Math 313 Final: June, 2002 C. Robinson

No books, no notes. You may use hand calculators

1. (35 Points) Consider the set
$$S = \{\,0\,\} \cup \{\,\frac{1}{k} : 1 \le k < \infty,\ k \text{ is an integer}\,\}.$$
Derive the box dimension of S.

2. (35 Points) Consider the iterated function system with the three maps on $\mathbb{R}^2$
$$\begin{aligned}\mathbf{F}_1(\mathbf{x}) &= 0.2\,\mathbf{x}\\ \mathbf{F}_2(\mathbf{x}) &= 0.2\,\mathbf{x} + (0.8, 0)\\ \mathbf{F}_3(\mathbf{x}) &= 0.2\,\mathbf{x} + (0, 0.8).\end{aligned}$$
   (a) Describe the attractor for the iterated function system.
   (b) How many boxes of size $(0.2)^3$ are needed to cover the attractor?
   (c) What is the box dimension of the attractor for the iterated function system? (Derive the answer, don't just write down a number.)

3. (40 Points) Consider the baker type map given by
$$\mathbf{F}\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases}\begin{pmatrix}\frac{1}{4}\,x + \frac{1}{8}\,\sin(2\pi\,y)\\ 2\,y\end{pmatrix} & \text{for } y < \frac{1}{2}\\[2ex] \begin{pmatrix}\frac{1}{4}\,x + \frac{3}{4} + \frac{1}{8}\,\sin(2\pi\,y)\\ 2\,y - 1\end{pmatrix} & \text{for } \frac{1}{2} \le y \le 1.\end{cases}$$
   (a) For a point $\mathbf{p}_0 = (x_0, y_0)$ with $0 \le x_0 \le 1$ and $0 \le y_0 \le 1$, and the vector $\mathbf{v} = (1, 0)$, what is the Lyapunov exponent $h(\mathbf{p}_0, \mathbf{v}; \mathbf{F})$?
   (b) What is the other Lyapunov exponent? Hint: You may use the following fact. For a map in two dimensions for which $\det(D\mathbf{F}_{\mathbf{p}}) = \Delta$ is a constant, the sum of the Lyapunov exponents satisfies
$$\ln(\Delta) = h_1(\mathbf{p}; \mathbf{F}) + h_2(\mathbf{p}; \mathbf{F}).$$
   (c) What is the Lyapunov dimension of the attracting set for the map $\mathbf{F}$?

4. (30 Points) Consider the baker type map given by
$$\mathbf{F}\begin{pmatrix}x\\y\end{pmatrix} = \begin{cases}\begin{pmatrix}\frac{1}{4}\,x + \frac{1}{8}\,\sin(4\pi\,y)\\ 4\,y\end{pmatrix} & \text{for } y < \frac{1}{2}\\[2ex] \begin{pmatrix}\frac{1}{4}\,x + \frac{3}{4} - \frac{1}{8}\,\sin(4\pi\,(y - 0.75))\\ 4\,y - 3\end{pmatrix} & \text{for } \frac{1}{2} \le y.\end{cases}$$
Define the rectangles $\mathbf{R}_0 = [0, 1] \times [0, 0.25]$ and $\mathbf{R}_1 = [0, 1] \times [0.75, 1]$. Show that $\{\,\mathbf{R}_0,\ \mathbf{R}_1\,\}$ is a Markov partition, i.e., show that the images of $\mathbf{R}_0$ and $\mathbf{R}_1$ by $\mathbf{F}$ are correctly aligned with $\mathbf{R}_0$ and $\mathbf{R}_1$.

5. (30 Points) Consider the Hénon map with $a = 4.27$ and $b = 0.4$,

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4.27 - x^2 + 0.4\,y \\ x \end{pmatrix}.$$

(a) Verify that the points $(2.3, -1.7)$ and $(-1.7, 2.3)$ form a period-2 orbit.
(b) Classify this period-2 orbit as sink, saddle, source, or none of the above.

6. (30 Points) Consider the solenoid defined as follows. Let

$$N = \{\, (t \pmod 1), z) : t \in \mathbb{R},\ z \in \mathbb{C}, |z| \leq 1 \,\}$$

and the solenoid map given by

$$\mathbf{F}(t, z) = (2\,t \pmod 1), \frac{1}{4}\, z + \frac{1}{2}\, e^{2\pi t\, i}).$$

(a) Explain why N is a trapping region.
(b) Explain why $\Lambda = \bigcap_{k \geq 0} \mathbf{F}^k(N)$ is a chaotic attractor.