Math 285-1 **Test 2: November 14, 2006** **C. Robinson**

1. (12 Points) Calculate the determinant $\det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 2 & -4 & 4 & 8 \\ 3 & -4 & 2 & 5 \\ 0 & 2 & -4 & -9 \end{bmatrix}$.

Answer:

$$\det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 2 & -4 & 4 & 8 \\ 3 & -4 & 2 & 5 \\ 0 & 2 & -4 & -9 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & -1 & -7 \\ 0 & 2 & -4 & -9 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & -4 & -9 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & -2 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 2 & -1 & -7 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$$= -(1)(2)(2)(\text{-}2) = 8.$$

2. (24 Points) The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & 4 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & -1 & 0 & 0 & 2 \\ 2 & 1 & 6 & 0 & 1 \end{bmatrix}$ has the reduced echelon form $\mathbf{U} = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. Find a basis for the nullspace of A.

b. Find a basis for the column space of A.

c. Find a basis for the row space of A.

Answer:

(a) Using the free variables and the reduced echelon form, we get the following basis of Nul($\mathbf{A}$):

$$\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(b) Using the pivot columns of the original matrix $\mathbf{A}$ we get the following basis of Col($\mathbf{A}$):

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(c) Using the nonzero rows of the reduced echelon matrix $\mathbf{U}$, we get the basis of Row($\mathbf{A}$) as

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

3. (32 Points) Indicate which of the following statements are always true and which are false (not always true). If the statement is true, give a SHORT justification. If the statement is false, give a SHORT counterexample or explanation. Use complete sentences. Refer to any theorem by an informal statement, not by a theorem number.

a. If $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices with $\det(\mathbf{B}) \neq 0$, then $\det(\mathbf{A}\mathbf{B}^{-1}) = \det(\mathbf{A})/\det(\mathbf{B})$.

Answer: True
By properties of the determinant of products and inverses, $\det(\mathbf{A}\mathbf{B}^{-1}) = \det(\mathbf{A})\det(\mathbf{B}^{-1}) = \det(\mathbf{A})/\det(\mathbf{B})$.

b. If the columns of a 3×3 matrix $\mathbf{A}$ determine a parallelepiped in $\mathbb{R}^3$ of volume 10, then the volume of the parallelepiped determined by the columns of $2\mathbf{A}$ is 20.

Answer: False
There are three rows that are multiplied by 2, so the $\det(2\mathbf{A}) = 8\ \det \mathbf{A})$, so the volume is 80 and not 20.

c. If $\mathbf{A}$ is a 3×5 matrix with rank 2, then the dimension of the null space of $\mathbf{A}$ is 2.

Answer: False
The $\dim(\text{Nul}(\mathbf{A})) = 5 - \text{Rank}(\mathbf{A}) = 5 - 2 = 3$ not 2.

d. If $\mathbf{A}$ is a 5×3 matrix, then the rows of $\mathbf{A}$ must be linearly dependent.

Answer: True
The rank can be at most 3, so there can be at most 3 linearly independent rows. Therefore, the 5 rows must be linearly dependent.

e. Let $\mathbf{W}$ be a subspace of $\mathbf{V}$, $\dim(\mathbf{W}) = 4$, and $\dim(\mathbf{V}) = 7$. Every basis of $\mathbf{W}$ can be extended to a basis of $\mathbf{V}$ by adding three more vectors to it.

Answer: True
The vectors in the basis of $\mathbf{W}$ are linearly independent as vectors in $\mathbf{V}$. By the theorem on the extension of linearly independent vectors, they can be extended to a basis of $\mathbf{V}$.

f. Let $\mathbf{W}$ be a subspace $\mathbf{V}$, $\dim(\mathbf{W}) = 4$, and $\dim(\mathbf{V}) = 7$. Every basis $\mathscr{B}$ of $\mathbf{V}$ can be reduced to a basis of $\mathbf{W}$ by removing three vectors from $\mathscr{B}$.

Answer: False
None of the vectors of the original basis needs to be an element of $\mathbf{W}$, so they cannot always be made into a basis of $\mathbf{W}$. For example, Let $\mathbf{W} = \{ (x_1, x_2, x_3, x_4, 0, 0, 0)^T \}$ and $\dim(\mathbf{V}) = \mathbb{R}^7$. The original basis of $\mathbf{V}$ could be made up of the vectors $(1, 0, 0, 0, 1, 0, 0)^T$, $(0, 1, 0, 0, 1, 0, 0)^T$, $(0, 0, 1, 0, 1, 0, 0)^T$, $(0, 0, 0, 1, 1, 0, 0)^T$, $(0, 0, 0, 0, 1, 0, 0)^T$, $(0, 0, 0, 0, 0, 1, 0)^T$, and $(0, 0, 0, 0, 0, 0, 1)^T$.

g. The set of all polynomials $p(t)$ with the property that $p(1) = 1$ is a subspace of the vector space $\mathbb{P}$ of all polynomials.

Answer: False
The zero polynomial is not in this set, $\mathscr{O}(1) = 0 \neq 1$. Also, if $p(t)$ and $q(t)$ are in this set then the sum is not in this set, $(p + q)(1) = p(1) + q(1) = 2 \neq 1$.

h. If $\mathbf{H}$ is a subspace of $\mathbb{R}^3$ that is not the zero subspace, $\mathbf{H} \neq \{\mathbf{0}\}$, then there is a matrix $\mathbf{A}$ such that $\mathbf{H} = \mathrm{Col}(\mathbf{A})$.

Answer: True

Take a basis of $\mathbf{H}$ and put them into the columns of a matrix $\mathbf{A}$. Then, for this matrix, $\mathrm{Col}(\mathbf{A}) = \mathbf{H}$.

4. (14 Points) Assume $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{V} \subset \mathbb{R}^n$ is a subspace. Show directly from the definition of a subspace that the set $\mathbf{A}(\mathbf{V}) = \{\mathbf{A}\mathbf{v} : \mathbf{v} \in \mathbf{V}\}$ is a subspace of $\mathbb{R}^m$.

Answer:

With the wording of the question, the matrix $\mathbf{A}$ and the subspace $\mathbf{V}$ are fixed. The vector in the subspace can vary.

The zero vector $\mathbf{0}_n$ is an element of the subspace $\mathbf{V}$, so $\mathbf{A}\mathbf{0}_n = \mathbf{0}_m$ is an element of $\mathbf{A}(\mathbf{V})$.

If $\mathbf{A}\mathbf{v}^1$ and $\mathbf{A}\mathbf{v}^2$ are two elements of $\mathbf{A}(\mathbf{V})$ and r and s are real numbers, then $r\mathbf{v}^1 + s\mathbf{v}^2$ is an element of $\mathbf{V}$ (because it is a subspace), so $r\mathbf{A}\mathbf{v}^1 + s\mathbf{A}\mathbf{v}^2 = \mathbf{A}(r\mathbf{v}^1 + s\mathbf{v}^2)$ is an element of $\mathbf{A}(\mathbf{V})$.

Thus, $\mathbf{A}(\mathbf{V})$ contains the zero vector and is closed under linear combinations. Therefore, it is a subspace of $\mathbb{R}^m$.

5. (18 Points) Consider the set of polynomials $\mathbf{S} = \{1+t^2, 1+2t, 1+3t^2\}$ in $\mathbb{P}_2$, the set of polynomials of degree ≤ 2.

a. Prove that the set $\mathbf{S}$ is linearly independent.

b. Prove that $\mathbf{S}$ is a basis for $\mathbb{P}_2$.

c. Find the coordinate vector $[p(t)]_{\mathbf{S}}$ of the polynomial $p(t) = 7 + 4t + 9t^2$ relative to the basis $\mathbf{S}$.

Answer:

(a) The standard basis of $\mathbb{P}_2$ is $\mathscr{B} = \{1, t, t^2\}$. Taking the image of the polynomials in $\mathbf{S}$ by the coordinate map for the standard basis, we get the three vectors $[1+t^2]_{\mathscr{B}} = (1, 0, 1)^T$, $[1+2t]_{\mathscr{B}} = (1, 2, 0)^T$, $[1+3t^2]_{\mathscr{B}} = (1, 0, 3)^T$. Putting these three vectors in as columns of a matrix, we get $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$. This matrix has determinant equal to 4, which is not equal to zero. Therefore, the columns of the vectors in $\mathbb{R}^3$ are linearly independent, and so their corresponding polynomials in $\mathbb{P}_2$ are also linearly independent.

(b) From part (a), the set of "vectors" (or polynomials) in $\mathbf{S}$ are linearly independent in $\mathbb{P}_2$. The dimension of $\mathbb{P}_2$ (or $\mathbb{R}^3$) is 3, so three linearly independent $\mathbf{S}$ forms a basis of $\mathbb{P}_2$.

(c) The coordinate vector with respect to the standard basis is $[p(t)]_{\mathscr{B}} = (7, 4, 9)^T$. So we need to row reduced the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 2 & 0 & 4 \\ 1 & 0 & 3 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 2 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Thus, the coordinate vector in $\mathbb{R}^3$ is $[p(t)]_{\mathbf{S}} = (3, 2, 2)^T$ and the relation in $\mathbb{P}_2$ is
$7 + 4t + 9t^2 = 3(1+t^2) + 2(1+2t) + 2(1+3t^2)$.