Math 285-1 **Test 2: November 19, 2007** **C. Robinson**

1. (16 Points) *Calculate* the determinant $\det \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$.

Answer:

$$\det \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$
$$= -(1)(2)(-1)(-2) = -4.$$

2. (24 Points)

The matrix $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 & 3 & 9 \\ -1 & 0 & -2 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 \\ 2 & 1 & 1 & 1 & 2 \end{bmatrix}$ has the reduced echelon form $\mathbf{U} = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. *Give* a basis for the nullspace of **A** and its dimension.

b. *Give* a basis for the column space of **A** and its dimension.

c. *Give* a basis for the row space of **A** and its dimension.

Answer:

(**a**) The dimension of the nullspace is 2 and a basis is $\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$.

(**b**) The dimension of the column space is 3 and a basis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \right\}$.

(**c**) The dimensionof the row space is 3 and a basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \right\}$.

3. (18 Points) Assume that $\mathbf{v}^1$, $\mathbf{v}^2$, and $\mathbf{v}^3$ are three nonzero vectors in $\mathbb{R}^n$ such that $5\mathbf{v}^1 + 3\mathbf{v}^2 - \mathbf{v}^3 = \mathbf{0}$ and such that no pair of vectors is parallel. *Find* a basis of $\mathbf{W} = \text{Span}\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\}$ and *explain* why it is a basis.

Answer:

Since $\mathbf{v}^3 = 5\mathbf{v}^1 + 3\mathbf{v}^2$ is a linear combination of the first two vectors, $\text{Span}\{\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3\} = \text{Span}\{\mathbf{v}^1, \mathbf{v}^2\}$. Since $\mathbf{v}^1$ and $\mathbf{v}^2$ are not parallel, they are linearly independent. Therefore a basis of $\mathbf{W}$ is $\{\mathbf{v}^1, \mathbf{v}^2\}$.

4. (18 Points) Assume that $T : \mathbf{V} \to \mathbf{W}$ is a one-to-one linear transformation between the vector spaces $\mathbf{V}$ and $\mathbf{W}$ and that $\{ \mathbf{v}^1, \dots, \mathbf{v}^k \}$ is a set of linearly independent vectors in $\mathbf{V}$. *Prove* that $\{ T(\mathbf{v}^1), \dots, T(\mathbf{v}^k) \}$ is a set of linearly independent vectors in $\mathbf{W}$.

Answer:
Assume that $\mathbf{0} = c_1 T(\mathbf{v}^1) + \cdots + c_k T(\mathbf{v}^k) = T(c_1\mathbf{v}^1 + \cdots + c_k\mathbf{v}^k)$. Since T is one-to-one, this implies that $\mathbf{0} = c_1\mathbf{v}^1 + \cdots + c_k\mathbf{v}^k$. Since the $\mathbf{v}^j$ are linearly independent, all the $c_j = 0$ for $1 \le j \le k$. Thus, if a linear combination of the $T(\mathbf{v}^j)$ equals zero, the coefficients are all zero. This proves that the set $\{ T(\mathbf{v}^1), \dots, T(\mathbf{v}^k) \}$ is linearly independent.

5. (24 Points) *Indicate* which of the following statements are always true (T) and which are false (F). *Justify* each answer by a counterexample or explanation. Refer to any theorem by an informal statement, not by a theorem numbers.

a. If $\mathbf{v}^1$ and $\mathbf{v}^2$ are vectors in $\mathbb{R}^2$ which determine a parallelogram of area 3 and $\mathbf{A}$ is a 2×2 matrix with determinant 5, then $\mathbf{A}\mathbf{v}^1$ and $\mathbf{A}\mathbf{v}^2$ determine a parallelogram of area 8.

Answer: *False*: The area is $5 \cdot 3 = 15$ not 8.

b. If $\mathbf{A}$ is an 3×3 matrix with $\mathbf{A}^3 = \mathbf{0}$, then $\det(\mathbf{A}) = 0$.

Answer: *True*: $0 = \det(\mathbf{A}^3) = [\det(\mathbf{A})]^3$, so $\det(\mathbf{A}) = 0$.

Note that $vA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ has $\mathbf{A}^3 = \mathbf{0}$ but is not the zero matrix.

c. If $\mathbf{A}$ is an 3×3 matrix, then $\det(-\mathbf{A}) = -\det(\mathbf{A})$.

Answer: *True*: $\det(-\mathbf{A}) = (-1)^3 \det(\mathbf{A}) = -\det(\mathbf{A})$.

d. Some subset of the rows of a matrix $\mathbf{A}$ form a basis of the row space of $\mathbf{A}$.

Answer: *True*: The rows span the row space so some subset is a basis.

e. There is a basis of $\mathbb{P}_5$, the polynomials of degree less than or equal to five, that includes the two polynomials $p_1(t) = 1 + t^2 + t^4$ and $p_2(t) = t + t^3$.

Answer: *True*: The two polynomials are linearly independent in $\mathbb{P}_5$, so they can be extended to a basis.

f. If $\mathbf{A}$ is an $m \times n$ matrix with $\text{rank}(\mathbf{A}) = m$, then the transformation $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is one-to-one.

Answer: *False*: To be one-to-one, we need the $\text{rank}(\mathbf{A}) = n$. For example, the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ has rank 2, but the transformation is not one-to-one. (It is onto.)