Math 285-2 **Test 1: January 30, 2007** **C. Robinson**

1. (22 Points) Let $\mathbf{W}$ be the subspace spanned by the vectors $\mathbf{v}^1 = (1, 0, \text{-}1, 1)^T$ and $\mathbf{v}^2 = (0, 1, 2, 2)^T$.

a. Find the orthogonal projection of $\mathbf{y} = (6, 3, 3, 0)^T$ onto $\mathbf{W}$.

b. Find the distance of the point $\mathbf{y}$ to the subspace $\mathbf{W}$. You may leave a square root in your answer.

Answer:

a. $\mathbf{y} \cdot \mathbf{v}^1 = 6 + 0 - 3 = 3$, $\mathbf{v}^1 \cdot \mathbf{v}^1 = 3$, $\mathbf{y} \cdot \mathbf{v}^2 = 0 + 3 + 6 = 9$, $\mathbf{v}^2 \cdot \mathbf{v}^2 = 9$,

$$\begin{aligned}\hat{\mathbf{y}} &= \left(\frac{\mathbf{y} \cdot \mathbf{v}^1}{\mathbf{v}^1 \cdot \mathbf{v}^1}\right)\mathbf{v}^1 + \left(\frac{\mathbf{y} \cdot \mathbf{v}^2}{\mathbf{v}^2 \cdot \mathbf{v}^2}\right)\mathbf{v}^2 \\ &= \left(\frac{3}{3}\right)\begin{bmatrix}1\\0\\\text{-}1\\1\end{bmatrix} + \left(\frac{9}{9}\right)\begin{bmatrix}0\\1\\2\\2\end{bmatrix} \\ &= \begin{bmatrix}1\\1\\1\\3\end{bmatrix}.\end{aligned}$$

b. The distance is $\|\mathbf{y} - \hat{\mathbf{y}}\| = \|(6 - 1, 3 - 1, 3 - 1, 0 - 3)^T\| = \sqrt{25 + 4 + 4 + 9} = \sqrt{42}$.

2. (18 Points) Consider the three vectors $\mathbf{v}^1 = (2, 2, 1, 0)^T$, $\mathbf{v}^2 = (0, 3, 3, 1)^T$, and $\mathbf{v}^3 = (3, 0, 3, \text{-}10)^T$. Construct an orthogonal set of vectors $\{\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3\}$ such that $\text{Span}(\mathbf{w}^1) = \text{Span}(\mathbf{v}^1)$, $\text{Span}(\mathbf{w}^1, \mathbf{w}^2) = \text{Span}(\mathbf{v}^1, \mathbf{v}^2)$, and $\text{Span}(\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3) = \text{Span}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3)$.

Answer:

Let $\mathbf{w}^1 = \mathbf{v}^1 = (2, 2, 1, 0)^T$. Then $\mathbf{w}^1 \cdot \mathbf{v}^2 = 0 + 6 + 3 + 0 = 9$, $\mathbf{w}^1 \cdot \mathbf{w}^1 = 4 + 4 + 1 + 1 = 9$, so $\mathbf{w}^2 = (0, 3, 3, 1)^T - \left(\frac{9}{9}\right)(2, 2, 1, 0)^T = (\text{-}2, 1, 2, 1)^T$. $\mathbf{w}^1 \cdot \mathbf{v}^3 = 6 + 0 + 3 + 0 = 9$, $\mathbf{w}^2 \cdot \mathbf{v}^3 = \text{-}6 + 0 + 6 - 10 = \text{-}10$, $\mathbf{w}^2 \cdot \mathbf{w}^2 = 4 + 1 + 4 + 1 = 10$, so $\mathbf{w}^3 = (0, 3, 3, 1)^T - \left(\frac{9}{9}\right)(2, 2, 1, 0)^T - \left(\frac{\text{-}10}{10}\right)(\text{-}2, 1, 2, 1)^T = (\text{-}1, \text{-}1, 4, \text{-}9)^T$.

3. (22 Points) Determine whether the following two quadratic forms are positive definite, negative definite, or indefinite. Show your work.

$$\begin{aligned}&(\mathbf{a}) \quad Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_3 + 4x_2x_3 \\ &(\mathbf{b}) \quad Q(x_1, x_2, x_3) = -x_1^2 - 2x_2^2 - 4x_3^2 + 2x_1x_3 + 4x_2x_3\end{aligned}$$

Answer:

a. The matrix of the quadratic form is

$$\begin{aligned}\begin{bmatrix}1 & 0 & 1\\0 & 2 & 2\\1 & 2 & 2\end{bmatrix} &\sim \begin{bmatrix}1 & 0 & 1\\0 & 2 & 2\\0 & 2 & 1\end{bmatrix} \\ &\sim \begin{bmatrix}1 & 0 & 1\\0 & 2 & 2\\0 & 0 & \text{-}1\end{bmatrix}.\end{aligned}$$

The pivots are 1, 2, and -1. They do not have all one sign, so the quadratic form is indefinite.

b. The matrix of the quadratic form is

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \\ 1 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 2 & -3 \end{bmatrix}$$
$$\sim \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{bmatrix}.$$

The pivots are -1, -2, and -1. They are all negative, so the quadratic form is negative definite.

4. (24 Points) Indicate which of the following statements are always true (T) and which are false (F). Justify each answer by referring to an theorem, fact, or counterexample.

a. If $\{\mathbf{v}^1, \ \mathbf{v}^2, \ \mathbf{v}^3\}$ is an orthogonal set of three nonzero vectors, then they are linearly independent.

b. If a 3×2 matrix $\mathbf{U}$ has orthonormal columns, then $\mathbf{U}\mathbf{U}^T = \mathbf{I}$.

c. Assume that $\mathbf{A}$ is a symmetric matrix with eigenvectors $\mathbf{v}^1$ and $\mathbf{v}^2$ for distinct eigenvalues $\lambda_1 \neq \lambda_2$. Then $\mathbf{v}^1$ and $\mathbf{v}^2$ are orthogonal.

d. Let $\mathbf{A}$ be an 5×5 symmetric matrix, $\mathbf{B}_1$ be 3×3 with $\det(\mathbf{B}_1) \neq 0$, $\mathbf{B}_2$ be 3×2, and $\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2]$. Let $\mathbf{H}_8 = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A} \end{bmatrix}$ be the bordered Hessian and $\mathbf{H}_7$ be the 7×7 principal submatrix. If $\det(\mathbf{H}_8) > 0$ and $\det(\mathbf{H}_7) > 0$, then the quadratic form of $\mathbf{A}$ is positive definite on the null space of $\mathbf{B}$.

Answer:

a. True. If $\mathbf{0} = c_1\mathbf{v}^1 + c_2\mathbf{v}^2 + c_3\mathbf{v}^3$, then taking the dot product with $\mathbf{v}^i$ we get $0 = c_1\mathbf{v}^i \cdot \mathbf{v}^i$. Since $\mathbf{v}^i \cdot \mathbf{v}^i = \|\mathbf{v}^i\|^2 > 0$, this implies that $c_i = 0$. Thus all the coefficients are necessarily zero.

b. False. For example take

$$\mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{so}$$
$$\mathbf{U}\mathbf{U}^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \mathbf{I}.$$

c. True. $(\mathbf{v}^1)^T\mathbf{A}\mathbf{v}_2 = (\mathbf{v}^1)^T\lambda_2\mathbf{v}^2 = \lambda_2(\mathbf{v}^1)^T\mathbf{v}^2$. But it also equals $(\mathbf{A}\mathbf{v}^1)^T\mathbf{v}^2 = (\lambda_1\mathbf{v}^1)^T\mathbf{v}^2 = \lambda_1(\mathbf{v}^1)^T\mathbf{v}^2$. Therefore, $(\lambda_1 - \lambda_2)(\mathbf{v}^1)^T\mathbf{v}^2 = 0$. Since $\lambda_1 - \lambda_2 \neq 0$, it follows that $(\mathbf{v}^1)^T\mathbf{v}^2 = 0$ and the two vectors are orthogonal.

d. False. $k = 3$ and $n = 5$. Thus, $(-1)^3 \det(\mathbf{H}_8) < 0$, and it would have to be positive to be positive definite.

5. (14 Points) Let $\mathbf{A}$ be an $k \times n$ matrix. *Prove* that $\mathrm{Col}(\mathbf{A})^\perp = \mathrm{Nul}(\mathbf{A}^T)$.

Answer:

A vector $\mathbf{v} \in \mathrm{Col}(\mathbf{A})^\perp$ iff $\mathbf{a}^i \cdot \mathbf{v} = 0$ for each column of $\mathbf{A}$ iff $(\mathbf{a}^i)^T\mathbf{v} = 0$ for each column of $\mathbf{A}$ iff $\mathbf{A}^T\mathbf{v} = \mathbf{0}$ iff $\mathbf{v} \in \mathrm{Nul}(\mathbf{A}^T)$. Therefore, these two subspaces are equal.