Math 313-1 Final: March 21, 2002 R. Clark Robinson

No books, no notes. You may use hand calculators
Start each problem on a new page in the blue book.

1. (30 Points) Let $D(x)$ be the doubling map given by

$$\begin{aligned} D(x) &= 2\,x \qquad \text{(modulo) } 1 \\ &= \begin{cases} 2\,x & \text{for } 0 \le x < 0.5 \\ 2\,x - 1 & \text{for } 0.5 \le x < 1 \\ 0 & \text{for } x = 1. \end{cases} \end{aligned}$$

   a. For each of the points in the set $\{\,1/3,\ 3/10,\ 1/9,\ \pi/4\,\}$ tell whether the point is periodic, eventually periodic, or neither for D. If it is periodic or eventually periodic, give its period.
   b. How many period-k points does D have for $0 \le k \le 6$?

2. (30 Points) Let $f(x) = x/2 + x^3/2$.
   a. Find all the fixed points and classify them as attracting, repelling, or neither.
   b. Use graphical analysis (cobweb method) to describe all the orbits of points on the real line. Describe the behavior in words as well as draw the figure.

3. (40 Points) Let

$$T(x) = \begin{cases} 5\,x & x \le 0.5 \\ 5\,(1 - x) & x \ge 0.5 \end{cases}$$

   be a tent map with slope ± 5.
   a. Describe the set of points x such that x, $T(x)$, and $T^2(x)$ are in $[0, 1]$,

$$\{x : T^j(x) \in [0, 1] \text{ for } 0 \le j \le 2\,\}.$$

      It is made up of how many intervals of what length? What is its total length?
   b. What is the Lebesgue measure of the set

$$K = \{x : T^j(x) \in [0, 1] \text{ for all } j \ge 0\,\}.$$

      Lebesgue measure is the usual measure which generalizes the length of intervals.
   c. Let

$$K = \{x : T^j(x) \in [0, 1] \text{ for all } j \ge 0\,\}.$$

      Explain which numbers in $[0, 1]$ belong to K in terms of the numbers expansion base 5, i.e.,

$$x = \sum_{j=1}^{\infty} \frac{a_j}{5^j}.$$

   d. Give a number in K that is not an end point of one of the intervals in the finite process defining K.

(over for problems 4 & 5.)

4. (30 Points) The map $g_a(x) = a\,x\,(1-x)$ for $a = 3\frac{1}{6} = \frac{19}{6}$ has a period-2 orbit at

$$q_1 = \frac{10}{19} \qquad q_2 = \frac{15}{19}.$$

   a. Calculate the Schwarzian derivative,

$$S_{g_a}(x) = \frac{g_a'''(x)}{g_a'(x)} - \frac{3}{2}\left(\frac{g_a''(x)}{g_a'(x)}\right)^2.$$

   b. Find the Lyapunov exponent for the point q_1 for $g_{19/6}$. Is the orbit attracting or repelling?

   c. Find the Lyapunov exponent for the point $x_0 = g_{19/6}(0.5)$. Why is this the correct answer?

5. (70 Points) Consider the map given by

$$f(x) = \begin{cases} 4 + \frac{4}{3}x & \text{for } 0 \le x \le 3 \\ 8 - 2(x-3) & \text{for } 3 \le x \le 4 \\ 6 - \frac{3}{2}(x-4) & \text{for } 4 \le x \le 6 \\ 3 - \frac{3}{2}(x-6) & \text{for } 6 \le x \le 8 \end{cases}$$

on the interval $[0, 8]$. Let

$$F(x) = \begin{cases} 4 + 2x & \text{for } -1 \le x \le 0 \\ f(x) & \text{for } 0 \le x \le 8 \\ 2(x-8) & \text{for } 8 \le x \le 9. \end{cases}$$

be the extension of $f(x)$ to the interval $[-1, 9]$.

   a. Sketch the graphs of f and F.

   b. Find a partition for f and explain why it has the Markov property. (i.e., Find a stretching partition for f.)

   c. For what periods does f have a periodic orbit?

   d. Find a natural measure for f on $[0, 8]$. Hint: The matrix $M = \left(\frac{L_i t_{ij}}{L_j s_j}\right)$ where (i) $t_{ij} = 1$ if and only if there is a transition from the j^{th}-interval to the i^{th}-interval (ii) s_j is the slope on the j^{th}-interval, and (iii) L_i and L_j are the lengths of the i^{th}-interval and j^{th}-interval respectively,

   e. Show that F has a trapping region for the attracting set $[0, 8]$.

   f. Does F have sensitive dependence on initial conditions on $[0, 8]$? You do not need to give a complete argument, but explain the basic reason why it has or does not have sensitive dependence on initial conditions.

   g. What can you say about the sign of Lyapunov exponents of points of the map F? You do not have to calculate an exact value, but what sign do they have to have?

   h. Is there a point x_0 whose forward limit set $\omega(x_0)$ is all of $[0, 8]$? Why or why or why not?

   i. Is the interval $[0, 8]$ a chaotic attractor for F? Explain what definition you are using for a chaotic attractor and why F satisfies the conditions of the definition.