Math 285-1 **Test 1: October 24, 2006** **C. Robinson**

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1. (24 Points) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 3 & 2 & 5 & 8 & -4 \\ 0 & 1 & 1 & 5 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 9 \\ 19 \end{bmatrix}.$$

Find the general parametric vector solution of $\mathbf{Ax} = \mathbf{b}$.

2. (16 Points) Let T be a linear transformation with $m \times n$ matrix $\mathbf{A}$. Complete the following sentences:

a. T is one-to-one if and only if $\mathbf{A}$ has __________ pivot positions.

b. T is onto if and only if $\mathbf{A}$ has __________ pivot positions.

c. The columns of $\mathbf{A}$ span the codomain of T if and only if $\mathbf{A}$ has __________ pivot positions.

d. The columns of $\mathbf{A}$ are linearly independent if and only if $\mathbf{A}$ has __________ pivot positions.

3. (28 Points) Indicate which of the following statements are always true and which are false (not always true). If the statement is true, give a SHORT justification. If the statement is false, give a SHORT counterexample or explanation. Use complete sentences. Refer to any theorem by an informal statement, not by a theorem number.

a. If the three vectors $\mathbf{v}^1$, $\mathbf{v}^2$, and $\mathbf{v}^3$ are linearly dependent in $\mathbb{R}^n$, then one of these three vectors can be written as a linear combination of the other two vectors.

b. If there exist $n \times n$ matrices $\mathbf{A}$ and $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}_n$, then there is a nontrivial solution of $\mathbf{Ax} = \mathbf{0}$.

c. If $\mathbf{C}$ is a diagonal 3×3 matrix with nonzero entries and $\mathbf{A}$ is another 3×3 matrix, then the matrix product $\mathbf{AC}$ scales the rows of $\mathbf{A}$.

d. If $\mathbf{A}$ is an $m \times n$ matrix such that the equation $\mathbf{Ax} = \mathbf{b}$ has at least two different solutions, and if the equation $\mathbf{Ax} = \mathbf{c}$ is consistent, then $\mathbf{Ax} = \mathbf{c}$ has infinitely many solutions.

4. (32 Points)

a. Let $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Find $\mathbf{A}^{-1}$.

b. Write the two vector equations $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as a single matrix equation with c_1, c_2, c_3, and c_4 as entries of a matrix.

c. Use the answers to parts (a) and (b) to write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

d. Let T be a linear transformation of $\mathbb{R}^2$ such that the images of the two vectors $(1, 1)^T$ and $(1, 2)^T$ by T are $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$. Use the answer to part (c) and the linearity of T to find $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$, and the matrix of T.