Math 285-1 **Final December 2006** **C. Robinson**

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1. (21 Points) The matrix $\mathbf{A} = \begin{bmatrix} 2 & 4 & 5 & 8 & 5 \\ 1 & 2 & 2 & 3 & 1 \\ 4 & 8 & 3 & 2 & 6 \\ 2 & 4 & 4 & 6 & 1 \end{bmatrix}$ has the reduced echelon form $\mathbf{U} = \begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

- **a**. *Find* a basis for the nullspace of A.
- **b**. *Find* a basis for the column space of A.
- **c**. *Find* a basis for the row space of A.

2. (27 Points) The matrix $\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 4 & -2 \end{bmatrix}$ has eigenvalues 1 and -1 $\pm$ $2i$. *Find* an eigenvector for each eigenvalue.

3. (24 Points) Consider the stochastic matrix **M** that has eigenvectors $\mathbf{v}^1 = (.3, .6, .1)^T$ for the eigenvalue 1, $\mathbf{v}^2 = (.1, -.3, .2)^T$ for the eigenvalue 0.5, and $\mathbf{v}^3 = (.2, -.1, -.1)^T$ for the eigenvalue 0.2.

- **a**. *Write* $\mathbf{p} = (.2, .4, .4)^T$ as a linear combination of $\mathbf{v}^1$, $\mathbf{v}^2$, and $\mathbf{v}^3$.
- **b**. For $\mathbf{p} = (.2, .4, .4)^T$, *what is* $\mathbf{M}^3\mathbf{p}$?
- **c**. *Give* the matrices **P** and **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{M}\mathbf{P}$ and **D** is a diagonal matrix.
- **d**. *What is* the $\det(\mathbf{M})$?

4. (48 Points) Indicate which of the following statements are always true and which are false, i.e., not always true. Justify each answer by referring to an theorem, fact, or counterexample.

- **a**. The nonpivot columns of a matrix are always linearly dependent.
- **b**. The dimension of the null space of a matrix **A** equals the rank of **A**.
- **c**. The column space of a matrix **A** is equal to the column space of its row reduced echelon matrix **U**.
- **d**. If **A** is a $m \times n$ matrix and **B** is a $n \times p$ matrix, then $\mathrm{Col}(\mathbf{AB}) \subset \mathrm{Col}(\mathbf{A})$
- **e**. For any $n \times m$ matrix **A**, both the matrix products $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are defined.
- **f**. Let **W** be a subspace of **V** with $\dim(\mathbf{W}) = 4$, and $\dim(\mathbf{V}) = 7$. Then, any basis of **W** can be expanded to a basis of **V** by adding three more vectors to it.
- **g**. If **A** is a square matrix with $\det(\mathbf{A}) \neq 0$, then $\det(\mathbf{A}^{-1}) = \left(\det(\mathbf{A}^T)\right)^{-1}$.
- **h**. Every diagonalizable $n \times n$ matrix has n distinct eigenvalues.
- **i**. If **A** is a 4×4 matrix with eigenvalues 3, -1, 2, and 5, then it is diagonalizable.
- **j**. If λ is an eigenvalue of a matrix **A**, then there is a unique eigenvector of **A** that corresponds to λ.
- **k**. Assume **A** and **B** are both $n \times n$. If **v** is an eigenvector of both **A** and **B** then **v** is an eigenvector of $\mathbf{A} + \mathbf{B}$.
- **l**. If **v** is an eigenvector of an invertible matrix **A** that corresponds to a nonzero eigenvalue, then **v** is also an eigenvector for $\mathbf{A}^{-1}$.

(**over** for problems 5-8)

5. (14 Points) Let $\mathbf{a}^1, \ldots, \mathbf{a}^n$ be vectors in $\mathbb{R}^m$ and the columns of the matrix $\mathbf{A}$.

a. If the vectors are linearly independent, what can you say about the rank of $\mathbf{A}$?

b. If the vectors span $\mathbb{R}^m$, what can you say about the rank of $\mathbf{A}$?

6. (22 Points) Assume that (i) $\mathbf{V} \subset \mathbb{R}^n$ is a subspace, (ii) $\{\mathbf{b}^1, \ \ldots, \ \mathbf{b}^r\}$ is a basis of $\mathbf{V}$, and (iii) $\mathbf{A}$ is an $m \times n$ matrix of rank n. *Prove* that $\{\mathbf{A}\mathbf{b}^1, \ \ldots, \ \mathbf{A}\mathbf{b}^r\}$ is a basis of $\mathbf{A}\mathbf{V}$.

7. (22 Points) The set of all 3×3 matrices with real entries, $\mathbf{M}_{3\times 3}$, is a vector space. A matrix $\mathbf{A}$ is said to be a magic square provided that its row sums and column sums all add up the same number. (The number can depend on the matrix.) *Prove* that the set of all 3×3 matrices that are magic squares is a subspace $\mathbf{M}_{3\times 3}$.

8. (22 Points) Suppose that $\mathbf{V}$ is a vector space with basis $\{\mathbf{v}^1, \mathbf{v}^2\}$.

a. Let $\mathbf{w}^1 = 2\mathbf{v}^1 + \mathbf{v}^2$ and $\mathbf{w}^2 = \mathbf{v}^1 + \mathbf{v}^2$. *Prove* that the set $\mathscr{B} = \{\mathbf{w}^1, \mathbf{w}^2\}$ is a basis for $\mathbf{V}$.

b. *Find* $[\mathbf{v}^1]_{\mathscr{B}}$, the coordinate vector of $\mathbf{v}^1$ with respect to the basis $\mathscr{B}$.