Math 285-2 **Test 1: January 30, 2007** **C. Robinson**

No calculators, no books, no notes
Show all your work in your bluebook. Start each problem on a new page.

1. (22 Points) Let $\mathbf{W}$ be the subspace spanned by the vectors $\mathbf{v}^1 = (1, 0, -1, 1)^T$ and $\mathbf{v}^2 = (0, 1, 2, 2)^T$.

a. Find the orthogonal projection of $\mathbf{y} = (6, 3, 3, 0)^T$ onto $\mathbf{W}$.

b. Find the distance of the point $\mathbf{y}$ to the subspace $\mathbf{W}$. You may leave a square root in your answer.

2. (18 Points) Consider the three vectors $\mathbf{v}^1 = (2, 2, 1, 0)^T$, $\mathbf{v}^2 = (0, 3, 3, 1)^T$, and $\mathbf{v}^3 = (3, 0, 3, -10)^T$. Construct an orthogonal set of vectors $\{\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3\}$ such that $\text{Span}(\mathbf{w}^1) = \text{Span}(\mathbf{v}^1)$, $\text{Span}(\mathbf{w}^1, \mathbf{w}^2) = \text{Span}(\mathbf{v}^1, \mathbf{v}^2)$, and $\text{Span}(\mathbf{w}^1, \mathbf{w}^2, \mathbf{w}^3) = \text{Span}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3)$.

3. (22 Points) Determine whether the following two quadratic forms are positive definite, negative definite, or indefinite. Show your work.

$$(\mathbf{a}) \quad Q(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_3 + 4x_2x_3$$

$$(\mathbf{b}) \quad Q(x_1, x_2, x_3) = -x_1^2 - 2x_2^2 - 4x_3^2 + 2x_1x_3 + 4x_2x_3$$

4. (24 Points) Indicate which of the following statements are always true (T) and which are false (F). Justify each answer by referring to an theorem, fact, or counterexample.

a. If $\{\mathbf{u}^1, \mathbf{u}^2, \mathbf{u}^3, \mathbf{u}^4\}$ is a basis of $\mathbb{R}^4$ with each $\mathbf{u}^i$ a unit vector, then any vector $\mathbf{y}$ in $\mathbb{R}^4$ can be written as $\mathbf{y} = \left(\mathbf{u}^1 \cdot \mathbf{y}\right)\mathbf{u}^1 + \left(\mathbf{u}^2 \cdot \mathbf{y}\right)\mathbf{u}^2 + \left(\mathbf{u}^3 \cdot \mathbf{y}\right)\mathbf{u}^3 + \left(\mathbf{u}^4 \cdot \mathbf{y}\right)\mathbf{u}^4$.

b. A least squares solution of $\mathbf{Ax} = \mathbf{b}$ is a vector $\hat{\mathbf{x}}$ that satisfies $\mathbf{A}\hat{\mathbf{x}} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto $\text{Col}(\mathbf{A})$.

c. Assume that $\mathbf{A}$ is a symmetric matrix with eigenvectors $\mathbf{v}^1$ and $\mathbf{v}^2$ for distinct eigenvalues $\lambda_1 \neq \lambda_2$. Then $\mathbf{v}^1$ and $\mathbf{v}^2$ are orthogonal.

d. Let $\mathbf{A}$ be an 5×5 symmetric matrix, $\mathbf{B}_1$ be 3×3 with $\det(\mathbf{B}_1) \neq 0$, $\mathbf{B}_2$ be 3×2, and $\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2]$. Let $\mathbf{H}_8 = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{A} \end{bmatrix}$ be the bordered Hessian and $\mathbf{H}_7$ be the 7×7 principal submatrix. If $\det(\mathbf{H}_8) > 0$ and $\det(\mathbf{H}_7) > 0$, then the quadratic form of $\mathbf{A}$ is positive definite on the null space of $\mathbf{B}$.

5. (14 Points) Let $\mathbf{A}$ be an $k \times n$ matrix. *Prove* that $\text{Col}(\mathbf{A})^{\perp} = \text{Nul}(\mathbf{A}^T)$.