Math 285-1 **Final: December 2007** **C. Robinson**

No calculators, no books, no notes
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1. (20 Points) Consider the three vectors

$$\mathbf{v}^1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \mathbf{v}^2 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 8 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}^3 = \begin{bmatrix} 5 \\ 5 \\ 0 \\ 5 \end{bmatrix}.$$

a. *Determine* whether $\{ \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3 \}$ linearly independent or not. *Explain* your answer.
b. What is the dimension of $\text{Span}\{ \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3 \}$?

2. (20 Points) *Find* the matrix $\mathbf{A}$ such that $\mathbf{A}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{A}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, i.e., the matrix that satisfies $\mathbf{A}\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}$.

3. (20 Points) Consider the vectors

$$\mathbf{v}^1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}^2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \qquad \mathbf{v}^3 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ -1 \end{bmatrix}, \qquad \text{and} \quad \mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the orthogonal projection of $\mathbf{y}$ onto $\mathbf{W} = \text{Span}\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$. *Remark:* You do not need to simplify any fractions in the resulting vector.

4. (20 Points) Consider the three vectors $\mathbf{v}^1 = (1, -1, 0, 1, 1)^T$, $\mathbf{v}^2 = (3, -3, 2, 5, 5)^T$, and $\mathbf{v}^3 = (5, -1, 3, 2, 8)^T$ and set $\mathbf{W} = \text{Span}\{ \mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3 \}$ in $\mathbb{R}^5$. Use the Gram-Schmidt process to construct an orthogonal basis for $\mathbf{W}$. *Remark:* You do not need to simplify any fractions in the resulting vectors.

5. (20 Points) The scores on two midterm tests and a final are give in the following table for 5 students.

Student	Test 1	Test 2	Final
Amy	76	24	86
Jessica	92	92	180
John	68	82	128
Noelle	86	68	138
Wynn	54	70	100

Write down the normal equation that gives the best fit of the form $F = c_0 + c_1 T_1 + c_2 T_2$, where F is the score on the final, T_1 is the score on the first midterm test, and T_2 is the score on the second midterm test. Put in the explicit numbers from the table, but you do not need to solve the equations nor multiply any matrices.

(**Over for problems 6-9**)

6. (20 Points) Let $\mathbb{P}_3$ be the set of all polynomials of degree less than or equal to three, which has a standard basis of $\mathscr{B} = \{1, t, t^2, t^3\}$. *Prove* that the set of polynomials
$\mathscr{S} = \{\, 1 + t^3,\ t,\ 1 + 7t + t^2,\ 1 + t + t^2 + t^3 \,\}$ forms another basis of $\mathbb{P}_3$. *Explain your answer.*

7. (20 Points) Assume that $\{\, \mathbf{v}^1, \ldots, \mathbf{v}^n \,\}$ is an orthonormal basis of $\mathbb{R}^n$ and $\mathbf{W} = \mathrm{Span}\{\, \mathbf{v}^1, \ldots, \mathbf{v}^k \,\}$ with $1 \le k < n$. *Prove* that $\{\, \mathbf{v}^{k+1}, \ldots, \mathbf{v}^n \,\}$ is a basis of $\mathbf{W}^\perp$.
Hint: You must show that $\mathbf{v}^j \in \mathbf{W}^\perp$ for $k + 1 \le j \le n$ and that they form a basis. You may use the Orthogonal Decomposition Theorem that says that any $\mathbf{y} \in \mathbb{R}^n$ can be uniquely written as $\mathbf{y} = \mathbf{z} + \mathrm{proj}_{\mathbf{W}}\, \mathbf{y}$ where $\mathbf{z} \in \mathbf{W}^\perp$ and $\mathrm{proj}_{\mathbf{W}}\, \mathbf{y} \in \mathbf{W}$.

8. (20 Points) Assume that $\mathbf{T} : \mathbf{V} \to \mathbf{W}$ is a linear transformation from the vector space $\mathbf{V}$ onto the vector space $\mathbf{W}$ and that $\{\, \mathbf{v}^1, \ldots, \mathbf{v}^k \,\}$ is a set of vectors that span $\mathbf{V}$. *Prove* that $\{\, \mathbf{T}(\mathbf{v}^1), \ldots, \mathbf{T}(\mathbf{v}^k) \,\}$ spans $\mathbf{W}$.

9. (40 Points) Indicate which of the following statements are always *true* and which are *false*. *Justify* each answer by a counterexample or explanation. Refer to any theorem by an informal statement, not by a theorem number.

- **a**. A square matrix with orthogonal columns is an orthogonal matrix.
- **b**. If $\mathbf{W}$ is a subspace of $\mathbb{R}^n$, then $\|\, \mathrm{proj}_{\mathbf{W}}\, \mathbf{v}\|^2 + \|\mathbf{v} - \mathrm{proj}_{\mathbf{W}}\, \mathbf{v}\|^2 = \|\mathbf{v}\|^2$ for a vector $\mathbf{v} \in \mathbb{R}^n$.
- **c**. If $\mathbf{A}$ is a 5×4 matrix with $\mathrm{Nul}(\mathbf{A}) = \mathrm{Span}\{(-1, 1, 1, 0)^T\}$, then the rank of $\mathbf{A}$ is 3.
- **d**. If $\mathbf{A}$ is a 5×4 matrix with $\mathrm{Nul}(\mathbf{A}) = \mathrm{Span}\{(-1, 1, 1, 0)^T\}$, then the dimension of the column space is 4.
- **e**. If $\mathbf{A}$ is a 5×4 matrix and $\mathbf{B}$ is a 4×5 matrix, then $\mathbf{AB}$ cannot be invertible.
- **f**. If $\mathbf{A}$ is a 4×4 matrix, then $\det(2\mathbf{A}) = 2\det(\mathbf{A})$.
- **g**. If $\mathbf{A}$ is $n \times n$ and $\det(\mathbf{A}) = 0$, then $\mathbf{Ax} = \mathbf{b}$ is consistent for every $\mathbf{B} \in \mathbb{R}^n$.
- **h**. If $\mathbf{A}$ is an $m \times n$ matrix with $m \ne n$ and $\mathrm{rank}(\mathbf{A}) = m$, then the linear transformation for $\mathbf{A}$ is one to one.
- **i**. The transpose of an invertible matrix is invertible.
- **j**. If $\mathbf{A}$ is an $n \times n$ matrix with $\det(\mathbf{A}) \ne 0$, then $\mathbf{A}^9$ has linearly independent columns.