Math 285-1 **Test 1: October 24, 2007** **C. Robinson**

1. (18 Points) Let

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & -2 \\ 2 & 2 & -1 & 0 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix}.$$

Find the general parametric vector solution of $\mathbf{Ax} = \mathbf{b}$.

Answer:

We row reduce the augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & -2 & -4 \\ 2 & 2 & -1 & 0 & 3 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & -2 & -4 \\ 0 & 0 & -1 & 2 & 3 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & -2 & -4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right].$$

These give the equation

$$\begin{aligned} x_1 &= -x_2 + x_4 + 1 \\ x_3 &= 2x_4 + 2 \\ x_5 &= 3 \end{aligned}$$

which gives the vector parametric form of the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix}.$$

2. (18 Points) Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Answer:
To find the inverse, we row reduced the following augmented matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ -2 & -3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & 2 & -1 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & -4 & 2 \\ 0 & 1 & 0 & 4 & 2 & -1 \\ 0 & 0 & 1 & -2 & -1 & 1 \end{array}\right].$$

Therefore, the inverse is $\begin{bmatrix} -7 & -4 & 2 \\ 4 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix}$.

3. (14 Points) Give the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $T(\mathbf{e}^1) = 2\mathbf{e}^1 + \mathbf{e}^2$ and $T(\mathbf{e}^1 + \mathbf{e}^2) = \mathbf{e}^1 + \mathbf{e}^2$.

Answer:
$\mathbf{e}^1+\mathbf{e}^2 = T(\mathbf{e}^1+\mathbf{e}^2) = T(\mathbf{e}^1)+T(\mathbf{e}^2) = 2\mathbf{e}^1+\mathbf{e}^2+T(\mathbf{e}^2)$ so $T(\mathbf{e}^2) = \mathbf{e}^1+\mathbf{e}^2-\left(2\mathbf{e}^1 + \mathbf{e}^2\right) = -\mathbf{e}^1$. Therefore, the standard matrix is $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$.

4. (20 Points) Complete the sentences below by defining the italicized term. Do not quote a theorem giving conditions equivalent to the definition; give the definition itself.

a. A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear transformation* provided that (**Answer**) $T(c_1\mathbf{u} + c_2\mathbf{v}) = c_1T(\mathbf{u}) + c_2T(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars c_1, c_2.

b. A set of vectors $\left\{\mathbf{v}^1, \ldots, \mathbf{v}^p\right\}$ in $\mathbb{R}^m$ is *linearly dependent* provided that (**Answer**) there exist scalars $c_1, \ldots, c_p$ that are not all zero such that $c_1\mathbf{v}^1 + \cdots + c_p\mathbf{v}^p = \mathbf{0}$.

5. (30 Points) Indicate which of the following statements are always true (T) and which are false (F). Justify each answer by a counterexample or explanation. Refer to any theorem by an informal statement, not by a theorem numbers.

a. If $\mathbf{A}$ is an $m \times n$ matrix and the equation $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b} \in \mathbb{R}^m$, then the columns of $\mathbf{A}$ span $\mathbb{R}^m$.

Answer: False. A counterexample is $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ for which the equation is consistent but the columns of $\mathbf{A}$ do not span $\mathbb{R}^2$.

b. If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ in some $\mathbb{R}^n$, then $\operatorname{span}\{\mathbf{u}, \mathbf{v}\} = \operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Answer: True. Assume $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$. If $\mathbf{x}$ is any vector in $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, then $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = (c_1 + c_3 a)\mathbf{u} + (c_2 + c_3 b)\mathbf{v}$ is in $\operatorname{span}\{\mathbf{u}, \mathbf{v}\}$. The other inclusion is obvious, so the two spans are equal.

c. If the system $\mathbf{Ax} = \mathbf{b}$ has a unique solution, then $\mathbf{A}$ must be a square matrix.

Answer: False. A counterexample is given by the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for which $\mathbf{Ax} = \mathbf{b}$ has a unique solution but $\mathbf{A}$ is not square.

d. Any set of three vectors $\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ in $\mathbb{R}^2$ are linearly dependent.

Answer: True. Let $\mathbf{A}$ be the 2×3 matrix that has the $\mathbf{v}^j$ as columns. This matrix can have at most a rank of 2. Therefore, not every column is a pivot column and the homogeneous equation for $\mathbf{A}$ has a nontrivial solution. This shows that the columns are linearly dependent.

e. If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrices for which $\mathbf{AB} = \mathbf{C}$ and $\mathbf{C}$ has 2 columns, then $\mathbf{A}$ has 2 columns.

Answer: False. If the matrix $\mathbf{A}$ is $m \times n$ and the matrix $\mathbf{B}$ is $n \times 2$ for any m and n, then $\mathbf{C}$ is $m \times 2$. Therefore, $\mathbf{A}$ can have any number of columns. (The matrix $\mathbf{B}$ must have two columns.)

f. If $\mathbf{A}$ is a 5×3 matrix and $\mathbf{C}$ is a 3×5 matrix such that $\mathbf{CA} = \mathbf{I}$, then the linear transformation $\mathbf{x} \mapsto \mathbf{Ax}$ is one-to-one.

Answer: True. Assume that $\mathbf{Ax}^1 = \mathbf{Ax}^2$, then $\mathbf{x}^1 = \mathbf{CAx}^1 = \mathbf{CAx}^2 = \mathbf{x}^2$. This shows that the transformation is one-to-one. (The transformation cannot be onto, since $\mathbf{A}$ can have at most rank 3. The transformation is not invertible.)