Math 285-1 **Test 1: October 24, 2006** **C. Robinson**

1. (24 Points) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 3 & 2 & 5 & 8 & -4 \\ 0 & 1 & 1 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 19 \end{bmatrix}.$$

Find the general parametric vector solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Answer:

We row reduce the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1 \\ 3 & 2 & 5 & 8 & -4 & 9 \\ 0 & 1 & 1 & 5 & 6 & 19 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 2 & 2 & 2 & -4 & 6 \\ 0 & 1 & 1 & 5 & 6 & 19 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 4 & 8 & 16 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & -4 & -7 \\ 0 & 1 & 1 & 0 & -4 & -1 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

These give the equation

$$\begin{aligned} x_1 &= -x_3 + 4x_5 - 7 \\ x_2 &= -x_3 + 4x_5 - 1 \\ x_4 &= -2x_5 + 4 \end{aligned}$$

which gives the vector parametric form of the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -7 \\ -1 \\ 0 \\ 4 \\ 0 \end{bmatrix}.$$

2. (16 Points) Let T be a linear transformation with $m \times n$ matrix $\mathbf{A}$. Complete the following sentences:

a. T is one-to-one if and only if $\mathbf{A}$ has __________ pivot positions.

b. T is onto if and only if $\mathbf{A}$ has __________ pivot positions.

c. The columns of $\mathbf{A}$ span the codomain of T if and only if $\mathbf{A}$ has __________ pivot positions.

d. The columns of $\mathbf{A}$ are linearly independent if and only if $\mathbf{A}$ has __________ pivot positions.

Answer: (**a**) n (**b**) m (**c**) m (**d**) n

3. (28 Points) Indicate which of the following statements are always true and which are false (not always true). If the statement is true, give a SHORT justification. If the statement is false, give a SHORT counterexample or explanation. Use complete sentences. Refer to any theorem by an informal statement, not by a theorem number.

a. If the three vectors $\mathbf{v}^1$, $\mathbf{v}^2$, and $\mathbf{v}^3$ are linearly dependent in $\mathbb{R}^n$, then one of these three vectors can be written as a linear combination of the other two vectors.

b. If there exist $n \times n$ matrices $\mathbf{A}$ and $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}_n$, then there is a nontrivial solution of $\mathbf{Ax} = \mathbf{0}$.

c. If $\mathbf{C}$ is a diagonal 3×3 matrix with nonzero entries and $\mathbf{A}$ is another 3×3 matrix, then the matrix product $\mathbf{AC}$ scales the rows of $\mathbf{A}$.

d. If $\mathbf{A}$ is an $m \times n$ matrix such that the equation $\mathbf{Ax} = \mathbf{b}$ has at least two different solutions, and if the equation $\mathbf{Ax} = \mathbf{c}$ is consistent, then $\mathbf{Ax} = \mathbf{c}$ has infinitely many solutions.

Answer:

(**a**) True. If three vectors are linearly dependent then there are three constants, c_1, c_2, and c_3, not all zero, such that $c_1\mathbf{v}^1 + c_2\mathbf{v}^2 + c_3\mathbf{v}^3 = \mathbf{0}$. Assume $c_3 \neq 0$. then $c_3\mathbf{v}^3 = -c_1\mathbf{v}^1 - c_2\mathbf{v}^2$ and $\mathbf{v}^3 = -{}^{c_1}/_{c_3}\mathbf{v}^1 - -{}^{c_2}/_{c_3}\mathbf{v}^2$. Thus, $\mathbf{v}^3$ is a linear combination of the other two. If a different constant is nonzero, a similar result holds for the corresponding vector.

(**b**) False. If there exists such a matrix $\mathbf{D}$, then $\mathbf{A}$ is invertible and has only trivial solutions of the homogeneous equation.

(**c**) False. The product scales the columns not the rows. For example,

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 20 & 49 \\ 6 & 25 & 56 \\ 9 & 30 & 63 \end{bmatrix}.$$

(**d**) True. If $\mathbf{Ax} = \mathbf{b}$ has two solutions, then $\mathbf{A}$ are fewer pivots than columns. Then $\mathbf{Ax} = \mathbf{c}$ has fewer pivots than columns and there is a free variable. Thus there are infinitely many solutions.

4. (32 Points)

a. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Find $\mathbf{A}^{-1}$.

b. Write the two vector equations $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a single matrix equation with c_1, c_2, c_3, and c_4 as entries of a matrix.

c. Use the answers to parts (a) and (b) to write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

d. Let T be a linear transformation of $\mathbb{R}^2$ such that the images of the two vectors $(1, 1)^T$ and $(1, 2)^T$ by T are $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Use the answer to part (c) and the linearity of T to find $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, and the matrix of T.

Answer:

(**a**) $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

(**b**) These two vector equations can be written as the following two matrix equations:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix}.$$

These can be combined into the single matrix equation

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 & c_3 \\ c_2 & c_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(**c**) The matrix of the c_k must be the inverse of $\mathbf{A}$, so $c_1 = 2$, $c_2 = c_3 = -1$, and $c_4 = 1$:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and}$$
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(**d**)

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = 2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = -T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}.$$

Thus, the matrix of T is

$$\begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}.$$