Math 285-1 **Final: December 2007** **C. Robinson**

1. (20 Points) Consider the three vectors

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \mathbf{v}^2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 8 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}^3 = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 5 \end{bmatrix}.$$

a. *Determine* whether $\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ linearly independent or not. *Explain* your answer.
b. What is the dimension of Span$\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$?

Answer: (**a**)

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 5 \\ 1 & 2 & 0 \\ 3 & 8 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & -5 \\ 0 & 0 & -5 \\ 0 & 2 & -10 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & -5 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rank is 3, so the (column) vectors are linearly independent.
(**b**) The dimension of the column space equals the rank, which is 3.

2. (20 Points) *Find* the matrix $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, i.e., the matrix that satisfies $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$.

Answer:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}$$

3. (20 Points) Consider the vectors

$$\mathbf{v}^1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}^2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \qquad \mathbf{v}^3 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{W} = \text{Span}\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$. *Remark:* You do not need to simplify any fractions in the resulting vector.

Answer:

$$\text{proj}_{\mathbf{W}}\,\mathbf{y} = \text{proj}_{\mathbf{v}^1}\,\mathbf{y} + \text{proj}_{\mathbf{v}^2}\,\mathbf{y} + \text{proj}_{\mathbf{v}^3}\,\mathbf{y} = \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{11}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix} + \frac{3}{11}\begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 12 \\ 8 \\ 12 \\ 11 \end{bmatrix}.$$

4. (20 Points) Consider the three vectors $\mathbf{v}^1 = (1, -1, 0, 1, 1)^T$, $\mathbf{v}^2 = (3, -3, 2, 5, 5)^T$, and $\mathbf{v}^3 = (5, -1, 3, 2, 8)^T$ and set $\mathbf{W} = \mathrm{Span}\{\, \mathbf{v}^1,\ \mathbf{v}^2,\ \mathbf{v}^3 \,\}$ in $\mathbb{R}^5$. Use the Gram-Schmidt process to construct an orthogonal basis for $\mathbf{W}$. *Remark:* You do not need to simplify any fractions in the resulting vectors.

Answer:

$$\mathbf{w}^1 = \mathbf{v}^1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{w}^2 = \mathbf{v}^2 - \mathrm{proj}_{\mathbf{w}^1} \mathbf{v}^2 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \frac{16}{4} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix},$$

$$\mathbf{w}^3 = \mathbf{v}^3 - \mathrm{proj}_{\mathbf{w}^1} \mathbf{v}^3 - \mathrm{proj}_{\mathbf{w}^2} \mathbf{v}^3 = \begin{bmatrix} 5 \\ -1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \frac{16}{4} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{10}{8} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 9 \\ 7 \\ 2 \\ -13 \\ 11 \end{bmatrix}.$$

$\{\, \mathbf{w}^1,\ \mathbf{w}^2,\ \mathbf{w}^3 \,\}$ is the orthogonal basis.

5. (20 Points) The scores on two midterm tests and a final are give in the following table for 5 students.

Student	Test 1	Test 2	Final
Amy	76	24	86
Jessica	92	92	180
John	68	82	128
Noelle	86	68	138
Wynn	54	70	100

Write down the normal equation that gives the best fit of the form $F = c_0 + c_1 T_1 + c_2 T_2$, where F is the score on the final, T_1 is the score on the first midterm test, and T_2 is the score on the second midterm test. Put in the explicit numbers from the table, but you do not need to multiply any matrices.

Answer: The normal equation is $\mathbf{A}^T\mathbf{A}\mathbf{c} = \mathbf{A}^T\mathbf{b}$, so the equation for this problem is the following:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 76 & 92 & 68 & 86 & 54 \\ 24 & 92 & 82 & 68 & 70 \end{bmatrix} \begin{bmatrix} 1 & 76 & 24 \\ 1 & 92 & 92 \\ 1 & 68 & 82 \\ 1 & 86 & 68 \\ 1 & 54 & 70 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 76 & 92 & 68 & 86 & 54 \\ 24 & 92 & 82 & 68 & 70 \end{bmatrix} \begin{bmatrix} 86 \\ 180 \\ 128 \\ 138 \\ 100 \end{bmatrix}.$$

6. (20 Points) Let $\mathbb{P}_3$ be the set of all polynomials of degree less than or equal to three, which has a standard basis of $\mathscr{B} = \{1, t, t^2, t^3\}$. *Prove* that the set of polynomials
$\mathscr{S} = \{\, 1 + t^3,\ t,\ 1 + 7t + t^2,\ 1 + t + t^2 + t^3 \,\}$ forms another basis of $\mathbb{P}_3$. *Explain your answer.*

Answer: To check that the polynomials are linearly independent, we look at the matrix of their coordinate representation in terms of the standard basis and find the rank.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is 3, so the coordinate representations are linearly independent and the polynomials are linearly independent.

Since there are 4 linearly independent polynomials in a vector space of dimension 4, they form a basis of $\mathbb{P}_3$.

7. (20 Points) Assume that $\{\, \mathbf{v}^1, \dots, \mathbf{v}^n \,\}$ is an orthonormal basis of $\mathbb{R}^n$ and $\mathbf{W} = \mathrm{Span}\{\, \mathbf{v}^1, \dots, \mathbf{v}^k \,\}$ with $1 \leq k < n$. *Prove* that $\{\, \mathbf{v}^{k+1}, \dots, \mathbf{v}^n \,\}$ is a basis of $\mathbf{W}^{\perp}$.
Hint: You must show that $\mathbf{v}^j \in \mathbf{W}^{\perp}$ for $k + 1 \leq j \leq n$ and that they form a basis. You may use the Orthogonal Decomposition Theorem that says that any $\mathbf{y} \in \mathbb{R}^n$ can be uniquely written as $\mathbf{y} = \mathbf{z} + \mathrm{proj}_{\mathbf{W}}\, \mathbf{y}$ where $\mathbf{z} \in \mathbf{W}^{\perp}$ and $\mathrm{proj}_{\mathbf{W}}\, \mathbf{y} \in \mathbf{W}$.

Answer: (i) For $k + 1 \leq j \leq n$, the vector $\mathbf{v}^j$ is orthogonal to the vectors $\{\, \mathbf{v}^1, \dots, \mathbf{v}^k \,\}$ and so to the subspace $\mathbf{W}$. This shows that each of these $\mathbf{v}^j$ is in $\mathbf{W}^{\perp}$.

(ii) The vectors $\{\, \mathbf{v}^{k+1}, \dots, \mathbf{v}^n \,\}$ are part of a basis of $\mathbb{R}^n$, so they are linearly independent.

(iii) Take any $\mathbf{z} \in \mathbf{W}^{\perp}$. This vector can be represented in terms of the basis of $\mathbb{R}^n$ as $\mathbf{z} = c_1 \mathbf{v}^1 + \cdots + c_n \mathbf{v}^n$. Since $\mathbf{z}$ is orthogonal to $\{\, \mathbf{v}^1, \dots, \mathbf{v}^k \}$, the coefficients $c_1 = \cdots = c_k = 0$. Thus, $\mathbf{z} = c_{k+1} \mathbf{v}^{k+1} + \cdots + c_n \mathbf{v}^n$. This shows that $\{\, \mathbf{v}^{k+1}, \dots, \mathbf{v}^n \,\}$ span $\mathbf{W}^{\perp}$. Combining, we have that this is a basis of $\mathbf{W}^{\perp}$.

8. (20 Points) Assume that $\mathbf{T} : \mathbf{V} \to \mathbf{W}$ is a linear transformation from the vector space $\mathbf{V}$ onto the vector space $\mathbf{W}$ and that $\{\, \mathbf{v}^1, \dots, \mathbf{v}^k \,\}$ is a set of vectors that span $\mathbf{V}$. *Prove* that $\{\, \mathbf{T}(\mathbf{v}^1), \dots, \mathbf{T}(\mathbf{v}^k) \,\}$ spans $\mathbf{W}$.

Answer: Take any $\bar{\mathbf{w}} \in \mathbf{W}$. Since $\mathbf{T}$ is onto, there is a $\bar{\mathbf{v}} \in \mathbf{V}$ such that $\mathbf{T}(\bar{\mathbf{v}}) = \bar{\mathbf{w}}$. Since the vectors span $\mathbf{V}$, there are scalars $c_1, \dots, c_k$ such that $\bar{\mathbf{v}} = c_1 \mathbf{v}^1 + \cdots + c_k \mathbf{v}^k$. Then, $\bar{\mathbf{w}} = \mathbf{T}(\bar{\mathbf{v}}) = \mathbf{T}(c_1 \mathbf{v}^1 + \cdots + c_k \mathbf{v}^k) = c_1 \mathbf{T}(\mathbf{v}^1) + \cdots + c_k \mathbf{T}(\mathbf{v}^k)$. This proves that $\{\, \mathbf{T}(\mathbf{v}^1), \dots, \mathbf{T}(\mathbf{v}^k) \,\}$ spans $\mathbf{W}$.

9. (40 Points) Indicate which of the following statements are always *true* and which are *false*. *Justify* each answer by a counterexample or explanation. Refer to any theorem by an informal statement, not by a theorem number.

a. A square matrix with orthogonal columns is an orthogonal matrix.

Answer: False. The columns must be orthonormal, i.e., of length one.

b. If $\mathbf{W}$ is a subspace of $\mathbb{R}^n$, then $\|\, \mathrm{proj}_{\mathbf{W}}\, \mathbf{v}\|^2 + \|\mathbf{v} - \mathrm{proj}_{\mathbf{W}}\, \mathbf{v}\|^2 = \|\mathbf{v}\|^2$ for a vector $\mathbf{v} \in \mathbb{R}^n$.

Answer: True. This is the Pythagorean Theorem for orthogonal vectors.

c. If $\mathbf{A}$ is a 5×4 matrix with $\mathrm{Nul}(\mathbf{A}) = \mathrm{Span}\{(-1, 1, 1, 0)^T\}$, then the rank of $\mathbf{A}$ is 3.

Answer: True. The dimension of the null space is 1 and the dimension of the domain is 4, so the rank is $4 - 1 = 3$.

d. If $\mathbf{A}$ is a 5×4 matrix with $\text{Nul}(\mathbf{A}) = \text{Span}\{(-1, 1, 1, 0)^T\}$, then the dimension of the column space is 4.

Answer: False. The rank of the matrix is 3, so the dimension of the column space is 3.

e. If $\mathbf{A}$ is a 5×4 matrix and $\mathbf{B}$ is a 4×5 matrix, then $\mathbf{AB}$ cannot be invertible.

Answer: True. The rank of the composition can at most be the minimum of the ranks of the individual matrices. Therefore, the rank is at most 4 and so cannot be invertible.

f. If $\mathbf{A}$ is a 4×4 matrix, then $\det(2\mathbf{A}) = 2\det(\mathbf{A})$.

Answer: False. $\det(2\mathbf{A}) = 2^4 \det(\mathbf{A})$.

g. If $\mathbf{A}$ is $n \times n$ and $\det(\mathbf{A}) = 0$, then $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{B} \in \mathbb{R}^n$.

Answer: False. The matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has determinant 0 and there is no solution for $\mathbf{Ax} = \mathbf{e}^2$.

h. If $\mathbf{A}$ is an $m \times n$ matrix with $m \neq n$ and $\text{rank}(\mathbf{A}) = m$, then the linear transformation for $\mathbf{A}$ is one to one.

Answer: False. For the linear transformation to be one to one, the rank must be the dimension of the domain, n.

i. The transpose of an invertible matrix is invertible.

Answer: True. The $\det(\mathbf{A}^T) = \det(\mathbf{A}) \neq 0$, so $\mathbf{A}^T$ is invertible.

j. If $\mathbf{A}$ is an $n \times n$ matrix with $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}^9$ has linearly independent columns.

Answer: True. $\det(\mathbf{A}^9) = \det(\mathbf{A})^9 \neq 0$, so the columns of $\mathbf{A}^9$ are linearly independent.