Math 285-1 **Final December 2006** **C. Robinson**

1. (21 Points) The matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 & 5 & 8 & 5 \\ 1 & 2 & 2 & 3 & 1 \\ 4 & 8 & 3 & 2 & 6 \\ 2 & 4 & 4 & 6 & 1 \end{bmatrix}$ has the reduced echelon form $\mathbf{U} = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. *Find* a basis for the nullspace of A.

b. *Find* a basis for the column space of A.

c. *Find* a basis for the row space of A.

Answer:
(**a**) Use the free variables to find the nullspace of **U**: $\{(-2, 1, 0, 0, 0)^T,\ (1, 0, -2, 1, 0)^T\}$.
(**b**) The basis of the column space is given by the pivot columns in the original matrix **A**:
$\{(2, 1, 4, 2)^T,\ (5, 2, 3, 4)^T,\ (5, 1, 6, 1)^T\}$.
(**c**) The basis of the row space is given by the nonzero rows of **U**:
$\{(1, 2, 0, -1, 0)^T,\ (0, 0, 1, 2, 0)^T,\ (0, 0, 0, 0, 1)^T\}$.

2. (27 Points) The matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 4 & -2 \end{bmatrix}$ has eigenvalues 1 and $-1 \pm 2i$. *Find* an eigenvector for each eigenvalue.

Answer:
For $\lambda = 1$ we row reduce as follows:

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & -1 \\ -1 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

so an eigenvector is $(1, 1, 1)^T$.
For $\lambda = -1 + 2i$ we row reduce as follows:

$$\mathbf{A} - (-1 + 2i)\mathbf{I} = \begin{bmatrix} 1 - 2i & 0 & 1 \\ 1 & 2 - 2i & -1 \\ -1 & 4 & -1 - 2i \end{bmatrix}$$

multiplying the first row by $1 + 2i$

$$\sim \begin{bmatrix} 51 & 2 - 2i & -1 \\ 1 & -4 & 1 + 2i \end{bmatrix}$$

interchanging rows 1 & 3 and clearing the first column

$$\sim \begin{bmatrix} 1 & -4 & 1 + 2i \\ 0 & 6 - 2i & -2 - 2i \\ 0 & 20 & -4 - 8i \end{bmatrix}$$

multiplying second row by $(3 + 2i)/2$

$$\sim \begin{bmatrix} 1 & -4 & 1 + 2i \\ 0 & 10 & -2 - 4i \\ 0 & 5 & -1 - 2i \end{bmatrix}$$

clearing the third column

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 5 & -1 - 2i \\ 0 & 0 & 0 \end{bmatrix},$$

so an eigenvector for $-1+2i$ is $(-1-2i, 1+2i, 5)^T$.
Other possible eigenvectors include $(-1, 1, 1-2i)^T$.
An eigenvector for $-1-2i$ is the complex conjugate of the eigenvector for $-1-2i$, $(-1+2i, 1-2i, 5)^T$.

3. (24 Points) Consider the stochastic matrix $\mathbf{M}$ that has eigenvectors $\mathbf{v}^1 = (.3, .6, .1)^T$ for the eigenvalue 1, $\mathbf{v}^2 = (.1, \text{-}.3, .2)^T$ for the eigenvalue 0.5, and $\mathbf{v}^3 = (.2, \text{-}.1, \text{-}.1)^T$ for the eigenvalue 0.2.

a. *Write* $\mathbf{p} = (.2, .4, .4)^T$ as a linear combination of $\mathbf{v}^1$, $\mathbf{v}^2$, and $\mathbf{v}^3$.
b. For $\mathbf{p} = (.2, .4, .4)^T$, *what is* $\mathbf{M}^3\mathbf{p}$?
c. *Give* the matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{M}\mathbf{P}$ and $\mathbf{D}$ is a diagonal matrix.
d. *What is* the det($\mathbf{M}$)?

Answer:
(**a**) To find the coefficients by a systematic method, we have to row reduce the augmented matrix as follows:

$$\left[\begin{array}{ccc|c} .3 & .1 & .2 & .2 \\ .6 & -.3 & -.1 & .4 \\ .1 & .2 & -.1 & .4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 6 & -3 & -1 & 4 \\ 3 & 1 & 2 & 2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -15 & 5 & -20 \\ 0 & -5 & 5 & -10 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -3 & 1 & -4 \\ 0 & 1 & -1 & 2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -2 & 2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right],$$

so $c_1 = 1$, $c_2 = 1$, and $c_3 = -1$: $\mathbf{p} = \mathbf{v}^1 + \mathbf{v}^2 - \mathbf{v}^3$.
(**b**) We know that a matrix acts linearly on combinations of vectors, so
$\mathbf{M}^3\mathbf{p} = \mathbf{M}^3\mathbf{v}^1 + \mathbf{M}^3\mathbf{v}^2 - \mathbf{M}^3\mathbf{v}^3 = (.3, .6, .1)^T + (.5)^3(.1, \text{-}.3, .2)^T - (.2)^3(.2, \text{-}.1, \text{-}.1)^T$.
(**c**) The matrix $\mathbf{P}$ has the eigenvectors as columns and the matrix $\mathbf{D}$ is the diagonal matrix with the eigenvalues as entries:

$$\mathbf{P} = \begin{bmatrix} .3 & .1 & .2 \\ .6 & \text{-}.3 & \text{-}.1 \\ .1 & .2 & \text{-}.1 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .2 \end{bmatrix}.$$

(**d**) The determinant of $\mathbf{M}$ is either equal to the product of the eigenvalues or the determinant of $\mathbf{D}$: either of these equals to 0.1.

4. (48 Points) Indicate which of the following statements are always true and which are false, i.e., not always true. Justify each answer by referring to an theorem, fact, or counterexample.

a. The nonpivot columns of a matrix are always linearly dependent.

Answer: False.
A counter example is $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, where $(1, 0)^T$ and $(1, 1)^T$ are not linearly dependent.

b. The dimension of the null space of a matrix $\mathbf{A}$ equals the rank of $\mathbf{A}$.

Answer: False
The matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has rank 2 and dimension of the nullspace equal to 1 (the number of free variables).

c. The column space of a matrix $\mathbf{A}$ is equal to the column space of its row reduced echelon matrix $\mathbf{U}$.

Answer: False
The column space of $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is spanned by $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and not by the columns of $\mathbf{U} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ which all have 0 in the second component.

d. If $\mathbf{A}$ is a $m \times n$ matrix and $\mathbf{B}$ is a $n \times p$ matrix, then $\mathrm{Col}(\mathbf{AB}) \subset \mathrm{Col}(\mathbf{A})$

Answer: True
The column space of $\mathbf{AB}$ is the set of all vectors of the form $\mathbf{ABv} = \mathbf{Aw}$ where $\mathbf{w} = \mathbf{Bv}$. These are contained in the set of all $\mathbf{Aw}$, where $\mathbf{w}$ is any vector in $\mathbb{R}^n$.

e. For any $n \times m$ matrix $\mathbf{A}$, both the matrix products $\mathbf{A}^T\mathbf{A}$ and $\mathbf{AA}^T$ are defined.

Answer: True.
The matrix $\mathbf{A}^T$ is $m \times n$, so $\mathbf{A}^T$ has the same number of columns as $\mathbf{A}$ has rows, so $\mathbf{A}^T\mathbf{A}$ is defined; also, $\mathbf{A}^T$ has the same number of rows as $\mathbf{A}$ has columns, so $\mathbf{AA}^T$ is defined.

f. Let $\mathbf{W}$ be a subspace of $\mathbf{V}$ with $\dim(\mathbf{W}) = 4$, and $\dim(\mathbf{V}) = 7$. Then, any basis of $\mathbf{W}$ can be expanded to a basis of $\mathbf{V}$ by adding three more vectors to it.

Answer: True
The theorem on extension of basis says that any basis of $\mathbf{W}$ can be expanded to a basis of $\mathbf{V}$. We need to add three more vectors because of the dimensions given.

g. If $\mathbf{A}$ is a square matrix with $\det(\mathbf{A}) \neq 0$, then $\det(\mathbf{A}^{-1}) = \left(\det(\mathbf{A}^T)\right)^{-1}$.

Answer: True
We have theorems that say that $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$ and $\det(\mathbf{A}) = \det(\mathbf{A}^T)$. Combining, we get the formula given.

h. Every diagonalizable $n \times n$ matrix has n distinct eigenvalues.

Answer: False
The matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ has repeated eigenvalue 2 and is diagonal (so diagonalizable).

i. If $\mathbf{A}$ is a 4×4 matrix with eigenvalues 3, -1, 2, and 5, then it is diagonalizable.

Answer: True
Because the eigenvalues are distinct, and eigenvectors for different eigenvalues are linearly independent, any set of eigenvectors are linearly independent and so a basis. There is a theorem that says that a matrix with a basis of eigenvectors is diagonalizable.

j. If λ is an eigenvalue of a matrix $\mathbf{A}$, then there is a unique eigenvector of $\mathbf{A}$ that corresponds to λ.

Answer: False
The eigenvalue can have an eigenspace of dimension 2. Also, if $\mathbf{v}$ is an eigenvector that $2\mathbf{v}$ is also an eigenvector. In any case, the eigenvector is not unique.

k. Assume $\mathbf{A}$ and $\mathbf{B}$ are both $n \times n$. If $\mathbf{v}$ is an eigenvector of both $\mathbf{A}$ and $\mathbf{B}$ then $\mathbf{v}$ is an eigenvector of $\mathbf{A} + \mathbf{B}$.

Answer: True
Use the definition of an eigenvector. $\mathbf{Av} = \lambda_A \mathbf{v}$ and $\mathbf{Bv} = \lambda_B \mathbf{v}$ for the eigenvalues λ_A and λ_B. Then $(\mathbf{A} + \lambda_B)\mathbf{v} = (\lambda_A + \lambda_B)\mathbf{v}$, so $\mathbf{v}$ is an eigenvector or $\mathbf{A} + \mathbf{B}$ for the eigenvalue $\lambda_A + \lambda_B$.

l. If $\mathbf{v}$ is an eigenvector of an invertible matrix $\mathbf{A}$ that corresponds to a nonzero eigenvalue, then $\mathbf{v}$ is also an eigenvector for $\mathbf{A}^{-1}$.

Answer: True
If $\mathbf{Av} = \lambda\mathbf{v}$, then $\mathbf{A}^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$ and so $\mathbf{v}$ is an eigenvector for $\mathbf{A}^{-1}$ corresponding to the eigenvalue λ^{-1}.

5. (14 Points) Let $\mathbf{a}^1, \ldots, \mathbf{a}^n$ be vectors in $\mathbb{R}^m$ and the columns of the matrix $\mathbf{A}$.

a. If the vectors are linearly independent, what can you say about the rank of $\mathbf{A}$?

b. If the vectors span $\mathbb{R}^m$, what can you say about the rank of $\mathbf{A}$?

Answer:
(**a**) $\text{rank}(\mathbf{A}) = n$.
(**b**) $\text{rank}(\mathbf{A}) = m$.

6. (22 Points) Assume that (i) $\mathbf{V} \subset \mathbb{R}^n$ is a subspace, (ii) $\{\mathbf{b}^1, \ \ldots, \ \mathbf{b}^r\}$ is a basis of $\mathbf{V}$, and (iii) $\mathbf{A}$ is an $m \times n$ matrix of rank n. *Prove* that $\{\mathbf{Ab}^1, \ \ldots, \ \mathbf{Ab}^r\}$ is a basis of $\mathbf{AV}$.

Answer:
We need to show that these vectors are linearly independent and span the subspace. Assume that $\mathbf{0} = c_1\mathbf{Ab}^1 + \cdots + c_r\mathbf{Ab}^r$. Then, $\mathbf{0} = \mathbf{A}(c_1\mathbf{b}^1 + \cdots + c_r\mathbf{b}^r)$. Since $\mathbf{A}$ has rank n, it has a trivial nullspace, so it follows that $\mathbf{0} = c_1\mathbf{b}^1 + \cdots + c_r\mathbf{b}^r$. Since the vectors $\mathbf{b}^j$ are linearly independent, it follows that all the $c_j = 0$. Thus, any linear combination of the vectors $\{\mathbf{Ab}^1, \ \ldots, \ \mathbf{Ab}^r\}$ giving the zero vector must have coefficients equal to zero. This shows that the set of vectors $\{\mathbf{Ab}^1, \ \ldots, \ \mathbf{Ab}^r\}$ is linearly independent.
Any vector $\mathbf{w}$ in $\mathbf{AV}$ can be written as $\mathbf{w} = \mathbf{Av}$ for some vector $\mathbf{v}$ in $\mathbf{V}$. But, the vectors $\{\mathbf{b}^1, \ \ldots, \ \mathbf{b}^r\}$ is a basis of $\mathbf{V}$, so $\mathbf{v}$ can be written as a linear combination of them, $\mathbf{v} = c_1\mathbf{b}^1 + \cdots + c_r\mathbf{b}^r$ for some $c_1, \ldots, c_r$. Thus, $\mathbf{w} = \mathbf{Av} = \mathbf{A}(c_1\mathbf{b}^1 + \cdots + c_r\mathbf{b}^r) = c_1\mathbf{Ab}^1 + \cdots + c_r\mathbf{Ab}^r$. This shows that the set of vectors $\{\mathbf{Ab}^1, \ \ldots, \ \mathbf{Ab}^r\}$ span $\mathbf{AV}$.
Combining, they are a basis of $\mathbf{AV}$.

7. (22 Points) The set of all 3×3 matrices with real entries, $\mathbf{M}_{3\times 3}$, is a vector space. A matrix $\mathbf{A}$ is said to be a magic square provided that its row sums and column sums all add up the same number. (The number can depend on the matrix.) *Prove* that the set of all 3×3 matrices that are magic squares is a subspace $\mathbf{M}_{3\times 3}$.

Answer:
We must show the zero "vector" is a magic square and that magic squares are closed under linear combinations.
The 3×3 with all zeroes (the zero "vector" in $\mathbf{M}_{3\times 3}$) as all row sums and columns sums equal to 0. Thus, it is a magic square.

Now assume that $\mathbf{A}$ and $\mathbf{B}$ are magic squares with row sums and columns sums equal to r_A and r_B respectively. Then, the row sums and columns sums of $c_1\mathbf{A} + c_2\mathbf{B}$ are

$$\begin{aligned}
(c_1a_{i1} + c_2b_{i1}) + (c_1a_{i2} + c_2b_{i2}) + (c_1a_{i3} + c_2b_{i3}) &= c_1(a_{i1} + a_{i2} + a_{i3}) + c_2(b_{i1} + b_{i2} + b_{i3}) \\
&= c_1r_A + c_2r_B \\
(c_1a_{1j} + c_2b_{1j}) + (c_1a_{2j} + c_2b_{2j}) + (c_1a_{3j} + c_2b_{3j}) &= c_1(a_{1j} + a_{i2j} + a_{3j}) + c_2(b_{1j} + b_{2j} + b_{3j}) \\
&= c_1r_A + c_2r_B.
\end{aligned}$$

Since these are equal for all i and j, $c_1\mathbf{A} + c_2\mathbf{B}$ is a magic square.
This shows that the set of magic squares is a subspace.

8. (22 Points) Suppose that $\mathbf{V}$ is a vector space with basis $\{\mathbf{v}^1, \mathbf{v}^2\}$.

a. Let $\mathbf{w}^1 = 2\mathbf{v}^1 + \mathbf{v}^2$ and $\mathbf{w}^2 = \mathbf{v}^1 + \mathbf{v}^2$. *Prove* that the set $\mathscr{B} = \{\mathbf{w}^1, \mathbf{w}^2\}$ is a basis for $\mathbf{V}$.

b. *Find* $[\mathbf{v}^1]_{\mathscr{B}}$, the coordinate vector of $\mathbf{v}^1$ with respect to the basis $\mathscr{B}$.

Answer:

(**a**) The easiest way to solve this problem is to use the coordinates of the $\mathbf{w}^j$ in terms of the original basis $\mathscr{B}_0$: $[\mathbf{w}^1, \mathbf{w}^2]_{\mathscr{B}_0} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. This matrix has nonzero determinant, so the columns are linearly independent. Since the columns of the $\mathscr{B}_0$-coordinates are linearly independent, the vectors $\mathbf{w}^1$ and $\mathbf{w}^2$ are linearly independent. Since the vector space $\mathbf{V}$ has dimension equal to 2 (a basis with 2 members), any set of 2 linearly independent vectors forms a basis and the set $\mathscr{B}$ is a basis.

(**b**) The inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ gives the vectors $\mathbf{v}^j$ in terms of the $\mathbf{w}^i$: so $\mathbf{v}^1 = \mathbf{w}^1 - \mathbf{w}^2$ and $\mathbf{v}^2 = -\mathbf{w}^1 + 2\mathbf{w}^2$. (This can also be obtained by solving the two equations for $\mathbf{v}^1$ and $\mathbf{v}^2$ in terms of $\mathbf{w}^1$ and $\mathbf{w}^2$.)