This critical value can be expressed in terms of an isoperimetric constant defined by (g,Ω), which coincides with Cheeger's isoperimetric constant when M is an oriented surface and Ω is the area form of g. When the magnetic flow of (g,Ω) is Anosov on the unit tangent bundle SM, we show that $1/2>c(g,\Omega)$ and any closed bounded form in $\widetilde{M}$ of degree ≥ 2 has a bounded primitive.

Next we consider the 1-parameter family of magnetic flows on SM associated with the pair $(g,\lambda\Omega)$ for $\lambda\geq 0$, where Ω is such that its lift to $\widetilde{M}$ has a bounded primitive.

We introduce a volume entropy $h_v(\lambda)$ defined as the exponential growth rate of the average volume of certain balls. We show that $h_v(\lambda)\leq h_{top}(\lambda)$, where $h_{top}(\lambda)$ is the topological entropy of the magnetic flow of $(g,\lambda\Omega)$ on SM and that equality holds if the magnetic flow of $(g,\lambda\Omega)$ is Anosov on SM. If $\lambda_1\leq\lambda_2$ and the magnetic flows for $(g,\lambda_1\Omega)$ and $(g,\lambda_2\Omega)$ are both Anosov on SM, then $h_v(\lambda_1)\geq h_v(\lambda_2)$.

We construct an example of a Riemannian metric of negative curvature on a closed oriented surface of higher genus such that if ϕ^λ is the magnetic flow associated to the area form with intensity λ, then there are values of the parameter $0<\lambda_1<\lambda_2$ with the property that ϕ^{λ_1} has conjugate points and ϕ^{λ_2} is Anosov. Variations of this example show that it is also possible to exit and reenter the set of Anosov magnetic flows arbitrarily many times along the one-parameter family. Moreover, we can start with a Riemannian metric with conjugate points and end up with an Anosov magnetic flow for some $\lambda>0$. Finally we have a version of the example (in which Ω is no longer the area form) such that the topological entropy of ϕ^{λ_1} is greater than the topological entropy of the geodesic flow, which in turn is greater than the topological entropy of ϕ^{λ_2}.

1. INTRODUCTION

Let M be a closed n-dimensional manifold endowed with a C^∞ Riemannian metric g, and let $\pi: TM\to M$ be the canonical projection. Let ω_0 be the symplectic form on TM obtained by pulling back the canonical symplectic form of T^*M via the Riemannian metric. Let Ω be a closed 2-form on M and consider the new symplectic

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form ω_1 defined as:

$$\omega_1 \stackrel{\text{def}}{=} \omega_0 + \pi^*\Omega.$$

The 2-form ω_1 is a symplectic form and defines what is called a *twisted symplectic structure* [3].

Let $E : TM \to \mathbb{R}$ be given by

$$E(x,v) = \frac{1}{2} g_x(v,v).$$

The magnetic flow of the pair (g,Ω) is the Hamiltonian flow of E with respect to ω_1. The magnetic flow models the motion of a particle of unit mass and charge under the effect of a magnetic field, whose Lorentz force $Y : TM \to TM$ is the bundle map defined by:

$$\Omega_x(u,v) = g_x(Y_x(u),v),$$

for all $x \in M$ and all u and v in T_xM. In other words, the curve

$$t \mapsto (\gamma(t),\dot\gamma(t)) \in TM$$

is an orbit of the Hamiltonian flow if and only if

$$\frac{D\dot\gamma}{dt} = Y_\gamma(\dot\gamma), \tag{1}$$

where D stands for the covariant derivative of g. The magnetic flow of the pair $(g,0)$ is the geodesic flow of the Riemannian metric g. A curve γ that satisfies (1) will be called a *magnetic geodesic*.

Magnetic flows were first considered by V.I. Arnold in [2] and by D.V. Anosov and Y.G. Sinai in [1]. Recent work on these flows has uncovered several remarkable properties, see [6, 20, 21, 22, 26, 37, 38, 41, 42, 47, 48].

1.1. **Critical values.** Let $\widetilde\Omega$ be the lift of Ω to the universal cover $\widetilde M$ of M. Suppose that $\widetilde\Omega$ is an exact form, i.e., there exists a smooth 1-form Θ such that $\widetilde\Omega = d\Theta$. Let us consider the Lagrangian on $\widetilde M$ given by

$$L(x,v) = \frac{1}{2}|v|_x^2 - \Theta_x(v).$$

It is well known that the extremals of L, i.e., the solutions of the Euler-Lagrange equations of L,

$$\frac{d}{dt}\frac{\partial L}{\partial v}(x,v) = \frac{\partial L}{\partial x}(x,v)$$

coincide with the lift to $\widetilde M$ of the magnetic geodesics. The *energy function*

$$E(x,v) = \frac{\partial L}{\partial v}(x,v)v - L(x,v)$$

is invariant under the Euler-Lagrange flow. It is easily checked that this definition of E coincides with the lift to $\widetilde M$ of the function E defined above. Being able to express the magnetic flow as a Lagrangian flow is an advantage since it allows us to use variational techniques to derive results for magnetic flows.

The magnetic flow shares with the geodesic flow the property that the level sets of the function E are invariant. There is, however, a significant difference. The geodesic flow is the same for all energy levels up to a uniform change of speed. For the magnetic flow, on the other hand, the behaviour of the flow depends in an essential way on the energy. In particular there is a *critical value* of the energy at which there is a decisive change in the behaviour of the flow. This critical value has been extensively studied, notably by A. Fathi [17], R. Mañé [29] and J. Mather [31, 32].

We now give Mañé's definition of the critical value in our context. The action of the Lagrangian L on an absolutely continuous curve $\gamma : [a, b] \to \widetilde{M}$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt.$$

The critical value is

$$c(L) := \inf\{k \in \mathbb{R} :\ A_{L+k}(\gamma) \geq 0 \text{ for any absolutely continuous closed curve } \gamma \\ \text{defined on any closed interval } [a, b]\ \}.$$

Like any Lagrangian flow, the magnetic flow for $T\widetilde{M}$ in the case when $\widetilde{\Omega}$ is exact can be viewed as the Hamiltonian flow defined by the canonical symplectic form on $T^*\widetilde{M}$ and a suitable Hamiltonian function $H : T^*\widetilde{M} \to \mathbb{R}$; in this case

$$H(x, p) = \frac{1}{2}|p + \Theta_x|^2.$$

The Legendre transform $\mathcal{L} : T\widetilde{M} \to T^*\widetilde{M}$ defined by

$$\mathcal{L}(x, v) = \frac{\partial L}{\partial v}(x, v)$$

carries orbits of the Lagrangian flow for L to orbits of the Hamiltonian flow defined by H and the canonical symplectic form.

The critical value can also be defined in Hamiltonian terms. We now introduce the *critical value of the pair* (g, Ω) as the real number:

$$\begin{aligned} c(g, \Omega) &= \inf_{u \in C^\infty(\widetilde{M}, \mathbb{R})} \sup_{x \in \widetilde{M}} H(x, d_x u) \\ &= \inf_{u \in C^\infty(\widetilde{M}, \mathbb{R})} \sup_{x \in \widetilde{M}} \frac{1}{2}|d_x u + \Theta_x|^2. \end{aligned}$$

As u ranges over $C^\infty(\widetilde{M}, \mathbb{R})$ the form $\Theta - du$ ranges over all primitives of $\widetilde{\Omega}$, because any two primitives differ by a closed 1-form which must be exact since $\widetilde{M}$ is simply connected.

We show in Appendix A that $c(L) = c(g, \Omega)$ whenever $\widetilde{\Omega}$ is exact, even if all primitives of $\widetilde{\Omega}$ are unbounded. This generalizes Theorem A in [15], which gives $c(L) = c(g, \Omega)$ when Ω itself is exact. Our proof closely follows the arguments of Fathi and Maderna in [18]. Clearly $c(g, \Omega) \geq 0$ and we prove in Lemma 2.2 that if Ω is non-trivial, then $c(g, \Omega) > 0$.

The Hamiltonian characterization of the critical value can also be expressed in Lagrangian terms. Let us call two Lagrangians L and $\overline{L}$ *equivalent* if there is a function $u \in C^\infty(\widetilde{M}, \mathbb{R})$ such that

$$L(x,v) - \overline{L}(x,v) = d_x u(v)$$

for all $(x,v) \in T\widetilde{M}$. If a Lagrangian is convex and superlinear, so are all Lagrangians equivalent to it. Equivalent Lagrangians have the same action on closed curves and define the same Euler-Lagrange equation and the same energy function on $T\widetilde{M}$. The Hamiltonian definition of $c(L) = c(g,\Omega)$ means that if $k > c(L)$ and we choose $k' \in (c(L), k)$, we can then choose $u \in C^\infty(\widetilde{M}, \mathbb{R})$ such that

$$H(x, d_x u) = \max_{v \in T_x \widetilde{M}} \{d_x u(v) - L(x,v)\} \leq k'$$

for all $x \in \widetilde{M}$, and hence

$$L(x,v) - d_x u(v) + k' \geq 0$$

for all $(x,v) \in T\widetilde{M}$. The Lagrangian $\overline{L}$ defined by $\overline{L}(x,v) = L(x,v) - d_x u(v)$ satisfies

$$\overline{L}(x,v) + k \geq k - k' > 0$$

for all $(x,v) \in T\widetilde{M}$. Thus if $k > c(L)$ we can find a Lagrangian $\overline{L}$ equivalent to L such that $\overline{L} + k$ is positive and uniformly bounded away from 0 throughout $T\widetilde{M}$.

The critical value is closely related to an isoperimetric constant. For a smooth map φ of the standard unit disk $\mathbb{D}^2$ into $\widetilde{M}$, let $\ell(\partial\varphi)$ be the length of $\varphi(\partial\mathbb{D}^2)$ and

$$a(\varphi) := \left| \int_{\mathbb{D}^2} \varphi^*(\widetilde{\Omega}) \right|.$$

We define our isoperimetric constant as

$$\operatorname{iso}(g,\Omega) = \inf_{\varphi} \frac{\ell(\partial\varphi)}{a(\varphi)}.$$

This constant is defined even when $\widetilde{\Omega}$ is not exact, but we show in Proposition 2.1 that it is always zero in that case. If M is an oriented surface and Ω is the area form then $\operatorname{iso}(g,\Omega)$ coincides with Cheeger's isoperimetric constant introduced in [11].

Theorem A. *If $\widetilde{\Omega}$ is exact, then*

$$\sqrt{2c(g,\Omega)} = \frac{1}{\operatorname{iso}(g,\Omega)}.$$

The proof is given in Section 2.

Using Theorem A and A. Katok's methods from [24], we can give a lower bound for the critical value when M is an orientable surface with Euler characteristic $\chi < 0$. Given a Riemmanian metric g, let a_g be the total g-area of M. By the conformal equivalence theorem there exists a unique positive scalar C^∞ function ρ such that the metric $\rho^2 g$ has constant negative curvature and $a_{\rho^2 g} = a_g$. Let ρ_g be the *conformality coefficient* given by

$$\rho_g := \int_M \rho \, d\mu_g,$$

where $d\mu_g$ is the normalized Riemmanian measure. By the Cauchy-Schwartz inequality, $\rho_g \leq 1$ and equality holds if and only if g itself is a metric of constant negative curvature. In Section 2 we show:

Theorem B. *Let M be a closed orientable surface with Euler characteristic $\chi < 0$. For any pair (g, Ω) we have:*

$$c(g, \Omega) \geq \frac{\left(\int_M \Omega\right)^2}{-4\pi\chi \, \rho_g^2 \, a_g}.$$

1.2. **Anosov magnetic flows.** We are especially interested in the case when the magnetic flow on the unit tangent bundle is Anosov. In the theory of Lagrangian systems, an Anosov energy level is a regular level set of the energy on which the Euler-Lagrange flow is Anosov. In our case $S\widetilde{M} = E^{-1}(1/2)$ and $1/2$ is a regular value of the energy function E.

Theorem C. *Suppose that the restriction to the unit tangent bundle of the magnetic flow of the pair (g, Ω) is Anosov. Then*

$$c(g, \Omega) < 1/2.$$

Moreover any closed bounded form of degree ≥ 2 on $\widetilde{M}$ has a bounded primitive.

The first statement in the theorem can be seen as a "twisted version" of Theorem B in [15]. The second statement means that the L^∞-cohomology of $\widetilde{M}$ vanishes in degree ≥ 2. This extends the observation, made by M. Gromov in [23] (see [35, Proposition 7.1] for a proof), that if M admits a metric of negative sectional curvature, then every closed bounded form of degree ≥ 2 on $\widetilde{M}$ has a bounded primitive. We remark that we do not know of any example of a manifold with an Anosov magnetic flow that does not admit a Riemannian metric of negative curvature.

Given a real number λ, we can consider the restriction to the unit tangent bundle of the magnetic flow associated with the pair $(g, \lambda\Omega)$ and we call this flow the *λ-magnetic flow* and denote it by $\phi^\lambda : SM \to SM$. Similarly, a *λ-magnetic geodesic* will be a unit speed magnetic geodesic of the pair $(g, \lambda\Omega)$. The 0-magnetic flow is the geodesic flow.

The structural stability of Anosov flows means that the set of λ for which the λ-magnetic flow is Anosov is open. We call a component of that set an *Anosov interval.* It is obvious that the λ-magnetic flow is Anosov if and only if the $-\lambda$-magnetic flow is Anosov.

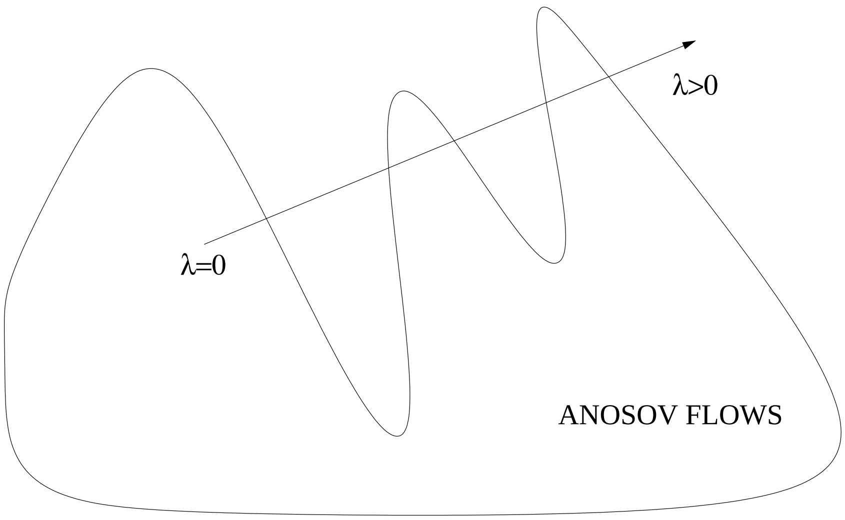

FIGURE 1. Escape from and reentry into the set of Anosov magnetic flows.

It follows from Theorem C that if the λ-magnetic flow is Anosov, then

$$1/2 > c(g, \lambda\Omega) = \lambda^2 c(g, \Omega) \geq \lambda^2 c(g, \Omega).$$

Unless $\Omega \equiv 0$ we have $c(g, \Omega) > 0$ and hence

$$\lambda^2 < \frac{1}{2\, c(g, \Omega)}$$

if the λ-magnetic flow is Anosov. In [41] G. and M. Paternain obtained a different bound on the λ for which the λ-magnetic flow is Anosov. At the end of Section 2 we use Theorem B to show that their bound is not as sharp as the above estimate.

The fact that the λ-magnetic flow must be non Anosov when λ is large enough naturally raises the question:

Question. Is it true that if the λ_0-magnetic flow is Anosov for some $\lambda_0 > 0$, then the λ-magnetic flow is Anosov for all $\lambda \in [0, \lambda_0]$?

We answer this question in the negative (see Figure 1). In Section 7 we construct a simple and explicit example with Anosov geodesic flow and more than one Anosov interval. Our example is a closed oriented surface with negative Gaussian curvature and Ω is the area form. We also exhibit a surface with non Anosov geodesic flow such that the λ-magnetic flow is Anosov for some $\lambda > 0$.

In [20, 21, 48], N. Gouda, S. Grognet and M. Wojtkowski established geometric conditions on the Riemannian metric and the form Ω to ensure that ϕ^λ is Anosov. For closed oriented surfaces with negative curvature and for Ω the area form, these conditions read:

$$K(x) + \lambda^2 < 0,$$

for all $x \in M$, where K is the Gaussian curvature of M. If we define the *magnetic curvature* as $K^{\lambda}_{mag}(x) := K(x) + \lambda^2$, then their result simply says that if the magnetic curvature is negative then ϕ^{λ} is Anosov. Our first example shows that their condition is not optimal since if $K^{\lambda_0}_{mag} < 0$ implies $K^{\lambda}_{mag} < 0$ for all $\lambda \in [0, \lambda_0]$.

Recently, M. Bialy [4] has shown that if we take a Riemannian metric on the n-torus which is conformally flat then for any non-trivial 2-form Ω the magnetic flow ϕ^1 has conjugate points. In Bialy's proof it is essential to assume that M is an n-torus. Our second example shows how different the situation is for surfaces of higher genus. The surface in that example has conjugate points because there is a closed geodesic along which the curvature is positive. But the Anosov λ-magnetic flow does not have conjugate points [36].

1.3. **Topological entropy.** Finally we discuss how topological entropy of the λ-magnetic flow changes with λ. Let $h_{top}(\lambda)$ denote the topological entropy of ϕ^{λ}_t on SM. Given a point $x \in \widetilde{M}$ and $T > 0$ set

$$B_{mag}(x, \lambda, T) = \{y \in \widetilde{M} : \text{ there is a } \lambda\text{-magnetic geodesic} \\ \text{from } x \text{ to } y \text{ with length } < T\}.$$

We call $B_{mag}(x, \lambda, T)$ a *magnetic ball* with center x and radius T. Our next theorem shows that we can define an average volume entropy by considering the exponential growth rate of the average volume of magnetic balls and that this quantity enjoys similar properties to those obtained by A. Manning in [27] and A. Freire and R. Mañé in [19] for geodesic flows. Let $\operatorname{Vol} B_{mag}(x, \lambda, T)$ be the Riemannian volume of $B_{mag}(x, \lambda, T)$; we shall see at the beginning of Section 4 that $x \mapsto \operatorname{Vol} B_{mag}(x, \lambda, T)$ is invariant under covering transformations and hence it defines a function on M, which we still denote by $\operatorname{Vol} B_{mag}(x, \lambda, T)$.

Theorem D. *Let $h_v(\lambda)$ be the exponential growth rate of the average volume of a magnetic ball, i.e.*

$$h_v(\lambda) := \limsup_{T \to \infty} \frac{1}{T} \log \int_M \operatorname{Vol} B_{mag}(x, \lambda, T)\, dx.$$

Then

$$h_{top}(\lambda) \geq h_v(\lambda).$$

If the λ-magnetic flow is Anosov, then

$$h_{top}(\lambda) = h_v(\lambda) = \lim_{T \to \infty} \frac{1}{T} \log \int_M \operatorname{Vol} B_{mag}(x, \lambda, T)\, dx.$$

A particular case of our theorem was obtained by S. Grognet in [22].

Theorem D applies to *any* magnetic flow; it is not even necessary to assume that the lift $\widetilde{\Omega}$ of the magnetic field to the universal cover has a primitive. In the case when $\widetilde{\Omega}$ does have a primitive Θ, the lifts to $\widetilde{M}$ of the λ-magnetic geodesics (which we still call λ-magnetic geodesics) are solutions to the Euler-Lagrange equation for the Lagrangian

$$L_\lambda(x,v) = |v|_x^2/2 - \lambda\Theta_x(v).$$

They are extremals for the action of $L_\lambda + 1/2$. A λ-magnetic geodesic γ between x and y is called *minimizing* if

$$A_{L_\lambda+1/2}(\gamma) \le A_{L_\lambda+1/2}(\overline{\gamma})$$

for any other curve $\overline{\gamma}$ joining x to y ($\overline{\gamma}$ is defined on arbitrary time intervals). We show in Lemma 2.3 that the action of $L_\lambda + 1/2$ along a curve which does not have unit speed will decrease if the curve is reparametrized to have unit speed. Thus a minimizer must have speed one.

A theorem due to Mañé [12, 29] ensures that any two distinct points in $\widetilde{M}$ are joined by a minimizing λ-magnetic geodesic, provided

$$1/2 > c(L_\lambda) = c(g, \lambda\Omega) = \lambda^2\, c(g,\Omega).$$

In this case, i.e. when $0 \le \lambda < 1/\sqrt{2c(g,\Omega)}$, we can define the *minimal ball*

$$B_{min}(x,\lambda,T) = \{y \in \widetilde{M} : \text{ there is a minimizing } \lambda\text{-magnetic geodesic}$$
$$\text{from } x \text{ to } y \text{ with length} < T\}.$$

It is obvious that

$$B_{min}(x,\lambda,T) \subset B_{mag}(x,\lambda,T).$$

In the case when the λ-magnetic flow is Anosov on SM, which entails $0 \le \lambda < 1/\sqrt{2c(g,\Omega)}$ by Theorem C, there is only one λ-magnetic geodesic from a point of $\widetilde{M}$ to another. It must perforce be the minimizing λ-magnetic geodesic guaranteed by Mañé's theorem, and hence

$$B_{min}(x,\lambda,T) = B_{mag}(x,\lambda,T)$$

when the λ-magnetic flow is Anosov on SM.

We show in Section 4 that $\mathrm{Vol}\, B_{min}(x,\lambda,T)$ is a nonincreasing function on the interval $0 \le \lambda < 1/\sqrt{2c(g,\Omega)}$ for any given x and T. It follows from this and Theorem D that $h_{top}(\lambda)$ is nonincreasing on the set of λ such that the λ-magnetic flow is Anosov on SM.

In [41, 42] G. and M. Paternain showed that if we start with an Anosov geodesic flow and $\Omega \not\equiv 0$ then, the function $\lambda \mapsto h_{top}(\lambda)$ is *strictly* decreasing in the Anosov interval containing zero. But their arguments in [42] in fact show more. They prove that if the λ-magnetic flow is Anosov, then $h'_{top}(\lambda) \ne 0$ and for this property 0 does not need to belong to the Anosov interval that contains λ. Combining this result with the above discussion gives

Theorem E. *Suppose that $0 \le \lambda_1 < \lambda_2$ and that the λ_i-magnetic flow is Anosov for $i = 1, 2$. Then*

$$h_{top}(\lambda_1) > h_{top}(\lambda_2).$$

Our earlier examples show that there can be more than one Anosov interval in $[0, 1/\sqrt{2\,c(g,\Omega)})$. This raises the following natural question.

Question. Can the topological entropy go up in between two Anosov intervals?

A further refinement of our examples shows that this is indeed possible.

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2. Proof of Theorems A and B

We first prove the following result that we mentioned in the introduction.

Proposition 2.1. *Suppose that $\widetilde{\Omega}$ is not exact. Then* $\mathrm{iso}(g,\Omega)=0$.

Proof. Since $\widetilde{\Omega}$ is not exact, there exists a cycle $c \in H_2(\widetilde{M},\mathbb{Z})$ such that

$$\int_c \widetilde{\Omega} \neq 0.$$

Since $\widetilde{M}$ is simply connected, the Hurewicz isomorphism theorem ensures that $\pi_2(M)$ is isomorphic to $H_2(\widetilde{M},\mathbb{Z})$. Hence there exists a smooth map $f : S^2 \to \widetilde{M}$ such that

$$\int_{S^2} f^*(\widetilde{\Omega}) \neq 0.$$

Endow S^2 with the canonical metric and fix a point $x \in S^2$. Consider a disk U_ε in S^2 which is given by the complement of an open geodesic disk with center at x and radius ε. As $\varepsilon \to 0$ we have

$$\int_{U_\varepsilon} f^*(\widetilde{\Omega}) \to \int_{S^2} f^*(\widetilde{\Omega}) \neq 0,$$

$$\ell(f|_{\partial U_\varepsilon}) \to 0,$$

and hence $\mathrm{iso}(g,\Omega)=0$.

□

In the rest of this section we assume that $\widetilde{\Omega}$ is exact.

Lemma 2.2. $c(g,\Omega) > 0$ *if Ω is non-trivial.*

Proof. Suppose that $c(g,\Omega)=0$. Since the image of γ is a compact set, there exists, for any $\varepsilon > 0$, a smooth function $u : \widetilde{M} \to \mathbb{R}$ such that $|d_x u + \Theta_x| < \varepsilon$ for all x in the image of γ. Hence

$$\left|\int_\gamma \Theta\right| = \left|\int_\gamma du + \Theta\right| \leq \varepsilon\,\ell,$$

where ℓ is the length of γ. Since this holds for all ε,

$$\int_\gamma \Theta = 0.$$

Since γ was arbitrary, this implies that Θ is exact and thus $\Omega \equiv 0$.

□

2.1. **Proof of Theorem A.** If $c(g,\Omega) = 0$, then Ω is trivial by the previous lemma and we have $\mathrm{iso}(g,\Omega) = \infty$. Thus we can assume that $c(g,\Omega) > 0$.

Suppose $0 < k < c(g,\Omega)$. Then there exists an absolutely continuous closed curve $\alpha : [0,T] \to \widetilde{M}$ such that $A_{L+k}(\alpha) < 0$.

Since $\alpha([0,T])$ is a compact set, by Theorem 3.5.2 in [17] there exists a constant $\delta_0 > 0$ such that if $x, y \in \alpha([0,T])$ and $\delta \in [0,\delta_0]$ satisfy $d(x,y) \leq \delta$, then there exists a magnetic geodesic defined in $[0,\delta]$ that connects x to y and that minimizes the action among all absolutely continuous curves from the fixed interval $[0,\delta]$ into $\widetilde{M}$ that connect x to y. Therefore by dividing the interval $[0,T]$ into sufficiently small subintervals we can find a curve $\gamma : [0,T] \to \widetilde{M}$ which is a piecewise magnetic geodesic and such that

$$A_{L+k}(\gamma) \leq A_{L+k}(\alpha) < 0.$$

Lemma 2.3. *Let $\sigma : [0,\ell] \to \widetilde{M}$ be an absolutely continuous curve parametrized by arc length. The reparametrization of σ that minimizes A_{L+k} has constant speed $\sqrt{2k}$.*

Proof. This is an immediate consequence of Lemma 3-3.2 in [14]. Alternatively, suppose the reparametrization has speed $v(t)$ at $\sigma(t)$. Then the action of $L+k$ along the reparametrization is

$$\int_0^\ell \frac{v(t)^2}{2} + k - \Theta(v(t)\dot{\sigma}(t))\,\frac{dt}{v(t)} = \int_0^\ell \frac{v(t)}{2} + \frac{k}{v(t)}\,dt - \int_\sigma \Theta.$$

Since the last integral is independent of the reparametrization and the function

$$v \mapsto \frac{v}{2} + \frac{k}{v}$$

has a unique minimum at $v = \sqrt{2k}$, the action is minimized when $v(t) \equiv \sqrt{2k}$. □

If necessary, we now reparametrize γ so that it has constant speed $\sqrt{2k}$. Since this reparametrization can only decrease the $L+k$ action, γ now has energy k and negative $L+k$ action. The curve γ is smooth except for a possible finite number of corners. By rounding off these corners if necessary we can obtain a curve, which we still denote by γ, with negative $L+k$ action that defines a smooth map of S^1 into $\widetilde{M}$ and has energy k. It is clear from the proof of the lemma that the new parametrization gives us

$$A_{L+k}(\gamma) = \sqrt{2k}\,\ell - \int_\gamma \Theta,$$

where ℓ is the length of γ. Since $A_{L+k}(\gamma) < 0$, we obtain

$$\sqrt{2k} < \frac{\int_\gamma \Theta}{\ell},$$

and since the left hand side is independent of the parametrization we can assume that γ is a smooth map from the unit circle in the plane into $\widetilde{M}$.

On the other hand, since $\widetilde{M}$ is simply connected γ can be extended to a smooth map $\varphi : \mathbb{D}^2 \to \widetilde{M}$ such that $\varphi|\partial\mathbb{D}^2 = \gamma$. For any such extension φ we have

$$\int_{\mathbb{D}^2} \varphi^*(\widetilde{\Omega}) = \int_\gamma \Theta.$$

We obtain:

$$\sqrt{2k} < \frac{\int_\gamma \Theta}{\ell} = \frac{a(\varphi)}{\ell(\partial\varphi)} \leq \frac{1}{\operatorname{iso}(g,\Omega)}.$$

Since $k < c(g,\Omega)$ was arbitrary, this yields:

$$\sqrt{2\,c(g,\Omega)} \leq \frac{1}{\operatorname{iso}(g,\Omega)}.$$

To prove that this inequality is in fact an equality we argue by contradiction. Suppose that there exists $r > 0$ such that

$$\sqrt{2\,c(g,\Omega)} < r < \frac{1}{\operatorname{iso}(g,\Omega)}.$$

This means that we can find a smooth regular closed curve $\gamma : S^1 \to \widetilde{M}$ with length ℓ such that

$$r < \frac{\int_\gamma \Theta}{\ell},$$

which implies:

$$r\,\ell - \int_\gamma \Theta < 0.$$

If we reparametrize γ to a curve $\tilde{\gamma} : [0,T] \to \widetilde{M}$ so that $\tilde{\gamma}$ has energy $k := r^2/2$ we obtain:

$$2kT - \int_\gamma \Theta < 0,$$

which by the Lagrangian definition of the critical value means that $r^2/2 = k < c(g,\Omega)$. This contradiction completes the proof of the theorem.

□

2.2. **Proof of Theorem B.** Let $g_0 = \rho^2 g$ be the metric of constant negative curvature such that $a_{g_0} = a_g$. Lift g and g_0 to metrics $\widetilde{g}$ and $\widetilde{g_0}$ in $\widetilde{M}$. Let $C(R)$ be a geodesic circle of radius R with respect to the metric $\widetilde{g_0}$ and let $D(R)$ be the disk bounded by $C(R)$. Let Ω_0 be the area form of g_0. Write:

$$\begin{aligned}
\Omega &= f\,\Omega_0, \text{ where } f \text{ is smooth function on } M,\\
\ell_0(R) &:= \widetilde{g_0}\text{-length of } C(R),\\
\ell(R) &:= \widetilde{g}\text{-length of } C(R),\\
a_0(R) &:= \widetilde{g_0}\text{-area of } D(R),\\
a(R) &:= \int_{D(R)} \widetilde{\Omega} = \int_{D(R)} \widetilde{f}\,\widetilde{\Omega}_0.
\end{aligned}$$

In the disk $D(R)$ we introduce coordinates (r, s), where r is the $\widetilde{g_0}$-arc length parameter along radial geodesics and s is the $\widetilde{g_0}$-arc length parameter along concentric circles. Let k be the square root of minus the curvature of g_0. We have:

$$\ell_0(R) = \frac{2\pi}{k}\sinh kR = \frac{da_0}{dR}(R), \tag{2}$$

$$a_0(R) = \frac{2\pi}{k^2}(\cosh(kR) - 1), \tag{3}$$

$$\ell(R) = \int_0^{\ell_0(R)} \widetilde{\rho}^{-1}(R, s)\,ds, \tag{4}$$

$$a(R) = \int_0^R \int_0^{\ell_0(r)} \widetilde{f}(r, s)\,dsdr. \tag{5}$$

The key observation is that the projection to M of a circle in $\widetilde{M}$ converges to a horocycle when the radius goes to infinity, and the projection to the unit sphere bundle of (M, g_0) of the normalized arc length measure weakly converges to an invariant measure for the horocycle flow. But the only invariant measure for the horocycle flow is the Liouville measure. Hence from (4) we obtain:

$$\lim_{R\to+\infty} \frac{\ell(R)}{\ell_0(R)} = \lim_{R\to+\infty} \frac{1}{\ell_0(R)} \int_0^{\ell_0(R)} \widetilde{\rho}^{-1}(R, s)\,ds = \int_M \rho^{-1}\,d\mu_0 = \rho_g. \tag{6}$$

Similarly, we obtain:

$$F := \lim_{R\to+\infty} \frac{1}{\ell_0(R)} \int_0^{\ell_0(R)} \widetilde{f}(R, s)\,ds = \int_M f\,d\mu_0 = \frac{1}{a_g}\int_M \Omega. \tag{7}$$

The last equality implies that given $\varepsilon > 0$ there exists $m > 0$ such that for all $R \geq m$ we have

$$(F + \varepsilon)\ell_0(R) \geq \int_0^{\ell_0(R)} \widetilde{f}(R, s)\,ds \geq (F - \varepsilon)\ell_0(R),$$

and hence from (2) and (5) we obtain:

$$\begin{aligned}\limsup_{R\to+\infty}\frac{a(R)}{\ell_0(R)} &\geq (F-\varepsilon)\limsup_{R\to+\infty}\frac{1}{\ell_0(R)}\int_m^R \ell_0(r)\,dr\\ &= (F-\varepsilon)\limsup_{R\to+\infty}\frac{1}{\ell_0(R)}(a_0(R)-a_0(m)).\end{aligned}$$

Now observe that from (2) and (3) we get:

$$\lim_{R\to+\infty}\frac{a_0(R)}{\ell_0(R)} = 1/k, \tag{8}$$

and hence

$$\limsup_{R\to+\infty}\frac{a(R)}{\ell_0(R)} \geq (F-\varepsilon)/k. \tag{9}$$

It follows from (6), (8) and (9) that:

$$\limsup_{R\to+\infty}\frac{a(R)}{\ell(R)} = \limsup_{R\to+\infty}\frac{a(R)}{\ell_0(R)}\frac{\ell_0(R)}{\ell(R)} \geq \frac{F}{k\,\rho_g}. \tag{10}$$

Combining Theorem A with (10) yields:

$$c(g,\Omega) \geq \frac{F^2}{2k^2\,\rho_g^2} = \frac{a_g\,F^2}{-4\pi\chi\,\rho_g^2}.$$

This inequality and equation (7) complete the proof of Theorem B.

□

We conclude this section with the following remarks. Let us normalize the metric g so that $\rho^2 g$ has curvature -1. Then $-2\pi\chi = a_g$ and hence Theorem B says that:

$$c(g,\Omega) \geq \frac{\left(\int_M\Omega\right)^2}{2\,\rho_g^2\,a_g^2} \geq \frac{\left(\int_M\Omega\right)^2}{2\,a_g^2}.$$

Assume further that the cohomology class of Ω is the same up to sign as that of the area form Ω_g or equivalently that

$$\int_M\Omega = \pm\int_M\Omega_g.$$

Then the last inequality implies that

$$c(g,\Omega)\geq 1/2,$$

and g has constant curvature -1 if equality holds. It follows that the functional

$$g\mapsto c(g,\Omega_g),$$

over the space of metrics g with $a_g = -2\pi\chi$ achieves its minimum $1/2$ if and only if g has constant curvature -1.

In [41, Theorem 1.3] Paternain and Paternain proved that if the magnetic flow of $(g, \lambda\Omega)$ is Anosov, then there is an upper bound for λ^2 in terms of the curvature tensor of g and the form Ω. When M is a surface and $\Omega \equiv \Omega_g$ this bound reads:

$$\lambda^2 < \frac{-\int_M K(x)\,dx}{a_g} = \frac{-2\pi\chi}{A_g},$$

where K is the Gaussian curvature. On the other hand our Theorems B and C give:

$$\lambda^2 < \frac{1}{2\,c(g,\Omega_g)} \leq \frac{-2\pi\chi\,\rho_g^2}{A_g} \leq \frac{-2\pi\chi}{a_g},$$

It follows that the bound in Theorem C is always sharper than Paternain and Paternain's bound unless g has constant negative curvature in which case they coincide.

3. Proof of Theorem C

We recall the following key fact proved by G. Paternain and M. Paternain in [40, 36].

Proposition 3.1. *Suppose that the magnetic flow ϕ of the pair (g, Ω) is Anosov. Then the weak stable foliation $\mathcal{W}^s$ of ϕ is transverse to the fibres of the fibration $\pi : SM \to M$.*

Let $\widetilde{M}$ be the universal covering of M with covering projection $p : \widetilde{M} \to M$. Let $\widetilde{\mathcal{W}^s}$ be the lifted foliation which is in turn a weak stable foliation for the lifted magnetic flow. The next observation appears also in [15, 43].

Lemma 3.2. *For any $(x, v) \in S\widetilde{M}$, $\widetilde{\mathcal{W}^s}(x, v)$ intersects each fibre of the fibration $\widetilde{\pi} : S\widetilde{M} \to \widetilde{M}$ at just one point.*

Proof. By Proposition 3.1 the foliation $\widetilde{\mathcal{W}^s}$ is transverse to the fibration $\widetilde{\pi} : S\widetilde{M} \to \widetilde{M}$. Since the fibres are spheres (which are compact) a result of Ehresman (cf. [10]) implies that for every $(x, v) \in S\widetilde{M}$ the map

$$\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,v)} : \widetilde{\mathcal{W}^s}(x, v) \to \widetilde{M},$$

is a covering map. Since $\widetilde{M}$ is simply connected, $\widetilde{\pi}|_{\widetilde{\mathcal{W}^s}(x,v)}$ is in fact a diffeomorphism and $\widetilde{\mathcal{W}^s}(x, v)$ is simply connected. Consequently, $\widetilde{\mathcal{W}^s}(x, v)$ intersects each fibre of the fibration $\widetilde{\pi} : S\widetilde{M} \to \widetilde{M}$ at just one point.

□

The lemma implies that each leaf $\widetilde{\mathcal{W}^s}(x, v)$ is diffeomorphic to $\widetilde{M}$ and hence $\widetilde{M}$ is diffeomorphic to $\mathbb{R}^n$. This implies that $\widetilde{\Omega}$ is an exact form, so we can write $\widetilde{\Omega} = d\Theta$ for some smooth 1-form Θ. As in the introduction, let us consider the Lagrangian on $\widetilde{M}$ given by

$$L(x, v) = \frac{1}{2}|v|_x^2 - \Theta_x(v).$$

The extremals of L, i.e., the solutions of the Euler-Lagrange equations of L coincide with the lift to $\widetilde{M}$ of the magnetic geodesics. The Hamiltonian associated with L is

$$H(x,p) = \frac{1}{2}|p + \Theta_x|^2.$$

The Legendre transform $\mathcal{L} : T\widetilde{M} \to T^*\widetilde{M}$ takes orbits of L to orbits of the Hamiltonian flow of H with respect to the canonical symplectic form of $T^*\widetilde{M}$.

Lemma 3.3. *There exists $\varepsilon > 0$ such that for any $k \in (1/2-\varepsilon, 1/2+\varepsilon)$, there exists a smooth function $u : \widetilde{M} \to \mathbb{R}$ such that for all $x \in \widetilde{M}$ we have*

$$H(x, d_x u) = k.$$

Proof. The Legendre transform $\mathcal{L} : T\widetilde{M} \to T^*\widetilde{M}$ associated to L carries $S\widetilde{M}$ diffeomorphically onto the level set $\{H = 1/2\}$. Using $\mathcal{L}$ we can push forward the weak stable foliation $\widetilde{\mathcal{W}}^s$ to obtain the weak stable foliation of the Hamiltonian flow of H with respect to the canonical symplectic form of $T^*\widetilde{M}$. By the last lemma, the leaves of this foliation, which we still denote by $\widetilde{\mathcal{W}}^s$, project diffeomorphically onto $\widetilde{M}$; and hence, for any (x,p) with $H(x,p) = 1/2$, $\widetilde{\mathcal{W}}^s(x,p)$ is the graph of a 1-form ω. On the other hand it is well known that the weak stable leaves are Lagrangian submanifolds and hence ω must be closed. Since any closed 1-form in the universal covering must be exact, it follows that each leaf $\widetilde{\mathcal{W}}^s(x,p)$ is the graph of an exact 1-form. This means that there exists a smooth function $u : \widetilde{M} \to \mathbb{R}$ such that $H(x, d_x u) = 1/2$. Finally by structural stability, if the magnetic flow is Anosov, there exists $\varepsilon > 0$ such that the restriction of the Hamiltonian flow of E with respect to ω_1 to the energy level $E^{-1}(k)$ is Anosov for any $k \in (1/2-\varepsilon, 1/2+\varepsilon)$. The previous argument can be reproduced to obtain a smooth function $u : \widetilde{M} \to \mathbb{R}$ such that $H(x, d_x u) = k$ for all $x \in \widetilde{M}$ and all $k \in (1/2-\varepsilon, 1/2+\varepsilon)$ as desired. □

By Lemma 3.3 there exist $\varepsilon > 0$ and a smooth function $u : \widetilde{M} \to \mathbb{R}$ such that $d_x u$ is in the level set $H = 1/2 - \varepsilon/2$ for all $x \in \widetilde{M}$. That is, for all $x \in \widetilde{M}$,

$$|d_x u + \Theta_x|^2 = 1 - \varepsilon.$$

Recall that

$$\begin{aligned} c(g,\Omega) &= \inf_{u \in C^\infty(\widetilde{M},\mathbb{R})} \sup_{x \in \widetilde{M}} H(x, d_x u) \\ &= \inf_{u \in C^\infty(\widetilde{M},\mathbb{R})} \sup_{x \in \widetilde{M}} \frac{1}{2}|d_x u + \Theta_x|^2. \end{aligned}$$

It follows that

$$c(g,\Omega) \leq 1/2 - \varepsilon/2 < 1/2,$$

which proves the first claim in Theorem C.

Given $(x,v) \in \widetilde{M}$, let $E^s(x,v) \subset T_{(x,v)}S\widetilde{M}$ be the weak stable subspace at (x,v) and let $E^{ss}(x,v)$ be the strong stable subspace at (x,v). Let $X(x,v)$ be the vector field of the magnetic flow at (x,v). The subspace $E^s(x,v)$ is the direct sum of $E^{ss}(x,v)$ and the 1-dimensional subspace spanned by $X(x,v)$. Since M is compact and E^s is transverse to the vertical subbundle $V := \ker d\widetilde{\pi}$ (cf. Proposition 3.1), the angle between $E^s(x,v)$ and $V(x,v)$ is uniformly bounded away from zero. Hence there exists a positive constant R_1 such that for all $(x,v) \in S\widetilde{M}$ and all $\xi \in E^s(x,v)$ we have

$$|d_{(x,v)}\widetilde{\pi}(\xi)|_{\widetilde{M}} \leq |\xi|_{SM} \leq R_1\,|d_{(x,v)}\widetilde{\pi}(\xi)|_{\widetilde{M}}, \tag{11}$$

where we consider in $S\widetilde{M}$ the Sasaki metric induced by the metric in $\widetilde{M}$.

Given $w \in T_x\widetilde{M}$ we can write in a unique way $w = z + u$ where $z \in d_{(x,v)}\widetilde{\pi}(E^{ss}(x,v))$ and u is a vector collinear with v. Since the angle between $E^{ss}(x,v)$ and the 1-dimensional subspace spanned by $X(x,v)$ is uniformly bounded away from zero there exists another positive constant R_2 such that

$$|z| \leq R_2\,|w|. \tag{12}$$

Now fix a weak stable leaf W in $S\widetilde{M}$. The restriction of $\widetilde{\pi}$ to W gives a diffeomorphism between W and $\widetilde{M}$ by Lemma 3.2. Using W we can define a smooth vector field Z on $\widetilde{M}$ by setting $Z(x) := v = d_{(x,v)}\widetilde{\pi}(X(x,v))$ where v is the unique unit vector such that $(x,v) \in W$. Let τ_s be the flow of Z. Clearly

$$\tau_s(x) = \widetilde{\pi} \circ \phi_s(x,v).$$

Lemma 3.4. *There exist positive constants C and κ such that for all $x \in \widetilde{M}$ and all $z \in d_{(x,v)}\widetilde{\pi}(E^{ss}(x,v))$ we have*

$$|d_x\tau_s(z)| \leq C\,e^{-\kappa s}|z|,$$

for all $s \geq 0$.

Proof. Since the magnetic flow is Anosov, there exist positive constants C_1 and κ such that for all $(x,v) \in S\widetilde{M}$ and all $\xi \in E^{ss}(x,v)$ we have

$$|d_{(x,v)}\phi_s(\xi)|_{S\widetilde{M}} \leq C_1\,e^{-\kappa s}|\xi|_{S\widetilde{M}},$$

for all $s \geq 0$. Combining this inequality with (11) and the definition of τ_s we obtain

$$|d_x\tau_s(z)|_{\widetilde{M}} \leq C_1\,R_1\,e^{-\kappa s}|z|_M,$$

for all $s \geq 0$. Set $C := C_1\,R_1$.

□

Let us complete the proof of Theorem C. We follow Pansu in Proposition 7.1 of [35]. Let τ_s be the flow of the vector field Z that we introduced above.

Let α be a closed bounded k-form on $\widetilde{M}$ with $k \geq 2$. To solve $d\beta = \alpha$ we use Poincaré's formula

$$\beta_T = -\int_0^T \tau_s^*(i_Z\alpha)\,ds.$$

Since α is closed, Cartan's formula $L_Z = i_Z d + d i_Z$ and the fact that d is linear and commutes with pullbacks give us

$$\alpha - \tau_T^*\alpha = -\int_0^T \frac{d}{ds}\tau_s^*\alpha\, ds = -\int_0^T \tau_s^* L_Z\alpha\, ds = -\int_0^T \tau_s^* d i_Z\alpha\, ds = d\beta_T.$$

Take $x \in \widetilde{M}$ and $w_1, \dots, w_{k-1} \in T_x\widetilde{M}$. Write

$$w_i = z_i + u_i,$$

where $z_i \in d_{(x,v)}\widetilde{\pi}(E^{ss}(x,v))$ and u_i is collinear with $v := Z(x)$ for $i = 1, \dots, k-1$. Then

$$\begin{aligned}\left[\tau_s^*(i_Z\alpha)\right]_x(w_1, \dots, w_{k-1}) &= \alpha_{\tau_s(x)}(d_x\tau_s(w_1), \dots, d_x\tau_s(w_{k-1}), d_x\tau_s(Z(x)) \\ &= \alpha_{\tau_s(x)}(d_x\tau_s(z_1), \dots, d_x\tau_s(z_{k-1}), Z(\tau_s(x))).\end{aligned}$$

Hence using Lemma 3.4 and (12) we obtain

$$\begin{aligned}|\left[\tau_s^*(i_Z\alpha)\right]_x(w_1, \dots, w_{k-1})| &\le \|\alpha\|_\infty\, |d_x\tau_s(z_1) \wedge \cdots \wedge d_x\tau_s(z_{k-1})| \\ &\le \|\alpha\|_\infty\, C^{k-1}\, e^{-\kappa(k-1)s}|z_1 \wedge \cdots \wedge z_{k-1}| \\ &\le \|\alpha\|_\infty\, C^{k-1}\, R_2^{k-1}\, e^{-\kappa(k-1)s}|w_1 \wedge \cdots \wedge w_{k-1}|.\end{aligned}$$

Hence, the form β_T converges as $T \to +\infty$ to a form β such that

$$|\beta_x(w_1, \dots, w_{k-1})| \le \frac{\|\alpha\|_\infty\, C^{k-1}\, R_2^{k-1}}{(k-1)\kappa}|w_1 \wedge \cdots \wedge w_{k-1}|,$$

and hence β is a bounded form. Also $\tau_T^*\alpha$ tends to zero so $d\beta = \alpha$.

□

4. Monotonicity of the volume of minimal balls

We suppose in this section that $\widetilde{\Omega}$ has a bounded primitive Θ. We begin by showing that $x \mapsto \operatorname{Vol} B_{min}(x, \lambda, T)$ is invariant under covering transformations. Let $\varphi : \widetilde{M} \to \widetilde{M}$ be a covering transformation. Since $\varphi^*\Theta - \Theta$ is closed and $\widetilde{M}$ is simply connected, there exists a smooth function f such that $\varphi^*\Theta - \Theta = df$. Hence

$$L_\lambda \circ d\varphi = L_\lambda + df,$$

and therefore if $\gamma : [0, T] \to \widetilde{M}$ is a curve connecting x and y we have

$$A_{L_\lambda + 1/2}(\varphi\,\gamma) = A_{L_\lambda + 1/2}(\gamma) + f(y) - f(x).$$

Hence φ takes minimizing λ-magnetic geodesics to minimizing λ-magnetic geodesics and λ-magnetic geodesics to λ-magnetic geodesics. It follows immediately that we have $\varphi\, B_{min}(x, \lambda, T) = B_{min}(\varphi(x), \lambda, T)$ and $\varphi\, B_{mag}(x, \lambda, T) = B_{mag}(\varphi(x), \lambda, T)$.

Lemma 4.1. *Suppose $0 \le \lambda_1 \le \lambda_2 < \sqrt{2c(g, \Omega)}$. Let $T(\lambda_i)$ be the length of a minimizing λ_i-magnetic geodesic γ_i from x to y for $i = 1, 2$. Then*

$$T(\lambda_2) \ge T(\lambda_1).$$

Proof. Using the minimization property we get

$$A_{L_{\lambda_1}+1/2}(\gamma_1) \leq A_{L_{\lambda_1}+1/2}(\gamma_2),$$
$$A_{L_{\lambda_2}+1/2}(\gamma_2) \leq A_{L_{\lambda_2}+1/2}(\gamma_1).$$

These are equivalent to

$$T(\lambda_1) - \lambda_1 \int_{\gamma_1} \Theta \leq T(\lambda_2) - \lambda_1 \int_{\gamma_2} \Theta,$$
$$T(\lambda_2) - \lambda_2 \int_{\gamma_2} \Theta \leq T(\lambda_1) - \lambda_2 \int_{\gamma_1} \Theta.$$

Hence

$$(\lambda_2 - \lambda_1)\left(\int_{\gamma_1} \Theta - \int_{\gamma_2} \Theta\right) \leq 0.$$

It follows that

$$T(\lambda_1) \leq T(\lambda_2) + \lambda_1 \left(\int_{\gamma_1} \Theta - \int_{\gamma_2} \Theta\right) \leq T(\lambda_2).$$

□

Proposition 4.2. *If* $0 \leq \lambda_1 \leq \lambda_2 < 1/\sqrt{2c(g,\Omega)}$, *then*

$$B_{min}(x, \lambda_2, T) \subseteq B_{min}(x, \lambda_1, T).$$

Proof. Take $y \in B_{min}(x, \lambda_2, T)$. By definition, there exists a minimizing λ_2-magnetic geodesic from x to y with length $T(\lambda_2) \leq T$. A theorem due to Mañé [29, 12] ensures that there exists a minimizing λ_1-magnetic geodesic from x to y with length $T(\lambda_1)$. By Lemma 4.1 $T(\lambda_1) \leq T(\lambda_2) \leq T$, and hence $y \in B_{min}(x, \lambda_1, T)$.

□

Remark 4.3. It might seem more natural to consider the balls

$$\{y \in \widetilde{M} : \Phi_{1/2}(x, y) < T\},$$

where $\Phi_{1/2}$ is the action potential for the Lagrangian L_λ as defined in Appendix A. But it is not clear how to relate the growth rate of volume of these balls to the entropy of the magnetic flow because there is no simple relationship beween time and the action potential. Peyerimhoff and Siburg [44] have shown that the ratio between the metric $d^\lambda_{mag}(x, y) = \Phi_{1/2}(x, y) + \Phi_{1/2}(y, x)$, which is defined by the action potential when $\lambda < 1/\sqrt{2c(g,\Omega)}$, and the Riemannian distance approaches 1 as the distance from x to y approaches 0. An immediate consequence is that d^λ_{mag} and the Riemannian metric define the same inner metric.

Since it is the action rather than the length that is minimized along a minimizing magnetic geodesic, it is possible that $B_{min}(x, \lambda, T)$ is strictly contained in $B_{mag}(x, \lambda, T)$. Note that we have Proposition 4.2 only for minimal balls, because Lemma 4.1 applies to *minimizing* magnetic geodesics.

In the case of the geodesic flow Manning [27] showed that

$$\lim_{T\to\infty} \frac{1}{T} \log \mathrm{Vol} B(x, 0, T) = h_v(0)$$

for all $x \in \widetilde{M}$. The proof is based on the fact that if d is the distance between x and y, then

$$B(x,0,T) \subset B(y,0,T+d),$$

which in turn comes from the triangle inequality. Since the triangle inequality applies only to geodesic triangles, it is not clear that Manning's result extends to the case when $\lambda \neq 0$.

5. Proof of Theorem D

To simplify the notation we omit the dependence on λ from the notation.

We use the following theorem, which is a variant of a theorem due to C. Niche [34]. A proof is sketched in Appendix B for the sake of completeness. The theorem generalizes Mañé's formula for geodesic flows [30]. Given $\theta = (x,v) \in SM$, let $X(\theta)$ be the vector field of the magnetic flow of the pair (g,Ω) and let

$$\alpha(\theta) = \ker d_\theta\pi \oplus \mathbb{R}X(\theta),$$

where $\pi : SM \to M$ is the canonical projection and d_θ denotes the derivative at θ. Let h_{top} be the topological entropy of the magnetic flow.

Theorem 5.1. *Let (g,Ω) be a C^∞ pair. Then we always have:*

$$h_{top} \geq \limsup_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det\, d_\theta(\pi\circ\phi_t)|_{\alpha(\theta)}\right| \, d\theta.$$

Suppose in addition that the magnetic flow ϕ admits a continuous invariant distribution of codimension one transversal to X. Then

$$h_{top} = \lim_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det\, d_\theta(\pi\circ\phi_t)|_{\alpha(\theta)}\right| \, d\theta.$$

We remark that in this theorem it is not important which metrics we choose in M and in SM to measure the absolute value of the determinant. Also note that when the magnetic flow is Anosov, the sum of the strong stable and strong unstable bundles provides a continuous invariant distribution of codimension one transversal to X.

Given $x \in \widetilde{M}$, we define the exponential map

$$\exp_x : T_x\widetilde{M} \to \widetilde{M}$$

of the magnetic flow as follows. If $v \in T_x\widetilde{M} - \{0\}$, then $\exp_x(v) = \widetilde{\pi}(\phi_t(x,u))$ where $t = |v|$ and $u = v/|v|$, and $\exp_x(0) = x$. This map is smooth [13].

Proposition 5.2. *If the magnetic flow is Anosov on SM, then the exponential map $\exp_x$ is a diffeomorphism for all $x \in \widetilde{M}$.*

Proof. As we already mentioned in the introduction, the results in [40, 36] ensure that there are no conjugate points. Since $S\widetilde{M}$ is a regular level set for the energy, Theorem F of [13] tells us that $\exp_x$ is a diffeomorphism provided

$$\inf_{(x,v)\in S\widetilde{M}} \Theta_{can}(d\mathcal{L}(X(x,v))) > 0,$$

where Θ_{can} is the canonical 1-form on $T^*\widetilde{M}$ and X is the vector field on $T\widetilde{M}$ that generates the magnetic flow, which is Euler-Lagrange flow for the Lagrangian

$$L(x,v) = \frac{1}{2}|v|_x^2 - \Theta_x(v).$$

Since the projection of $X(x,v)$ to $\widetilde{M}$ is v, we see that

$$\begin{aligned}
\Theta_{can}(d\mathcal{L}(X(x,v))) &= \mathcal{L}(x,v)(d\pi_{T^*\widetilde{M}}(d\mathcal{L}(X(x,v)))) \\
&= \frac{\partial L}{\partial v}(x,v)v \\
&= L(x,v) + E(x,v) \\
&= L(x,v) + 1/2
\end{aligned}$$

on $S\widetilde{M} = E^{-1}(1/2)$. Since the magnetic flow is Anosov on SM, it follows from Theorem C that $1/2 > c(g,\Omega)$. As we explained in the introduction, there must be a smooth function $u : T\widetilde{M} \to \mathbb{R}$ such that the Lagrangian $\overline{L}(x,v) = L(x,v) - d_x u(v)$ satisfies

$$\inf_{(x,v)\in T\widetilde{M}} \overline{L}(x,v) + 1/2 > 0.$$

The Lagrangians L and $\overline{L}$ have the same Euler-Lagrange equation, the same energy function and the same minimal trajectories, so we may replace L by $\overline{L}$.

□

We now show:

Lemma 5.3. *For a suitably chosen Riemannian metric on SM and for all $T > 0$ we have*

$$\int_M \operatorname{Vol} B_{mag}(x,T)\, dx \leq \int_0^T dt \int_{SM} \left|\det\, d_\theta(\pi \circ \phi_t)|_{\alpha(\theta)}\right|\, d\theta$$

and equality holds if the magnetic flow is Anosov.

Proof. Take $x \in \widetilde{M}$. Let $B(0,T)$ be the ball of radius T in $T_x\widetilde{M}$. By the definition of $B_{mag}(x,T)$ it is clear that

$$B_{mag}(x,T) = \widetilde{\pi}\left(\phi_{[0,T)}(S_x)\right) = \exp_x(B(0,T)),$$

where S_x is the unit sphere in $T_x\widetilde{M}$. Endow SM with a Riemannian metric g_0 defined as follows:

1. on the subspace $S(\theta)$ given by those $\xi \in T_\theta SM$ for which $\langle d_\theta\pi(\xi), v\rangle = 0$, we let g_0 coincide with the Sasaki metric of SM;
2. $S(\theta)$ is orthogonal to $X(\theta)$ for all θ;
3. $X(\theta)$ has norm one.

Let $\psi : [0,T) \times S_x \to \widetilde{M}$ be the map

$$\psi(t,\theta) = \widetilde{\pi}(\phi_t(\theta)).$$

We endow $[0,T) \times S_x$ with the product of the canonical metrics on its factors. Since $\widetilde{\pi}\left(\phi_{[0,T)}(S_x)\right) = \psi([0,T) \times S_x)$ we have

$$\operatorname{Vol} \widetilde{\pi}\left(\phi_{[0,T)}(S_x)\right) \leq \int_0^T dt \int_{S_x} \left|\det d_{(t,\theta)}\psi\right| \, d\theta,$$

with equality if the magnetic flow is Anosov since in that case $\exp_x$ is a diffeomorphism by Proposition 5.2. Now observe that using g on M and g_0 on SM and their lifts $\widetilde{g}$ and $\widetilde{g}_0$ to $\widetilde{M}$ and $S\widetilde{M}$ we have:

$$\left|\det d_{(t,\theta)}\psi\right| = \left|\det d_\theta(\widetilde{\pi} \circ \phi_t)|_{T_\theta S_x \oplus \mathbb{R}X(\theta)}\right|,$$

hence

$$\operatorname{Vol} \widetilde{\pi}\left(\phi_{[0,T)}(S_x)\right) \leq \int_0^T dt \int_{S_x} \left|\det d_\theta(\widetilde{\pi} \circ \phi_t)|_{T_\theta S_x \oplus \mathbb{R}X(\theta)}\right| \, d\theta.$$

Using that the maps $p : (\widetilde{M}, \widetilde{g}) \to (M, g)$ and $dp : (S\widetilde{M}, \widetilde{g}_0) \to (SM, g_0)$ are local isometries and Fubini's theorem the lemma follows.

□

Let us complete the proof of Theorem D. By the previous lemma and Theorem 5.1 we have:

$$\begin{aligned} h_v &= \limsup_{T\to\infty} \frac{1}{T} \log \int_M \operatorname{Vol} B_{mag}(x,T)\, dx \\ &\leq \limsup_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det d_\theta(\pi \circ \phi_t)|_{\alpha(\theta)}\right| \, d\theta \\ &\leq h_{top}. \end{aligned}$$

Suppose now that the magnetic flow is Anosov. Then by the previous lemma and Theorem 5.1 we have:

$$\begin{aligned} h_{top} &= \liminf_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det d_\theta(\pi \circ \phi_t)|_{\alpha(\theta)}\right| \, d\theta \\ &= \liminf_{T\to\infty} \frac{1}{T} \log \int_M \operatorname{Vol} B_{mag}(x,T)\, dx \\ &\leq h_v. \end{aligned}$$

□

We conclude this section with a discussion about the relation of h_v with $\pi_1(M)$. Let (g, Ω) be a pair with $\widetilde{\Omega} = d\Theta$ and Θ a bounded 1-form. Suppose that $1/2 > c(g, \Omega)$.

Proposition 5.4. *Suppose that $1/2 > c(g,\Omega)$. Then h_v is positive if and only if $\pi_1(M)$ has exponential growth.*

Proof. Since $1/2 > c(g,\Omega)$, the discussion in the introduction tells us that given a sufficiently small $\kappa > 0$ there exists a smooth function $u : \widetilde{M} \to \mathbb{R}$ such that

$$L + 1/2 - du > \kappa.$$

Replacing Θ by $\Theta + du$ gives another bounded form and a new Lagrangian with the same orbits as L and the same energy function. The balls $B_{mag}(x,T)$ are the same as before, so we might as well assume that L satisfies:

$$L + 1/2 > \kappa.$$

Let d be the distance function on $\widetilde{M}$ induced by the Riemannian metric and let B_{geo} denote a ball in this metric. Obviously given $y \in B_{mag}(x,T)$ we have $d(x,y) \leq T$ and hence $B_{mag}(x,T) \subseteq B_{geo}(x,T)$.

On the other hand, suppose that $y \in B_{geo}(x,T)$ and let $\gamma : [0,L] \to \widetilde{M}$ be a d-minimizing geodesic connecting x to y with length $L \leq T$. Let $\gamma_m : [0,R] \to \widetilde{M}$ be a minimizing *magnetic* geodesic connecting x to y; such a minimizing geodesic exists by Mañé's theorem since $1/2$ is bigger than the critical value. Hence:

$$\kappa R < A_{L+1/2}(\gamma_m) \leq A_{L+1/2}(\gamma) \leq KL \leq KT,$$

where

$$K := \max_{(x,v)\in S\widetilde{M}} \left(L(x,v) + 1/2\right).$$

It follows that $B_{geo}(x,T) \subseteq B_{min}(x, KT/\kappa) \subseteq B_{mag}(x, KT/\kappa)$. This implies

$$\operatorname{Vol} B_{mag}(x,T) \leq \operatorname{Vol} B_{geo}(x,T) \leq \operatorname{Vol} B_{mag}(x, KT/\kappa),$$

and therefore

$$h_v \leq \limsup_{T\to\infty} \frac{1}{T} \log \int_M \operatorname{Vol} B_{geo}(x,T)\, dx \leq \frac{K}{\kappa} h_v.$$

But

$$\limsup_{T\to\infty} \frac{1}{T} \log \int_M \operatorname{Vol} B_{geo}(x,T)\, dx$$

is the volume entropy of the Riemannian metric and it is well known that this quantity is positive if and only if $\pi_1(M)$ grows exponentially.

□

Corollary 5.5. *Suppose that the 1-parameter family of magnetic flows ϕ^λ exits the set of Anosov magnetic flows at $\lambda = \lambda_c$ and $h_{top}(\lambda_c) = 0$. Then*

$$\lambda_c^2 = \frac{1}{2\, c(g,\Omega)}.$$

Proof. This follows from Theorems C, D, Proposition 5.4 and the fact that Anosov flows have positive topological entropy. □

6. Magnetic flows on surfaces

Let M be an oriented surface endowed with a Riemannian metric g. Given $(x,v) \in TM$, let iv be the unique vector in T_xM such that $\{v, iv\}$ is a positively oriented orthonormal basis of T_xM. The area form Ω_g is given by

$$\Omega_g(u,v) = g(iu,v).$$

Any closed 2-form Ω can be written as $\Omega = f\,\Omega_g$ for some smooth function $f : M \to \mathbb{R}$. The Lorentz force Y associated with Ω is given by

$$Y_x(v) = f(x)\, iv,$$

where $\pi : TM \to M$ is the canonical projection. It follows from equation (1) that $t \mapsto \gamma(t)$ is a λ-magnetic geodesic if and only if:

$$\frac{D\dot{\gamma}}{dt} = \lambda\, f(\gamma)\, i\dot{\gamma}.$$

Note that if $f \equiv 1$, then γ is a λ-magnetic geodesic if and only if γ has constant geodesic curvature λ.

Given $(x, v) \in SM$ and $\xi \in T_{(x,v)}TM$, let

$$J_\xi(t) = d_{(x,v)}(\pi \circ \phi_t^\lambda)(\xi).$$

We call J_ξ a *λ-magnetic Jacobi field* with initial condition ξ. It was shown in [41] that J_ξ satisfies the following Jacobi equation:

$$\ddot{J}_\xi + R(\dot{\gamma}, J_\xi)\dot{\gamma} - \lambda[Y(\dot{J}_\xi) + (\nabla_{J_\xi} Y)(\dot{\gamma})] = 0, \tag{13}$$

where $\gamma(t) = \pi \circ \phi_t^\lambda(x, v)$ and R is the curvature tensor of g with sign convention used by Milnor in [33].

Let us express J_ξ as follows:

$$J_\xi(t) = x(t)\dot{\gamma}(t) + y(t) i\dot{\gamma}(t),$$

and suppose in addition that $\xi \in T_{(x,v)}SM$, which implies

$$g_\gamma(\dot{J}_\xi, \dot{\gamma}) = 0. \tag{14}$$

A straightforward computation using (13) and (14) shows that x and y must satisfy the scalar equations:

$$\dot{x} = \lambda\, f(\gamma)\, y \tag{15}$$

$$\ddot{y} + \left[K(\gamma) - \lambda\, \langle \nabla f(\gamma), i\dot{\gamma}\rangle + \lambda^2\, f^2(\gamma)\right] y = 0. \tag{16}$$

We call the last equation the *scalar Jacobi equation* of γ and we define the *magnetic curvature* as

$$K^\lambda_{mag}(x, v) = K(x) - \lambda\, \langle \nabla f(x), i\, v\rangle + \lambda^2\, f^2(x).$$

We say that equation (16) has an *exponential dichotomy* if there exist constants $C, \mu > 0$, and solutions y^s, y^u of the scalar Jacobi equation (16) along each λ-magnetic geoedesic such that

$$|y^s(t)| \leq C\, e^{-\mu t}, \quad \text{for all } t \geq 0,$$

$$|y^u(t)| \leq C\, e^{\mu t}, \quad \text{for all } t \leq 0.$$

It is easily seen that the magnetic flow is Anosov if such an exponential dichotomy holds. Since f is uniformly bounded, we can associate to each pair of solutions y^s, y^u as above, a pair of solutions x^s, x^u of equation (15) by setting

$$x^s(t) = -\int_t^{\infty} \lambda f(\gamma(\tau)) y^s(\tau)\, d\tau,$$

$$x^u(t) = \int_{-\infty}^{t} \lambda f(\gamma(\tau)) y^u(\tau)\, d\tau.$$

Then $x^s(t)$ and $x^u(-t)$ converge exponentially fast to 0 as $t \to \infty$. Let J^s denote the unique Jacobi field determined by the initial conditions

$$(x^s(0), y^s(0), f(\pi(x,v))y^s(0), \dot{y}^s(0))$$

and let J^u denote the unique Jacobi field determined by the initial conditions

$$(x^u(0), y^u(0), f(\pi(x,v))y^u(0), \dot{y}^u(0)).$$

Then

$$E^{ss}(v) = \mathbb{R}(J^s(0), \dot{J}^s(0)),$$
$$E^{su}(v) = \mathbb{R}(J^u(0), \dot{J}^u(0))$$

are clearly the strong stable and unstable spaces.

The existence of an exponential dichotomy can be verified using the cone method pioneered by Alexeev and Lewowicz and refined by Wojtkowski. In our context it involves studying the magnetic Riccati equation

$$\dot{u}_{mag}(t) + u_{mag}^2(t) + K_{mag}^{\lambda}(\dot{\gamma}(t)) = 0 \tag{17}$$

along a λ-magnetic geodesic $\gamma(t)$.

Lemma 6.1. *Suppose there are constants $T > 0$ and $H > 1$ such that $1/H \leq u_{mag}(T) \leq H$ whenever $u_{mag}(t)$ is a solution of (17) with $u_{mag}(0) \geq 0$. Then equation (16) has an exponential dichotomy and the λ-magnetic geodesic flow is Anosov.*

Proof. Consider the quadratic form $Q(y, \dot{y}) = y\dot{y}$. The hypotheses of the lemma imply that there is an $\eta > 0$ such that $Q(y(T), \dot{y}(T)) > \eta[y(0)^2 + \dot{y}(0)^2]$ whenever $y(t)$ is a solution of the scalar Jacobi equation (16) along a λ-magnetic geodesic $\gamma(t)$ with $Q(y(0), \dot{y}(0)) \geq 0$. Wojtkowski [46] showed that this condition in turn implies that there is an $\eta' > 0$ such that

$$Q(y(T), \dot{y}(T)) - Q(y(0), \dot{y}(0)) > \eta'[y(0)^2 + \dot{y}(0)^2]$$

for *any* solution $y(t)$ of (16) along any λ-magnetic geodesic $\gamma(t)$. It is well known that this last condition implies the existence of an exponential dichotomy. □

The magnetic Riccati equation has the same geometric significance as the Riccati equation for the geodesic flow. Suppose that $(x(t), y(t))$ is the solution of the magnetic Jacobi equations (15) and (16) along a λ-magnetic geodesic γ_0 that is defined by a 1-parameter family $\gamma_s(t)$ of λ-magnetic geodesics. Then $u_{mag}(t) = \dot{y}(t)/y(t)$ is a solution of the magnetic Riccati equation (17) and $u_{mag}(t)$ is the geodesic curvature at $\gamma_0(t)$ of the curve through $\gamma_0(t)$ orthogonal to the family γ_s.

6.1. **Rotationally symmetric surfaces.** Suppose now that $M = \mathbb{R} \times S^1$ is a rotationally symmetric surface and $\Omega = \Omega_g$, i.e. $f \equiv 1$. If (s, φ) are the obvious coordinates on M, then the Riemannian metric of M in these coordinates has the expression:

$$g = ds^2 + r(s)^2 d\varphi^2,$$

where $r : \mathbb{R} \to (0, \infty)$ is a smooth function. Note that r satisfies the scalar Jacobi equation

$$r''(s) + K(s) r(s) = 0, \tag{18}$$

where $K(s)$ is the Gaussian curvature on $\{s\} \times S^1$. The λ-magnetic curvature on $\{s\} \times S^1$ is $K^{\lambda}_{mag}(s) = K(s) + \lambda^2$.

The function $u(s) = r'(s)/r(s)$ satisfies the Riccati equation

$$u'(s) + u^2(s) + K(s) = 0. \tag{19}$$

The geometric significance of $u(s)$ is that the geodesic curvature of the parallel of latitude $\{s\} \times S^1$ is $\pm u(s)$ depending on the direction in which we traverse it.

We orient M so that $\{\partial/\partial s, \partial/\partial\varphi\}$ is a positively oriented basis of M and the area form Ω_g is given by

$$\Omega_g(s, \varphi = r(s)\, ds \wedge d\varphi.$$

We saw earlier that the λ-magnetic geodesics defined by Ω_g have constant geodesic curvature λ. This means that the parallel $\{s\} \times S^1$ can be the trace of a λ-magnetic geodesic only if $\lambda = \pm u(s)$. More precisely we have:

Lemma 6.2. *The parallel $t \mapsto (s, at)$ is a unit speed λ-magnetic geodesic if and only if*

$$|a| = r(s)^{-1} \qquad \textit{and} \qquad u(s) = \pm\lambda,$$

where the positive sign holds if $a > 0$ and the negative sign holds if $a < 0$. If $t \mapsto (s, at)$ is a λ-magnetic geodesic, then $K^{\lambda}_{mag}(s) = -u'(s)$.

Before proving this lemma, we introduce the Clairaut integral for the magnetic flow. Define

$$R(s) := \int_0^s r(\sigma)\, d\sigma.$$

Proposition 6.3 (Clairaut integral)**.** *If $t \mapsto (s(t), \varphi(t))$ is a λ-magnetic geodesic, then*

$$t \mapsto r^2(s(t))\dot{\varphi}(t) - \lambda\, R(s(t))$$

is constant.

The proposition is an easy consequence of the following:

Lemma 6.4. *A curve $t \mapsto (s(t), \varphi(t))$ is a λ-magnetic geodesic if and only if*

$$\ddot{s}(t) = r(s(t))\dot{\varphi}(t)\,[r'(s(t))\dot{\varphi}(t) - \lambda] \quad \textit{and} \quad \frac{d}{dt}\big(r^2(s(t))\dot{\varphi}(t) - \lambda\, R(s(t))\big) = 0,$$

where dot indicates derivative with respect to t and prime indicates derivative with respect to the s-parameter.

Proof. Note that $R \circ s\, d\varphi$ is a primitive of Ω, since $d(R \circ s\, d\varphi) = r \circ s\, ds \wedge d\varphi = \Omega$. This means that the λ-magnetic geodesics are extremals for the Lagrangian

$$L(x, v) = \frac{1}{2}|v|_x^2 - R(s(x))\, d\varphi(v).$$

In terms of our local coordinates,

$$L(s, \varphi, \dot{s}, \dot{\varphi}) = \frac{1}{2}\left(\dot{s}^2 + r(s)^2\dot{\varphi}^2\right) - \lambda\, R(s)\, \dot{\varphi}.$$

The two equations in the lemma are just the Euler-Lagrange equations for L,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = 0 \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0.$$

□

Proof of Lemma 6.2. The parallel has unit speed if and only if $r^2(s)\, a^2 = 1$, which is equivalent to $|a| = r(s)^{-1}$. Using Lemma 6.4, we see that the parallel is a λ-magnetic geodesic if and only if $r'(s_0)\, a = \lambda$. When $\lambda = \pm u(s)$, we have

$$K^{\lambda}_{mag}(s) = K(s) + \lambda^2 = -u'(s) - u^2(s) + u^2(s) = -u'(s)$$

by (19). □

The functions

$$r_{+,\lambda}(s) = r(s) - \lambda R(s) \qquad \text{and} \qquad r_{-,\lambda}(s) = -r(s) - \lambda R(s)$$

govern the behaviour of the λ-magnetic geodesics in much the same way that the function $r(s)$ governs the behaviour of the geodesics. If $r_{-,\lambda}(s) < c < r_{+,\lambda}(s)$, there will be two unit vectors at each point of $\{s\} \times S^1$ tangent to λ-magnetic geodesics along which the Clairaut integral is equal to c; these vectors will make equal angles with the positive direction along the parallel. The following is an easy consequence of Lemma 6.2:

Lemma 6.5. *The parallel $t \mapsto (s, r(s)^{-1}t)$ is a unit speed λ-magnetic geodesic if and only if $r'_{+,\lambda}(s) = 0$. If $r'_{+,\lambda}(s) = 0$, the λ-magnetic curvature along $\{s\} \times S^1$ is $-r''_{+,\lambda}(s)/r(s)$.*

Proof. Since $r'_{+,\lambda}(s) = r'(s) - \lambda r(s) = r(s)[u(s) - \lambda]$, we have $r'_{+,\lambda}(s) = 0$ if and only if $u(s) = \lambda$. Differentiating again gives

$$-\frac{r''_{+,\lambda}(s)}{r(s)} = -\frac{r''(s)}{r(s)} + \lambda\frac{r'(s)}{r(s)} = K(s) + \lambda^2 = K^{\lambda}_{mag}(s)$$

when $\lambda = u(s)$. □

In particular, if $r'_{+,\lambda}(s) = 0$ and $r''_{+,\lambda}(s) > 0$, then $t \mapsto (s, r(s)^{-1}t)$ is a closed λ-magnetic geodesic along which the λ-magnetic curvature is negative; the corresponding periodic orbit of the λ-magnetic flow is hyperbolic. Suppose now that we have $s' < s''$ and c with the following properties, which are indicated in Figure 2:

(1) $r_{+,\lambda}(s') = r_{+,\lambda}(s'') = c$;
(2) $r_{-,\lambda}(s) < c < r_{+,\lambda}(s)$ for $s' < s < s''$;

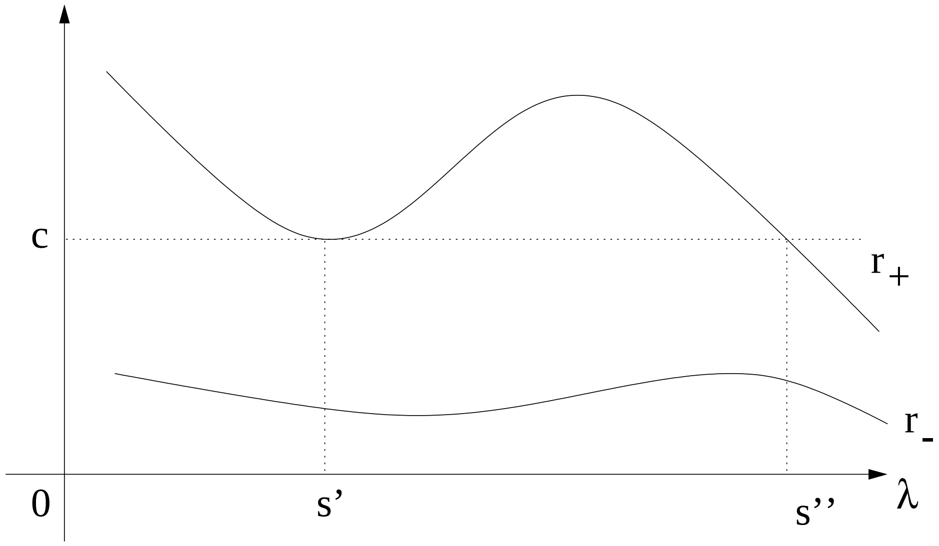

FIGURE 2. Properties of a homoclinic interval.

(3) $r'_{+,\lambda}(s') = 0$ and $r'_{+,\lambda}(s'') < 0$;
(4) $r''_{+,\lambda}(s') > 0$.

There will be two unit vectors at each point of $(s', s'') \times S^1$ that are tangent to λ-magnetic geodesics with Clairaut integral c. These geodesics will be tangent to the parallel $\{s''\} \times S^1$ and both forward and backward asymptotic to the λ-magnetic geodesic $t \mapsto (s', r(s')^{-1}t)$. This situation is analogous to what happens for the geodesic flow. The hyperbolic closed orbit of the λ-magnetic flow corresponding to $t \mapsto (s', r(s')^{-1}t)$ has a homoclinic connection.

We say that $[s', s'']$ is a *homoclinic interval* for the λ-magnetic flow on M if properties 1–4 hold.

7. EXAMPLES WITH ANOSOV INTERVALS THAT DO NOT CONTAIN 0

Let M be an oriented surface and let Ω_g be the area form of the Riemannian metric g. At the level of the differential of the magnetic flow ϕ^λ, increasing the intensity λ of the field amounts to increasing the curvature like term $K^\lambda_{mag} = K + \lambda^2$ in the corresponding Jacobi equation. This effect is obviously monotonic as λ increases and works to make the magnetic flow non Anosov. It can be overcome, however, by a second effect: the geodesic curvature of the λ-magnetic geodesics increases as λ increases and consequently the location of the λ-magnetic geodesics changes.

This possibility can be realized in the following way. Consider a rotationally symmetric surface as shown in Figure 3.

The Gaussian curvature of the surface is everywhere negative. It is less strongly negative in the annulus indicated in the figure and, as the intensity λ increases, K^λ_{mag} becomes positive in this annulus, but stays negative elsewhere. The figure shows how the shape of a λ-magnetic geodesic changes as λ increases and the geodesic curvature of the λ-magnetic geodesics increases. When $\lambda = \lambda_1$, the magnetic curvature in the

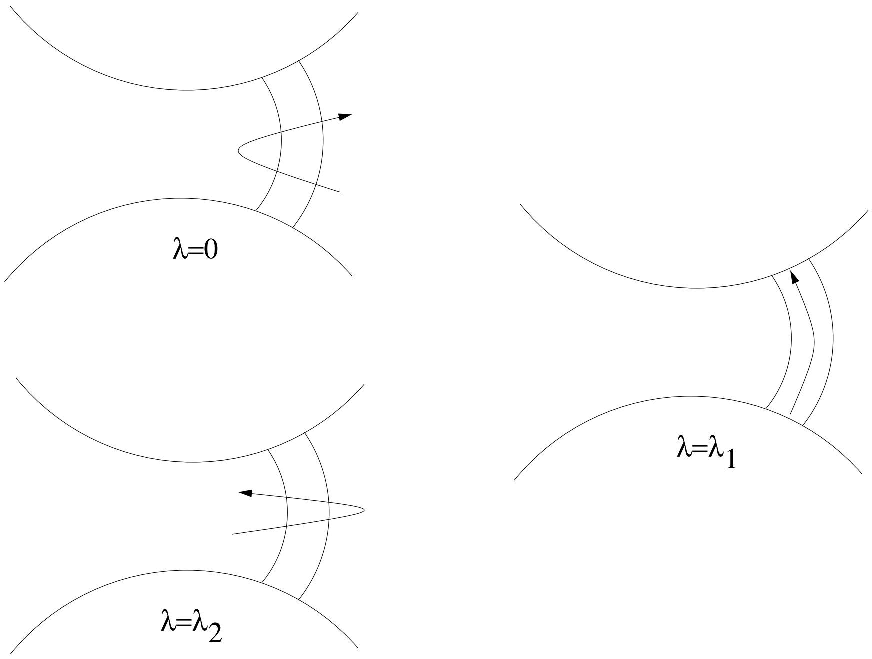

FIGURE 3. Magnetic geodesics move as intensity increases.

annulus has become positive and one of the parallels of latitude in the annulus is a closed λ-magnetic geodesic along which the λ-magnetic curvature is constant and positive. This ensures that the magnetic flow is not Anosov. When $\lambda = \lambda_2$, the magnetic curvature in this annulus has become more positive, but the λ_2-magnetic geodesics have stronger geodesic curvature than the parallels of latitude in the annulus, and so the λ_2-magnetic geodesics only stay in the annulus for a short time. The negative magnetic curvature outside the annulus is then able to overcome the effects of the positive magnetic curvature and the magnetic flow is again Anosov.

It is also possible to start with positive Gaussian curvature everywhere along a closed geodesic and make the magnetic flow become Anosov as the magnetic intensity increases. Of course in both examples the magnetic flow will eventually be non Anosov if the magnetic intensity is increased enough.

We now describe these examples in detail. Given a smooth function $u : \mathbb{R} \to \mathbb{R}$ and a constant $r_0 > 0$, we define the function $r : \mathbb{R} \to \mathbb{R}^+$ to be the solution of $r'(s) = u(s)r(s)$ with $r(0) = r_0$. In other words,

$$r(s) = r_0 \exp\left(\int_0^s u(t)\, dt\right).$$

Let $M = \mathbb{R} \times S^1$ be the rotationally symmetric surface determined by r. As we saw in Section 6.1, M has Gaussian curvature $K(s) = -u'(s) - u^2(s)$ on the parallel $\{s\} \times S^1$ and this parallel has geodesic curvature $\pm u(s)$ depending on the direction in which it is traversed.

We choose $u(s) = \tanh(s)$ except for a C^0 small perturbation in a very short interval $[s_1, s_2] \subset [-1/8, 1/8]$. We require that $|u(s)| < \tanh(1/4)$ and $K(s) = -u'(s) - u^2(s) < 1/4$ when $s_1 \leq s \leq s_2$. These properties ensure that the λ-magnetic geodesic flow is Anosov for $\lambda = 1/2$. In order to prove this, we check that the hypothesis of Lemma 6.1 holds. It suffices to show that there is a $T > 0$ such that every 1/2-magnetic geodesic segment with length T has the property that the magnetic curvature is at most $-3/4$ except for a sufficiently short subset where the magnetic curvature is at most $1/2$.

When $\lambda = 1/2$, we have $K^\lambda_{mag} = K + 1/4$. Hence the λ-magnetic curvature is at most $-3/4$ outside $A = [s_1, s_2] \times S^1$ and is at most $1/2$ in A. Let $B = [-1/4, 1/4] \times S^1$. Since the geodesic curvatures of the parallels in B lie in the interval $[-\tanh(1/4), \tanh(1/4)] \subset [-1/4, 1/4]$ and the λ-magnetic geodesics have geodesic curvature $1/2$ when $\lambda = 1/2$, we see that a 1/2-magnetic geodesic segment in B can be tangent to the s-direction at most once and cannot be asymptotic to the s-direction. A maximal 1/2-magnetic geodesic segment in B must begin and end with a segment that crosses one of the two components of $B \setminus A$ and can contain at most two connected subsegments that lie in A.

Any curve that crosses a component of $B \setminus A$ has length at least $1/8$. We now choose $T = 1/8$. Let t_A be the length of the longest magnetic geodesic segment in A. A 1/2-magnetic geodesic with length $1/8$ must have magnetic curvature at most $-3/4$ everywhere except for a subset with length at most $2t_A$ in which the curvature is at most $1/2$. It follows easily from the next lemma, with $\lambda = 1/2$ and $\delta = 1/4$, that we can make t_A as small as we wish by making $s_2 - s_1$ small enough. Thus $T = 1/8$ has the desired property described above.

Lemma 7.1. *There is a constant $C > 0$ with the following property. Suppose that for some λ and $\delta > 0$ the geodesic curvature of the parallel of latitude $\{s\} \times S^1$ lies in the interval $[-\lambda + \delta, \lambda - \delta]$ for $s_1 \leq s \leq s_2$. Let γ be a connected λ-magnetic geodesic segment that lies in the region $[s_2 - s_1] \times S^1$. Then the length of γ is at most $C\sqrt{(s_2 - s_1)/\delta}$.*

Proof. Let $\psi(t)$ be the angle between $\dot\gamma(t)$ and the unit vector field $U := \partial/\partial s$. It will suffice to show that $|\dot\psi(t)| > \delta$ when $\gamma(t)$ lies in the region $s_1 < s < s_2$. But

$$|\dot\psi(t)| \geq \left| \left|\frac{D\dot\gamma}{dt}\right| - \left|\frac{DU}{dt}\right| \right|,$$

and $|\frac{D\dot\gamma}{dt}| = \lambda$ because λ is the geodesic curvature of γ. Since the vector field U is tangent to the meridian geodesics, $\nabla_U U = 0$ and

$$\frac{DU}{dt} = \nabla_{\sin\psi(t) V(t)} U,$$

where $V(t)$ is a unit vector orthogonal to $U(t)$ and therefore tangent to the parallel through $\gamma(t)$. The latter means that $|\nabla_{V(t)} U|$ is the absolute value of the geodesic curvature of the parallel through $\gamma(t)$. Hence

$$\left|\left|\frac{D\dot\gamma}{dt}\right| - \left|\frac{DU}{dt}\right|\right| \geq \lambda - |\sin\psi(t)|(\lambda-\delta) \geq \delta$$

in the region $[s_1, s_2] \times S^1$.

□

We now choose the interval $[s_1, s_2]$ and the behaviour of the function u in this interval.

For the first example, in which the geodesic flow is Anosov and the magnetic flow becomes non Anosov and switches back to Anosov as the intensity increases, we choose $[s_1, s_2]$ to be a small neighbourhood of $1/16$ and arrange that $u'(1/16) < 0$ but that $-u'(s) < u^2(s)$ for all $s \in [s_1, s_2]$. Since $K(s) = -u'(s) - u^2(s)$ by the Riccati equation (19), we have $K(s) < 0$ for all $s \in [s_1, s_2]$. Outside of $[s_1, s_2]$ we have $u(s) = \tanh(s)$ and (19) gives $K(s) = -1$. When $\lambda = u(1/16)$ the parallel $t \mapsto (1/16, r(1/16)^{-1}t)$ is a magnetic geodesic along which the magnetic curvature is identically

$$K(1/16) + \lambda^2 = -u'(1/16) - u^2(1/16) + \lambda^2 = -u'(1/16) > 0.$$

The existence of a closed magnetic geodesic along which the magnetic curvature is constant and positive means that the $u(1/16)$-magnetic geodesic flow is not Anosov.

For the second example, in which the geodesic flow is non Anosov and the magnetic flow is Anosov for some positive intensity, we choose $[s_1, s_2]$ to be a small neighbourhood of 0 and arrange that $u(0) = 0$ and $u'(0) < 0$. This ensures that $\{0\} \times S^1$ is a closed geodesic and $K(0) = -u'(0) - u^2(0) > 0$, so the geodesic flow is not Anosov.

In both examples the argument given earlier in this section shows that the $1/2$-magnetic flow is Anosov.

It is possible to compactify the examples. We use the same construction as in [7] and [8]. Let us cut off an end of M along a parallel of latitude and then slit the end along a meridian geodesic. We obtain a fan shaped subset of the Poincaré disc. This subset is bounded by two geodesic rays η' and η'' and a curve corresponding to the parallel of latitude. For any large enough n, it is possible to draw a sequence of $4n+1$ hyperbolic geodesic segments

$$d_n', c_n', d_{n-1}', \dots, d_1', c_1', d_0, c_1'', d_1'', \dots, c_n'', d_n''$$

such that

- d_n' and d_n'' are orthogonal to η' and η'' respectively,
- adjacent geodesics in the sequence are orthogonal,
- c_i' and c_i'' have the same length for $1 \leq i \leq n$.

By identifying η' with η'' and c_i' with c_i'', $1 \leq i \leq n$, we obtain a hyperbolic metric on the sphere with $n+2$ punctures. One of the holes is bounded by a copy of the parallel of latitude where we cut off the end and the others by closed geodesics $d_0, \dots, d_n$. Now we can adjoin handles with curvature -1 along $d_0, \dots, d_n$.

It is easy to modify the examples so that there is an arbitrarily long finite sequence $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ such that the λ-magnetic flow is Anosov when $\lambda = \lambda_k$ for even k

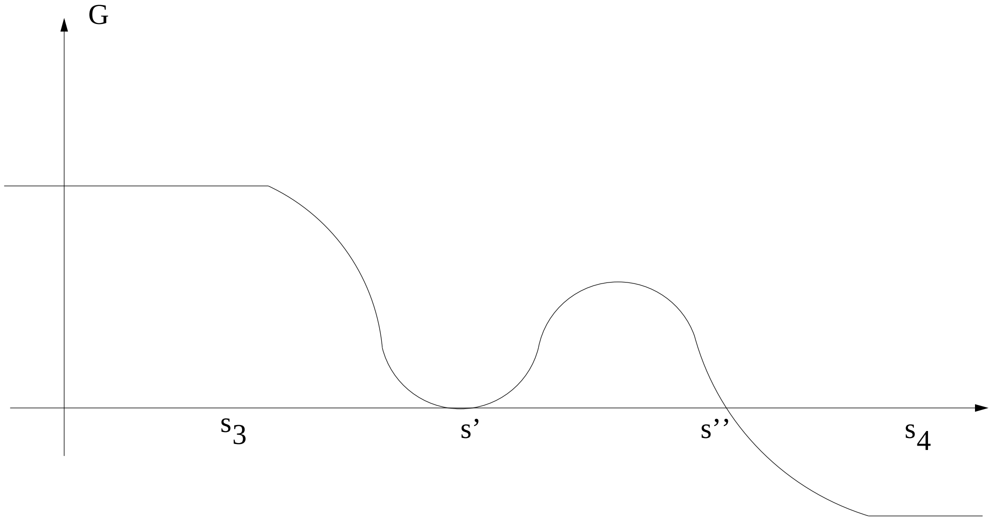

FIGURE 4. Graph of G.

and is non Anosov when $\lambda = \lambda_k$ for odd k. However we do not know how to arrange for an infinite number of changes from Anosov to non Anosov and back again.

8. Increasing topological entropy between Anosov intervals

The compactified version of the first example of the previous section can be modified so that the topological entropy for the $u(1/16)$-magnetic flow, which is not Anosov, is greater than the topological entropy for the geodesic flow.

We again choose $[s_1, s_2]$ to be a small neighbourhood of $1/16$. Now we also choose an even smaller neighbourhood of $1/16$, $[s_3, s_4] \subset (s_1, s_2)$. We want to create a homoclinic interval $[s', s'']$ as defined at the end of Section 6 inside the interval (s_3, s_4). To this end we set $\lambda_1 = \tanh(1/16)$ and consider a function $w(s)$ such that $w(s) = \tanh(s)$ for $s \notin [s_1, s_2]$, $w(s) = \lambda_1$ for $s \in [s_3, s_4]$, and w is monotonic in the intervals $[s_1, s_3]$ and $[s_2, s_4]$. We also ensure that $-w'(s) - w^2(s) < 0$ for all s.

We also choose a function $G(s)$ whose graph has the form shown in Figure 4 and let $g(s) = G'(s)$. The function G is constant and positive on $(-\infty, s_3]$ and is constant and negative on $[s_4, \infty)$. Inside (s_3, s_4) are two points $s' < s''$ where G vanishes; we have $G > 0$ on $(s_3, s') \cup (s', s'')$ and $G < 0$ on (s'', s_4). Also $G''(s') > 0$ and $G'(s'') < 0$. Both $G(s)$ and $g(s)$ are small for all s. In particular, we ensure that $G(s) \leq r_0$ for all s, where r_0 is the radius for the surface at $s = 0$ that was chosen at the beginning of Section 7.

We choose $R(s)$ to be the solution of the differential equation

$$R''(s) - w(s)R'(s) = g(s) \tag{20}$$

with $R(0) = 0$ and $R'(0) = r_0$. We set $r(s) = R'(s)$ and consider the surface of revolution defined by the function $r(s)$. In terms of r, equation (20) becomes

$$r'(s) - w(s)r(s) = g(s). \tag{21}$$

Inside $[s_3, s_4]$ we have $r'(s) - \lambda_1 R'(s) = g(s)$, which gives $r_{+,\lambda_1}(s) := r(s) - \lambda_1 R(s) = G(s) + c$ for some constant c. We see immediately from the graph of G that $[s', s'']$ satisfies all of the properties of a homoclinic interval, except for the condition $r_{-,\lambda_1}(s) < c$ for $s' < s < s''$. But it is clear from (21) that $r(s) \geq r_0$ for $s \geq 0$ and we have $r_{+,\lambda_1}(s) \leq c + r_0$ when $s_3 \leq s \leq s_4$ because $G(s) \leq r_0$. Hence $r_{-,\lambda_1}(s) = r_{+,\lambda_1}(s) - 2r(s) \leq c - r_0$ when $s_3 \leq s \leq s_4$. Thus $[s', s'']$ satisfies all of the conditions to be a homoclinic interval. As explained in Section 6, the orbit of the λ_1-magnetic flow corresponding to the parallel $\gamma_0(t) = (s', r(s')^{-1}t)$ is a hyperbolic closed orbit with a homoclinic connection.

Both the geodesic flow and the 1/2-magnetic flow are Anosov. Recall that the curvature is $K(s) = -u'(s) - u^s(s)$ where $u(s) = r'(s)/r(s)$. Outside $[s_1, s_2]$ equation (21) is $r'(s) - \tanh(s) r(s) = 0$ and hence $K(s) = -1$ for $s \notin [s_1, s_2]$, as in the examples of the previous section. The argument used in the previous section applies to show that the 1/2-magnetic flow is Anosov. The geodesic flow will be Anosov if we have $K(s) < 0$ for $s \in [s_1, s_2]$ also. If we had $g(s) = 0$ for all s, equation (21) would give us $u = w$ and $K(s)$ would be everywhere negative since we chose w so that $-w'(s) - w^2(s) < 0$ for all s. As long as we choose g close enough to 0, we will have $K(s) < 0$ for all $s \in [s_1, s_2]$.

Now we make a small change in the magnetic field in order to break the homoclinic connection for the λ_1-magnetic flow and create a transverse homoclinic point. Let γ_{su} be a λ_1-magnetic geodesic that is both positively and negatively asymptotic to the hyperbolic closed geodesic $\gamma_0(t) = (s', r(s')^{-1}t)$. Parametrize γ_{su} so that it does not have a self intersection at $\gamma_{su}(0)$. Then we can choose a function f on M with the following properties:

(1) $f = 1$ outside a small neighbourhood of $\gamma_{su}(0)$ that does not intersect γ_0;
(2) $f = 1$ at all points on γ_{su};
(3) $\nabla f \neq 0$ at $\gamma_{su}(0)$;
(4) ∇f points to the same side of γ_{su} at all points on γ_{su} where $\nabla f \neq 0$.

The new magnetic field is $f\Omega_g$. Both γ_0 and γ_{su} are λ_1-magnetic geodesics for the new field because $f = 1$ along them. The stable and unstable manifolds for γ_0 will have a *transverse* intersection at $\dot{\gamma}_{su}(0)$. The argument was given by Donnay [16] in the context of geodesic flows and used in [9]. Let u^- and u^+ be the solutions to the magnetic Riccati equation along γ_{su} that give the geodesic curvatures of the curves orthogonal to the λ_1-geodesics that respectively forwards and backwards asymptotic to γ_0. Before the perturbation we have $u^-_{old} \equiv u^+_{old}$. Let t_1 be the time when γ_{su} enters and t_2 the time when γ_{su} leaves the support of the perturbation to the magnetic field. Then $u^-_{new}(t) = u^-_{old}(t)$ for $t \geq t_2$ and $u^+_{new}(t) = u^+_{old}(t)$ for $t \leq t_1$. All that one needs to arrange is that $u^+_{new}(t_2) \neq u^-_{new}(t_2) = u^+_{old}(t_2)$. It is clear from the magnetic Jacobi equation (16) that this will be the case if f is close to 1 and property 4 above holds.

By Smale's theorem, the existence of a transverse homoclinic orbit means that there is a horseshoe in the λ_1-magnetic geodesic flow. Let h be the entropy of this horseshoe.

We now compactify the example in such a way that the geodesic flow has topological entropy less than h and it is still true that the geodesic and 1/2-magnetic flows are Anosov. Let $u(s) = r'(s)/r(s)$ where $r(s)$ is the function chosen above. Then $u(s) = \tanh(s)$ if $|s| \geq 1$. We choose a large constant R. We change $u(s)$ for $|s| > 2R$ in such a way that $K(s) = -u'(s) - u^2(s) < 0$ for all s and $u(s) \equiv h/2$ if $|s| \geq 3R$. We also change the function f in the region where $|s| > R$ in such a way that f is constant on the parallels of latitude and f decreases slowly from 1 to a very small positive value f_0 as $|s|$ increases from R to $2R$ and $f \equiv f_0$ when $|s| \geq 2R$. If we make R large enough, f_0 small enough, and $|\nabla f|$ small enough, both the geodesic flow and the 1/2-magnetic flow will still be Anosov.

We now use the procedure explained in the previous section to compactify the ends $(-\infty, -3R] \times S^1$ and $[3R, \infty) \times S^1$ by adding handles with constant curvature $-h^2/4$ to obtain a compact Riemannian surface with everywhere negative curvature. The geodesic flow of this surface is Anosov. If R is large enough, most of the surface will have curvature $-h^2/4$ and the average value of the square root of minus the curvature will be less than $3h/4$ and, by a result of Manning [28], the topological entropy of the geodesic flow will be less than $3h/4$. It follows from Theorem E in the introduction that the λ-magnetic flow also has topological entropy less than $3h/4$ for all $\lambda \geq 0$ such that the λ-magnetic flow is Anosov. In particular the 1/2-magnetic flow has topological entropy less than $3h/4$.

9. Appendix A

Let (g, Ω) be a pair with $\widetilde{\Omega} = d\Theta$ with Θ a smooth 1-form not necessarily bounded. Recall that

$$L(x, v) = \frac{1}{2}|v|_x^2 - \Theta_x(v),$$

and

$$H(x, p) = \frac{1}{2}|p + \Theta_x|^2.$$

We also recall the two definitions of critical value:

$$c(L) := \inf\{k \in \mathbb{R} : \ A_{L+k}(\gamma) \geq 0 \text{ for any absolutely continuous closed curve } \gamma\}$$

and

$$c(g, \Omega) := \inf_{u \in C^\infty(\widetilde{M}, \mathbb{R})} \max_{x \in \widetilde{M}} H(x, d_x u).$$

Proposition 9.1. $c(L) = c(g, \Omega)$.

Proof. We begin by showing that $c(L) \leq c(g, \Omega)$. This is obvious if $c(g, \Omega) = \infty$. It therefore suffices to show that $c(L) \leq k$ if there exists a smooth function $u : \widetilde{M} \to \mathbb{R}$ such that $H(x, d_x u) \leq k < \infty$ for all $x \in \widetilde{M}$. Observe that

$$H(x, p) = \max_{v \in T_x \widetilde{M}} \{p(v) - L(x, v)\}.$$

Since $H(x, d_x u) \leq k$ for all $x \in \widetilde{M}$ it follows that for all $(x, v) \in T\widetilde{M}$,

$$d_x u(v) - L(x, v) \leq k.$$

Therefore, along any absolutely continuous closed curve $\gamma : [0,T] \to \widetilde{M}$, we have

$$A_{L+k}(\gamma) = \int_0^T (L(\gamma,\dot\gamma)+k)\,dt = \int_0^T (L(\gamma,\dot\gamma)+k-d_\gamma u(\dot\gamma))\,dt \geq 0,$$

and thus $k \geq c(L)$ as desired.

We now turn to proving the reverse inequality $c(L) \geq c(g,\Omega)$. Since this is obvious if $c(L)=\infty$, we may assume that $c(L)$ is finite. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k : \widetilde{M} \times \widetilde{M} \to \mathbb{R}$ by

$$\Phi_k(x,y) = \inf\{A_{L+k}(\gamma) : \gamma \text{ is an absolutely continuous curve } x \text{ to } y\}.$$

It is obvious from its definition that

$$\Phi_k(x_1,x_3) \leq \Phi_k(x_1,x_2) + \Phi_k(x_2,x_3)$$

for all $x_1,x_2,x_3 \in \widetilde{M}$. When $k \geq c(L)$, we have $\Phi_k(x,y) > -\infty$ for all x and y and the function Φ_k is locally Lipschitz. This is proved in the case of a Lagrangian on a closed manifold in [29, 12, 14]. The only difference in our situation is that we cannot claim that Φ_k is uniformly Lipschitz since we are not assuming that Θ is bounded.

Lemma 9.2. *Suppose that $c(L)$ is finite and $k \geq c(L)$. If $u : \widetilde{M} \to \mathbb{R}$ is differentiable at $x \in \widetilde{M}$ and satisfies*

$$u(y) - u(x) \leq \Phi_k(x,y)$$

for all y in a neighbourhood of x, then $H(x,d_x u) \leq k$.

Proof. Let $\gamma(t)$ be a differentiable curve on $\widetilde{M}$ with $(\gamma(0),\dot\gamma(0)) = (x,v)$. Then

$$\limsup_{t\to 0^+} \frac{u(\gamma(t))-u(x)}{t} \leq \limsup_{t\to 0^+} \frac{\Phi_k(\gamma(0),\gamma(t))}{t} = \limsup_{t\to 0^+} \frac{1}{t}\int_0^t \big[L(\gamma,\dot\gamma)+k\big]\,ds.$$

Since the integral on the right is a differentiable function of t, we obtain

$$d_x u(v) \leq L(x,v) + k.$$

Since v was an arbitrary vector in $T_x\widetilde{M}$, it follows that

$$H(x,d_x u) = \max_{v\in T_x\widetilde{M}} \{d_x u(v) - L(x,v)\} \leq k.$$

□

We now complete the proof that $c(L) \geq c(g,\Omega)$. Fix a point $x_0 \in \widetilde{M}$ and define a function $u : \widetilde{M} \to \mathbb{R}$ by $u(x) = \Phi_{c(L)}(x_0,x)$. By the previous lemma, $H(x,d_x u) \leq c(L)$ at every point $x \in \widetilde{M}$ where $u(x)$ is differentiable. Since u is locally Lipschitz, u is differentiable almost everywhere by Rademacher's theorem. We now need to show that for any $k > c(L)$ there is a smooth function $\widehat{u} : \widetilde{M} \to \mathbb{R}$ that approximates u well enough so that $H(x,d_x\widehat{u}) \leq k$ for all $x \in \widetilde{M}$. The necessary approximation argument is presented in detail in Section 6 of [18]. The first step is to choose a locally finite U_i covering of $\widetilde{M}$ by relatively compact open sets that are the domains of charts. Next one uses a convolution argument to create a smooth approximation $\widehat{u}_i$ to u_i on each U_i. The convexity of the Hamiltonian $H(x,p)$ in the second variable and Jensen's inequality allow one to show that for any given $\epsilon_i > 0$ we can choose $\widehat{u}_i$ so that $H(x, d_x\widehat{u}_i) \leq c(L) + \epsilon_i$ for all $x \in U_i$. Then these local approximations are combined using a partition of unity subordinate to the cover to form $\widehat{u}$. The convexity of $H(x,p)$ as a function of p is used again in showing that if the ϵ_i are chosen small enough, then one can obtain $H(x, d_x\widehat{u}) \leq k$ for all $x \in \widetilde{M}$.

□

10. Appendix B

In this appendix we sketch a proof of the following theorem which is a variation of a theorem in [34]. This theorem was used for the proof of Theorem D and generalizes Mañé's formula for geodesic flows [30].

Given $\theta = (x,v) \in SM$, let $X(\theta)$ be the vector field of the magnetic flow ϕ_t of the pair (g,Ω) and let

$$\alpha(\theta) = \ker d_\theta\pi \oplus \mathbb{R}X(\theta),$$

where $\pi : SM \to M$ is the canonical projection. Let h_{top} be the topological entropy of the magnetic flow.

Theorem 10.1. *Let (g,Ω) be a C^∞ pair. Then we always have:*

$$h_{top} \geq \limsup_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det\, d_\theta(\pi\circ\phi_t)|_{\alpha(\theta)}\right|\, d\theta.$$

Suppose in addition that the magnetic flow ϕ admits a continuous invariant distribution of codimension one transversal to X. Then

$$h_{top} = \lim_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det\, d_\theta(\pi\circ\phi_t)|_{\alpha(\theta)}\right|\, d\theta.$$

A crucial ingredient in the proof of Theorem 10.1 is Kozlovski's formula which we now recall. Given a linear map $L : E \to F$ between finite dimensional vector spaces with inner products, we define its *expansion* $\mathrm{ex}(L)$ by

$$\mathrm{ex}(L) = \max_S |\det(L|_S)|,$$

where the maximum is taken over all subspaces $S \subset E$.

Theorem 10.2 ([25]). *Let X be a closed Riemannian manifold and let $\phi_t : X \to X$ be a flow of class C^∞. Then*

$$h_{top} = \lim_{T\to\infty} \frac{1}{T} \log \int_X \mathrm{ex}(d_x\phi_T)\, dx,$$

where h_{top} is the topological entropy of the flow ϕ.

The predecesor of this formula is *Przytycki's inequality* [45]. Kozlovski's proof in [25] for the equality case is based on Yomdin's work [49].

The first inequality in Theorem 10.1 is an immediate consequence of Kozlovski's formula since $|\det\, d_\theta\pi|_E| \leq 1$ for any subspace $E \subset T_\theta SM$ and hence

$$\begin{aligned}\limsup_{T\to\infty}\frac{1}{T}\log\int_0^T dt\int_{SM}\left|\det\, d_\theta(\pi\circ\phi_t)|_{\alpha(\theta)}\right|\,d\theta \\ \leq \limsup_{T\to\infty}\frac{1}{T}\log\int_0^T dt\int_{SM}\left|\det\, d_\theta\phi_t|_{\alpha(\theta)}\right|\,d\theta \\ \leq \max\left\{0,\limsup_{T\to\infty}\frac{1}{T}\log\int_{SM}\left|\det\, d_\theta\phi_T|_{\alpha(\theta)}\right|\,d\theta\right\} \\ \leq h_{top}.\end{aligned}$$

To prove the second statement in the theorem we certainly need more work.

Let X be a closed Riemannian manifold and let $\phi_t : X \to X$ be a flow without singularities of class C^∞ which preserves a volume form dx. Let $\tau : S \to X$ be a symplectic vector bundle over X and let $\phi_t^* : S \to S$ be a *symplectic cocycle over* ϕ. By this we mean:

1. $\tau\circ\phi_t^* = \phi_t\circ\tau$;
2. for all $x \in X$ and $t \in \mathbb{R}$, $\phi_t^*(x) := \phi_t^*|_{S(x)} : S(x) \to S(\phi_t(x))$ is a symplectic linear isomorphism;
3. for all s and t in $\mathbb{R}$ we have $\phi_{t+s}^*(x) = \phi_s^*(\phi_t(x))\circ\phi_t^*(x)$.

Let $\Lambda \to X$ be the Grassmannian bundle of Lagrangian subspaces, i.e., for each $x \in X$, the fibre $\Lambda(x)$ consists of all Lagrangian subspaces of $S(x)$. Suppose that the symplectic vector bundle S is endowed with a smooth Lagrangian distribution α^*. This means that we have a smooth map $x \mapsto \alpha^*(x)$ which is a section of the bundle $\Lambda \to X$.

The following definitions are taken from [5]. Given a Lagrangian subspace $\lambda \in \Lambda$, we identify $T_\lambda\Lambda$ with the space $\mathcal{S}(\lambda)$ of symmetric bilinear forms in the following way. Every curve $\lambda(t)$ in Λ with $\lambda(0) = \lambda$ can be written as $\lambda(t) = \Phi_t\lambda$ where Φ_t is a path of linear symplectic transformations with $\Phi_0 = Id$. Then the symmetric bilinear form corresponding to the vector $\lambda'(0)$ is given by

$$(\xi,\eta) \to \omega\,(\,\xi,\left.\frac{d}{dt}\right|_{t=0}\Phi_t\eta\,)\,,$$

for all $\xi,\eta \in \lambda$. One easily checks that this correspondence is well defined.

Given a pair (ϕ,ϕ^*) where ϕ^* is a symplectic cocycle over ϕ, let us consider the tangent vector

$$l_x \stackrel{\text{def}}{=} \left.\frac{d}{dt}\right|_{t=0}\phi_t^*(\alpha(\phi_{-t}x)) \in T_{\alpha(x)}\Lambda.$$

We shall say that the pair (ϕ,ϕ^*) is α^*-*optical* if the bilinear form in $\mathcal{S}(\alpha(x))$ that corresponds to l_x by the identification described above is positive definite for all $x \in X$.

Now choose an almost complex structure $J : S \to S$ which is compatible with the symplectic structure ω, i.e., for each $x \in X$, $\omega_x(\cdot, J_x\cdot)$ defines a Riemannian metric on the bundle S. We shall measure the expansion of ϕ^* with respect to this metric.

The next proposition is proved exactly as Proposition 4.18 in [39]. Our framework has been set up precisely for the proof to go through.

Proposition 10.3. *Let (ϕ, ϕ^*) be an α^*-optical pair. Then there exists a constant $D > 0$ such that for all t we have*

$$D \int_X \operatorname{ex}(\phi_t^*(x))\, dx \leq \int_X \left|\det\ \phi_t^*(x)|_{\alpha^*(x)}\right|\, dx.$$

Now let $\tau : S \to SM$ be the symplectic vector bundle given by

$$S(\theta) = T_\theta SM/\mathbb{R}X(\theta),$$

with the symolectic structure induced by $\omega_1 = \omega_0 + \pi^*\Omega$. The derivative $d_\theta\phi_t$ factors naturally to the quotient spaces and induces a symplectic cocycle ϕ^* over S. If we take the projection of $\alpha(\theta)$ to $S(\theta)$ we obtain a smooth Lagrangian distribution that we denote by α^* and it is easy to check that the pair (ϕ, ϕ^*) is α^*-optical [5].

We shall also need the following proposition which is proved exactly as Lemma 4.7 in [39].

Proposition 10.4. *There exists a constant $C > 0$ such that for all $T > 0$ we have:*

$$\int_0^{T+1} dt \int_{SM} \left|\det\ d_\theta(\pi \circ \phi_t)|_{\alpha(\theta)}\right|\, d\theta \geq C \int_0^T dt \int_{SM} \left|\det\ d_\theta\phi_t|_{\alpha(\theta)}\right|\, d\theta.$$

We now show:

Lemma 10.5. *Suppose that ϕ admits a continuous invariant distribution of codimension one transversal to X. Then there exist positive constants A and B such that for all $\theta \in SM$ and all $t \in \mathbb{R}$ we have:*

$$\operatorname{ex}(d_\theta\phi_t) \leq A \operatorname{ex}(\phi_t^*(\theta))$$
$$\left|\det\ \phi_t^*(\theta)|_{\alpha^*(\theta)}\right| \leq B \left|\det\ d_\theta\phi_t|_{\alpha(\theta)}\right|.$$

Proof. Let $\theta \mapsto T(\theta)$ be the continuous invariant distribution of codimension one transversal to X. Define a continuous Riemannian metric g_T on SM as follows:

(1) on the subspace $T(\theta)$, we let g_T coincide with the Sasaki metric of SM,
(2) $T(\theta)$ is g_T-orthogonal to $X(\theta)$ for all θ,
(3) $X(\theta)$ has g_T-norm 1.

Let ex^g denote expansion measured with respect to a Riemannian metric g. We clearly have

$$\operatorname{ex}^{g_T}(d_\theta\phi_t) = \operatorname{ex}^{g_T}(d_\theta\phi_t|_{T(\theta)}),$$

because $T(\theta)$ and $X(\theta)$ are $d_\theta\phi_t$-invariant and $T(\theta)$ is the g_T-orthogonal complement of $\mathbb{R}X(\theta)$. Now induce a continuous Riemannian metric $\widetilde{g}_T$ on S in such a way that the restriction of the projection map to $T(\theta)$ is an isometry. Then clearly:

$$\operatorname{ex}^{g_T}(d_\theta\phi_t|_{T(\theta)}) = \operatorname{ex}^{\widetilde{g}_T}(\phi_t^*(\theta)).$$

The first inequality in the lemma follows from the following easy lemma whose proof we omit:

Lemma 10.6. *Let $V \to X$ be a vector bundle over a compact manifold X. Given two continuous Riemannian metrics g_1 and g_2 on V there exists a constant $c > 0$ such that for any two points x and y in X and any linear map $L : V(x) \to V(y)$ we have*

$$\mathrm{ex}^{g_1}(L) \leq c\,\mathrm{ex}^{g_2}(L).$$

The second inequality in Lemma 10.5 is proved in a similar way.

□

Let us complete now the proof of Theorem 10.1. Suppose that ϕ admits a continuous invariant distribution of codimension one transversal to X. Combining Theorem 10.2, Lemma 10.5, Proposition 10.3 and Proposition 10.4 we obtain:

$$\begin{aligned}
h_{top} &= \lim_{T\to\infty} \frac{1}{T} \log \int_{SM} \mathrm{ex}(d_\theta \phi_T)\, d\theta \\
&\leq \liminf_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \mathrm{ex}(d_\theta \phi_t)\, d\theta \\
&\leq \liminf_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \mathrm{ex}(\phi_t^*(\theta))\, d\theta \\
&\leq \liminf_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det\ \phi_t^*(\theta)|_{\alpha^*(\theta)}\right|\, d\theta \\
&\leq \liminf_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det\ d_\theta \phi_t|_{\alpha(\theta)}\right|\, d\theta. \\
&\leq \liminf_{T\to\infty} \frac{1}{T} \log \int_0^T dt \int_{SM} \left|\det\ d_\theta (\pi \circ \phi_t)|_{\alpha(\theta)}\right|\, d\theta
\end{aligned}$$

thus concluding the proof of Theorem 10.1.

□

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