CHEEGER-GROMOLL SPLITTING THEOREM

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Contents

1. Cheeger-Gromoll Splitting Theorem

There is a version of the splitting theorem for Riemannian manifolds that have a Ricci curvature lower bound, i.e. $\text{Ric}_{M^n} \geq -(n-1)\delta$ for $\delta > 0$. For intuition, suppose $\text{Ric}_{M^n} \geq -(n-1)$. If we rescale the metric to $\delta^{-2}g$, then $\text{Ric}_{M^n} \geq -(n-1)\delta^2$. Let $\gamma : [\frac{-L}{2}, \frac{-L}{2}] \to M$ be a geodesic segment of length $L \geq 1$. The distance is rescaled by a factor of δ^{-1}, where $0 < \delta << 1, 1 << \delta^{-1}L$. Now the ball $B_\delta(\gamma(0))$ in the original metric is rescaled to be $B_1(\gamma(0))$. Thus, γ now looks like a line in a noncompact manifold with non-negative Ricci curvature, the setting in which the Cheeger-Gromoll splitting theorem applies. This suggests M^n should split in some sense. The almost splitting theorem says that if the Ricci curvature is "almost nonnegative", and one has a "long enough, minimizing" geodesic, then a ball $B_R(p)$ centered at $p \in M$ is Gromov-Hausdorff close to a ball in a product space $\mathbb{R} \times X$, where X can be taken to be a length space.

For notation, let $\Psi = \Psi(\epsilon_1, \ldots, \epsilon_k | c_1, \ldots, c_k)$ denote a non-negative function such that $\lim_{\epsilon_1,\ldots,\epsilon_k \to 0} \Psi = 0$ for fixed $c_1, \ldots, c_k$. Fix two points $q_\pm \in M$. Define the excess function

$$E(x) = d(x, q_+) + d(x, q_)) - d(q_+, q_-)$$

E is non-negative with Lip $E \leq 2$. The excess function measure how much the segments connecting $q_\pm$ to x fail to be length minimizing. We will work under the following assumptions,

$$\text{Ric} \geq -(n-1)\delta, \tag{1.1}$$

$$d(p, q_\pm) \geq L, \tag{1.2}$$

$$E(p) \leq \epsilon, \tag{1.3}$$

The second and third assumptions together suggest the existence of a "long enough, minimizing" geodesic. We first prove the following useful theorem due to Abresch-Gromoll.

Theorem 1.1. ***(Abresch-Gromoll)*** *Assuming (1)-(3), then*

$$E \leq \Psi(\delta, L^{-1}, \epsilon | n, R) \quad (on\ B_R(p))$$

Proof: Let $\Psi_1 = \Psi(\delta, L^{-1}|n, R)$. By Laplacian comparison ($\Delta r(x) \leq (n-1)\frac{sn'_{-\delta}(r)}{sn_{-\delta}(r)}$), we have

$$\Delta E \leq \Psi_1 \quad (on\ B_{2R+1}(p))$$

Set $d(x,p) = r$. Fix $0 < \eta < R$. We can assume ϵ is chosen to satisfy

$$\epsilon \leq \Psi_1 \underline{L}_{R+1}(R) \leq \Psi \underline{L}_{R+1}(\eta)$$

where $\underline{L}$ is the comparison function of Ch. 4 in [1]*. In particular, $\underline{L}_{R+1}(R+1) = 0, \underline{L}'_{R+1} \leq 0$ on $[0, R+1]$ implies $\underline{L}_{R+1}(R) \geq 0$ is nonnegative. The second inequality follows since $\underline{L}$ is monotonically decreasing. Notice that $p \in A_{\eta, R+1}(x)$. We see that*

$$E(p) \leq \epsilon \leq \Psi_1 \underline{L}_{R+1}(\eta) \leq \Psi \underline{L}_{R+1}(r)$$

By Theorem 8.12 of [1]*, for all r with $\eta \leq r < R$, we have*

$$E(x) \leq \Psi_1 \underline{L}_{R+1}(\eta) + 2\eta \tag{1.4}$$

Since Lip $E \leq 2$, for all r we have $E(x) \leq E(p) + 2r$. Since $E(p) \leq \epsilon \leq \Psi_1 \underline{L}_{R+1}(\eta)$, we have

$$E(x) \leq E(p) + 2r \leq \Psi_1 \underline{L}_{R+1}(\eta) + 2r \leq \Psi_1 \underline{L}_{R+1}(\eta) + 2\eta$$

Therefore, we have (4) *for all $r \leq \eta$ and hence for all $r \leq R$. If we choose η to satisfy*

$$\Psi_1 \underline{L}_{R+1}(\eta) = 2\eta$$

(since $\underline{L}_{R+1}(\eta) \geq 0$), then $\Psi_1 \to 0$ implies $\eta \to 0$ (furthermore as $\eta \to 0$, since $\underline{L}'_{R+1} \leq 0$ on $[0, R+1]$, this means $\Psi_1 \to 0$). Thus, the desired statement follows from (1)*.* □

Thus, if the excess function is sufficiently small at $p \in M$, then it is also small in the ball $B_R(p)$. The reason for taking $\Psi = \Psi(\delta, L^{-1}, \epsilon|n, R)$ is that we will eventually consider a sequence of manifolds M_i^n with $\mathrm{Ric}_{M_i^n} \geq -(n-1)\delta_i$ and $\delta_i \to 0$, and the M_i containing longer and longer geodesics ($L^{-1} \to 0$).

Let $\gamma_\pm$ denote minimal geodesics from $q_\pm$ to p. Define $b_\pm(x) = d(x, q_\pm) - d(p, q_\pm)$, a function that is similar in spirit to the Busemann function. Let $\mathbf{b}_\pm$ be the harmonic function satisfying

$$\Delta \mathbf{b}_\pm = 0 \quad (\text{on } B_R(p))$$
$$\mathbf{b}_\pm|_{\partial B_R(p)} = b_\pm$$

The function $\mathbf{b}_\pm$ will serve as our Busemann function-equivalent in the almost setting. We will prove various average integral estimates relating $b_\pm$ to $\mathbf{b}_\pm$ on balls centered at p. The first lemma shows that $b_\pm$ can be uniformly approximated by $\mathbf{b}_\pm$ on $B_R(p)$.

Lemma 1.2. *Assuming (1)-(3), then*

$$|b_\pm - \boldsymbol{b}_\pm| \leq \Psi \quad (\text{on } B_R(p)).$$

Proof: By Laplacian comparison, $\Delta(b_\pm - \boldsymbol{b}_\pm) = \Delta b_\pm \leq \Psi$. *By Lemma 8.5 of* [1], *setting* $t = 0$,

$$b_\pm - \boldsymbol{b}_\pm \geq \Psi \underline{L}_{R_2}(R) + \max_{\partial B_R(p)}(b_\pm - \boldsymbol{b}_\pm - \Psi \underline{L}_{R_2}) = \underline{L}_{R_2}(R) + \max_{\partial B_R(p)}(-\Psi \underline{L}_{R_2}) \geq -\Psi.$$

We have $b_+(x) + b_-(x) = E(x) - E(p)$. *By Theorem 1, this gives* $-\epsilon \leq b_+ - b_- \leq \Psi$. *Therefore, by the minimum principle,* $-\epsilon \leq \boldsymbol{b}_+ + \boldsymbol{b}_-$. *Combining these observations,*

$$\begin{aligned} \boldsymbol{b}_+ - \Psi &\leq b_+ \\ &\leq -b_- + \Psi \\ &\leq -\boldsymbol{b}_- + 2\Psi \\ &\leq \boldsymbol{b}_+ + 2\Psi + \epsilon \end{aligned}$$

Thus, $b_+ - \boldsymbol{b}_+ \leq 2\Psi + \epsilon = \Psi(\delta, L^{-1}, \epsilon | n, R)$ □

Recall that in the splitting theorem, we used the minimum principle to show $b_+ + b_- \equiv 0$. In the almost splitting theorem, we showed $\epsilon \leq \mathbf{b}_+ + \mathbf{b}_-$ above. We have the following L^2 gradient estimate.

Lemma 1.3. *Assuming (1)-(3), then*

$$_{B_R(p)}|\nabla b_+ - \nabla \mathbf{b}_+|^2 \leq \Psi$$

Proof: Using integration by parts and $\mathbf{b}_+ = b_+$ *on* $\partial B_R(p)$, *we have*

$$\begin{aligned} _{B_R(p)}|\nabla b_+ - \nabla \mathbf{b}_+|^2 &= -_{B_R(p)}\Delta(b_+ - \mathbf{b}_+)(b_+ - \mathbf{b}_+) \\ &\leq_{B_R(p)} |\Delta(b_+ - \mathbf{b}_+)(b_+ - \mathbf{b}_+)| \\ &\leq \Psi_{B_R(p)}|\Delta(b_+ - \mathbf{b}_+)|, \quad (\textit{Lemma 1}) \\ &= \Psi_{B_R(p)}|\Delta b_+| \\ &\leq \Psi \end{aligned}$$

□

In the splitting theorem, we proved the Busemann function was linear, i.e. Hess $b_+ \equiv 0$. In the almost setting, we instead provide an average L^2 estimate on $\text{Hess}_{\mathbf{b}_+}$.

Lemma 1.4. *Assuming (1)-(3), then*

$$ {}_{B_{R/2}(p)}|Hess_{\mathbf{b}_+}|^2 \leq \Psi $$

Proof: By Bochner's formula,

$$ \frac{1}{2}\Delta(|\nabla \mathbf{b}_+|^2) = |Hess_{\mathbf{b}_+}|^2 + Ric(\nabla \mathbf{b}_+, \nabla \mathbf{b}_+) $$

Using the cutoff function ϕ constructed in Theorem 8.16 of [1], *with $\phi|_{B_{R/2}(p)} \equiv 1, |\Delta\phi| \leq c(n)$, we have*

$$ \begin{aligned} {}_{B_{R/2}(p)}|Hess_{\mathbf{b}_+}|^2 &\leq_{B_R(p)} \phi|Hess_{\mathbf{b}_+}|^2 \\ &\leq_{B_R(p)} \frac{1}{2}\phi\Delta(|\nabla \mathbf{b}_+|^2 - 1) + (n-1)\delta|\nabla \mathbf{b}_+|^2, \quad (Ric\ bound) \\ &\leq_{B_R(p)} \frac{1}{2}|\Delta\phi|||\nabla \mathbf{b}_+|^2 - 1| + (n-1)\delta|\nabla \mathbf{b}_+|^2, \quad (integration\ by\ parts) \\ &\leq c(n)_{B_R(p)}||\nabla \mathbf{b}_+|^2 - 1| + (n-1)\delta|\nabla \mathbf{b}_+|^2 \\ &\leq \Psi, \quad (Lemma\ 2) \end{aligned} $$

□

Next, we show a quantitative version of the Pythagorean theorem.

Lemma 1.5. *Assume (1)-(3). Let $x, z, w \in B_{\frac{R}{8}}(p)$, with $x \in \mathbf{b}_+^{-1}(a)$, and z a point on $\mathbf{b}_+^{-1}(a)$ closest to w. Then*

$$ |d(x,z)^2 + d(z,w)^2 - d(x,w)^2| \leq \Psi $$

Proof: We apply the iterated segment inequality, volume comparison, and Lemma 3 to show there exist x^*, z^*, w^* such that,

$$ \begin{aligned} d(x^*, x) &\leq \Psi \\ d(z^*, z) &\leq \Psi \\ d(w^*, w) &\leq \Psi \end{aligned} $$

and in addition, if $\sigma : [0, e] \to M$ is minimal from z^* to w^*, then,

$$ \int_U \int_0^{l(s)} |Hess_{\mathbf{b}_+}(\tau_s(t))| dt ds \leq \Psi \tag{1.5} $$

, where $U \subset [0,e]$ is of full measure, such that for all $s \in U$, the minimal geodesic $\tau_s : [0, l(s)] \to M$ from x^* to $\sigma(s)$ is unique. By the segment inequality,

$$\int_{B(x,\epsilon)\times B(p,\frac{R}{4})} \mathcal{F}_{|Hess\ \mathbf{b}_+|}(x,r)dxdr \leq CR(|B(x,\epsilon)| + |B(p,\frac{R}{4})|)\int_{B(p,\frac{R}{2})} |Hess\ \mathbf{b}_+|$$

By Markov's inequality, there exists $x^* \in B(x,\epsilon)$ such that,

$$\int_{B(p,\frac{R}{4})} \mathcal{F}_{|Hess\ \mathbf{b}_+|}(x^*,r)dr \leq \frac{CR(|B(x,\epsilon)| + |B(p,\frac{R}{4})|)}{|B(x,\epsilon)|}\int_{B(p,\frac{R}{2})} |Hess\ \mathbf{b}_+|$$

Now, again by the segment inequality,

$$\int_{B(z,\epsilon)\times B(w,\epsilon)} \mathcal{F}_{\mathcal{F}_{|Hess\ \mathbf{b}_+|}(x^*,\cdot)}(z,w)dzdw \leq CR(|B(z,\epsilon)|+|B(w,\epsilon)|)\int_{B(p,\frac{R}{4})} \mathcal{F}_{|Hess\ \mathbf{b}_+|}(x^*,r)dr$$

Combined with the above and Markov's inequality again, there exists $z^* \in B(z,\epsilon), w^* \in B(w,\epsilon)$ such that,

$$\mathcal{F}_{\mathcal{F}_{|Hess\ \mathbf{b}_+|}(x^*,\cdot)}(z^*,w^*) \leq \frac{C^2R^2(|B(y,\epsilon)| + |B(z,\epsilon)|)(|B(x,\epsilon)| + |B(p,\frac{R}{2})|)}{|B(x,\epsilon)||B(z,\epsilon)||B(w,\epsilon)|}\int_{B(p,\frac{R}{2})} |Hess\ \mathbf{b}_+|$$

By relative volume comparison and Lemma 3, we therefore have,

$$\mathcal{F}_{\mathcal{F}_{|Hess\ \mathbf{b}_+|}(x^*,\cdot)}(z^*,w^*) = \int_U \int_0^{l(s)} |Hess_{\mathbf{b}_+}(\tau_s(t))|dtds \leq \Psi$$

Therefore, we have the desired x^*, z^*, w^*. Similarly, we apply the segment inequality to the function, $\mathcal{F}_{||\nabla \mathbf{b}_+|-1|}$ to get,

$$\int_0^e ||\nabla \mathbf{b}_+(\sigma(s))| - 1|ds \leq \Psi \tag{1.6}$$

The Abresch-Gromoll inequality implies $|E(z) - E(x)| \leq \Psi$, which means $|b_+(z) - b_+(x)| - d(z,x) \leq \Psi$. By Lemma 1,

$$|d(z,x) - (\mathbf{b}_+(z) - \mathbf{b}_+(x))| \leq \Psi \tag{1.7}$$

Equation (7), Lemma 1, and the Cheng-Yau gradient estimate ($\sup_{B_R(p)} |\nabla \mathbf{b}_+| \leq C$) *give*

$$\int_0^e |\nabla \mathbf{b}_+(\sigma(s)) - \sigma'(s)|ds \leq \Psi \tag{1.8}$$

Recall $\sigma'(s) = \nabla b_+(\sigma(s))$*, since* b_+ *is a distance function. So* (5) *provides an integral estimate of the gradients along a geodesic. Furthermore, notice that for all* $t \in [0, l(s)]$*,*

$$\begin{aligned}
|\langle \nabla \mathbf{b}_+(\tau_s(t)), \tau_s'(t)\rangle - \langle \nabla \mathbf{b}_+(\tau_s(l(s))), \tau_s'(l(s))\rangle| &= |\int_t^{l(s)} \frac{d}{du}\langle \nabla \mathbf{b}_+(\tau_s(u)), \tau_s'(u)\rangle du| \\
&= |\int_t^{l(s)} \tau_s' \cdot \langle \nabla \mathbf{b}_+(\tau_s(u)), \tau_s'(u)\rangle ds|, \quad (\tau_s' = \frac{d}{du}) \\
&= |\int_t^{l(s)} \mathit{Hess}\ _{\mathbf{b}_+}(\tau_s'(u), \tau_s'(u)) du|, \quad (\mathit{since}\ \nabla_{\tau_s'}\tau_s' = 0) \\
&\leq \int_0^{l(s)} |\mathit{Hess}\ _{\mathbf{b}_+}(\tau_s(u))| du
\end{aligned}$$

Integrating both sides by U*, we get*

(1.9)
$$\int_U |\langle \nabla \mathbf{b}_+(\tau_s(t), \tau_s'(t)\rangle - \langle \nabla \mathbf{b}_+(\tau_s(l(s))), \tau_s'(l(s))\rangle| \leq \int_U \int_0^{l(s)} |\mathit{Hess}\ _{\mathbf{b}_+}(\tau_s(u))| du ds \leq \Psi$$

We now have the tools to prove the quantitative Pythagorean theorem,

$$\begin{aligned}
\frac{1}{2}d(z, w)^2 = \frac{1}{2}d(z^*, w^*)^2 \pm \Psi &= \int_0^e s ds \pm \Psi \\
&= \int_0^e \mathbf{b}_+(\sigma(s)) - \mathbf{b}_+(\sigma(0)) ds \pm \Psi, \quad (\mathit{Lemma\ 1}) \\
&= \int_U \mathbf{b}_+(\tau_s(l(s))) - \mathbf{b}_+(\tau_s(0)) ds \pm \Psi \\
&(\tau(s) = \sigma(s), [0, e] \subset U \ \mathit{full\ measure}) \\
&= \int_u \int_0^{l(s)} \langle \nabla \mathbf{b}_+(\tau_s(t)), \tau_s'(t)\rangle dt ds \pm \Psi \\
&(\langle \mathbf{b}_+(\tau_s(t)), \tau_s'(t)\rangle = \frac{d}{dt}\mathbf{b}_+(\tau_s(t))) \\
&= \int_u \int_0^{l(s)} \langle \nabla \mathbf{b}_+(\tau_s(l(s))), \tau_s'(l(s))\rangle dt ds \pm \Psi, \quad (\mathit{by}\ (9)) \\
&= \int_u \langle \nabla \mathbf{b}_+(\tau_s(l(s))), \tau_s'(l(s))\rangle l(s) ds \pm \Psi
\end{aligned}$$

The above quantity in the last line is,

$$
\begin{aligned}
\int_u \langle \nabla \mathbf{b}_+(\tau_s(l(s))), \tau_s'(l(s))\rangle l(s)ds &= \int_u \langle \nabla \mathbf{b}_+(\sigma(s))), \tau_s'(l(s))\rangle l(s)ds, \quad (\tau_s(l(s)) = \sigma(s)) \\
&= \int_u \langle \sigma'(s), \tau_s'(l(s))\rangle l(s)ds, \quad (\textit{by } (8)) \\
&= \int_U l'(s)l(s)ds \pm \Psi \\
&\quad (\textit{1st variation of arc length} \Rightarrow l'(s) = \langle \sigma'(s), \tau_s'(l(s))\rangle) \\
&= \frac{1}{2}l^2(e) - \frac{1}{2}l^2(0) \pm \Psi \\
&= \frac{1}{2}d(x,w)^2 - \frac{1}{2}d(x,z)^2 \pm \Psi
\end{aligned}
$$

□

The quantitative Pythagorean theorem allows us to prove the quantitative version of the almost splitting theorem.

Theorem 1.6. *Assuming (1)-(3), there is a length space X such that for some ball $B_{R/4}((0,x)) \subset \mathbb{R} \times X$ with the product metric, we have,*

$$d_{GH}(B_{R/4}(p), B_{R/4}((0,x))) \leq \Psi$$

Proof: By the quantitative Pythagorean theorem, $B_{\frac{R}{4}}(p)$ is Ψ-Gromov-Hausdorff close to a subset of $B_{\frac{R}{4}}((0,x)) \subset \mathbb{R} \times \mathbf{b}_+^{-1}(0)$. By the Abresch-Gromoll inequality, the subset can be taken to be the whole ball $B_{\frac{R}{4}}((0,x))$. However, the metric space $\mathbf{b}_+^{-1}(0)$ with the inherited metric from M is not a length space. To get a length space X, take $B_{\frac{R}{4}}(p_i) \in M_i^n$, where $Ric_{M_i^n} \geq -(n-1)\delta_i$ and $\delta_i \to 0$; let $M_i^n = (M, \delta_i^{-1}g)$. By Gromov's compactness theorem, the sequence $B_{\frac{R}{4}}(p_i)$ subconverges. It must subconverge to a ball in a product space $\mathbb{R} \times X$ by the theorem. Since $B_{\frac{R}{4}}(p_i)$ is a length space, and the limit of length spaces is a length space, X must be a length space. □

Theorem 2 is equivalent to the splitting theorem extending to Gromov-Hausdorff limit spaces.

Theorem 1.7. *Let $M_i^n \xrightarrow{d_{GH}} Y$ satisfy $Ric_{M_i^n} \geq -(n-1)\delta_i$, where $\delta_i \to 0$. If Y contains a line, then Y splits as an isometric product $Y = \mathbb{R} \times X$, for some length space X.*

Proof: If Y contains a line, the M_i^n must contain minimizing geodesics γ_i of length L_i, where $L_i \to \infty$. By Theorem 2, there exists a ball $B_{R_i}(p_i) \in M_i^n$ that is Ψ-GH close to $B_{R_i}((0,x_i)) \subset \mathbb{R} \times X_i$, where X_i is some length space. Since the $R_i \to \infty$, in the limit Y splits isometrically as $\mathbb{R} \times X$, for some length space X. □

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