Math 313-1 Test 1 October 21, 2003.

① $f(x) = \frac{1}{3}x^3 - \frac{1}{3}x^2 + \frac{1}{3}x.$

$f'(x) = x^2 - \frac{2}{3}x + \frac{1}{3}.$

ⓐ Fixed Points.

$\frac{1}{3}x^3 - \frac{1}{3}x^2 + \frac{1}{3}x - x = 0 \Rightarrow x(x^2 - x - 2) = 0$

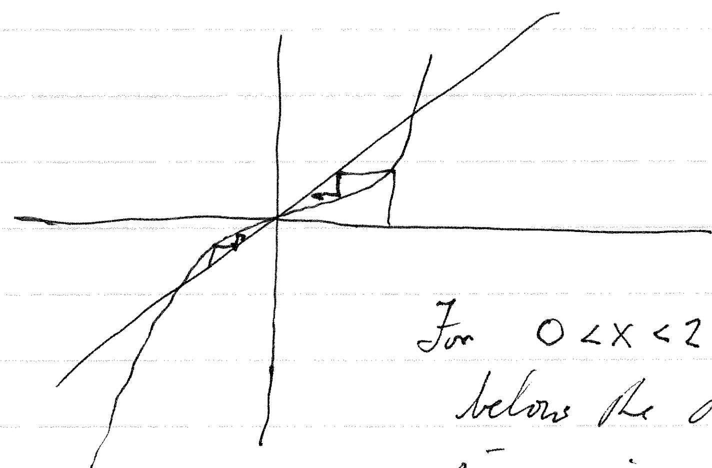
$x = 0, 2, -1.$

$f'(0) = \frac{1}{3} \quad \therefore x = 0$ attracting.

$f'(2) = 4 - \frac{4}{3} + \frac{1}{3} = 3 > 1 \quad x = 2$ repelling.

$f'(-1) = 1 + \frac{2}{3} + \frac{1}{3} = 2 > 1 \quad x = -1$ is repelling.

ⓑ



For $0 < x < 2$ the graph is below the diagonal and increasing. (no critical points).

Therefore, for $0 < x_0 < 2.$, & $x_n = f^n(x_0)$

$2 > x_0 > x_1 > x_2 > \cdots > 0.$

Therefore x_n converges to 0. Fixed Pt.

For $-1 < x < 0$, the graph is below the diagonal and increasing. (no critical points.).

For $-1 < x_0 < 0$, $x_n = f^n(x_0)$. $-1 < x_0 < x_1 < \cdots < x_n < 0.$

Therefore, $f^n(x_0) = x_n$ converges to fixed point $x = 0$.

For $x_0 > 2$, $f^n(x_0) \to \infty$.

For, $x_0 < -1$, $f^n(x_0) \to -\infty$.

Therefore $\mathcal{B}(0, f) = (-1, 2).$

(2) $f(x) = 1 - \frac{10}{9}x^2$.

$f'(x) = -\frac{20}{9}x$.

$$f'\left(\frac{9}{20} + \frac{3\sqrt{13}}{20}\right) = -\frac{20}{9}\left(\frac{9}{20} + \frac{3\sqrt{13}}{20}\right) = -1 - \frac{1}{3}\sqrt{13}$$

$$f'\left(\frac{9}{20} - \frac{3\sqrt{13}}{20}\right) = -\frac{20}{9}\left(\frac{9}{20} - \frac{3\sqrt{13}}{20}\right) = -1 + \frac{1}{3}\sqrt{13}.$$

$$\left|f'\left(\frac{9}{20} + \frac{3\sqrt{13}}{20}\right) f'\left(\frac{9}{20} - \frac{3\sqrt{13}}{20}\right)\right| = \left|\left(-1 - \frac{1}{3}\sqrt{13}\right)\left(-1 + \frac{1}{3}\sqrt{13}\right)\right|$$

$$= \left|1 - \frac{1}{9}\cdot 13\right| = \left|-\frac{4}{9}\right| = \frac{4}{9}.$$

Therefore, the orbit is attracting.

(b) $C(x) = \frac{1}{3}x + \frac{1}{2}$ $\quad g(y) = \frac{10}{3}y(1-y)$

$$C \circ f(x) = C\left(1 - \frac{10}{9}x^2\right) = \frac{1}{3}\left(1 - \frac{10}{9}x^2\right) + \frac{1}{2}$$

$$= \frac{5}{6} - \frac{10}{27}x^2.$$

$$g \circ C(x) = g\left(\frac{1}{3}x + \frac{1}{2}\right) = \frac{10}{3}\left(\frac{1}{3}x + \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{3}x\right)$$

$$= \frac{10}{3}\left(\frac{1}{4} - \frac{1}{9}x^2\right)$$

$$= \frac{5}{6} - \frac{10}{27}x^2.$$

They are equal. C is a homeomorphism.

(3) $f_r(x) = r \sin x.$

$f_r'(x) = r \cos x$

$f_r''(x) = -r \sin x$

$f_r'''(x) = -r \cos x.$

$$S_f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

$$= \frac{-r\cos x}{r \cos x} - \frac{3}{2}\left(\frac{-r \sin x}{r \cos x}\right)^2$$

$$= -1 - \frac{3}{2}\left[\tan x\right]^2 < 0.$$

f has negative Schwarzian derivative

(b) The only critical point in $[0,\pi]$ is when $r\cos x = 0$, or $x_c = \pi/2$.

~~Any attracti~~

The end points are both mapped to $x=0$. Therefore an interval in the basin of attraction is contained in $[0,\pi]$. It must contain a critical point, i.e. $\pi/2$ is in the basin of attraction.

④ $f_\mu(x) = \mu x^2 - 1$

$f_\mu'(x) = 2\mu x.$

Fixed points at $\mu x^2 - 1 - x = 0$,

$$x_\mu^\pm = \frac{1 \pm \sqrt{1+4\mu}}{2\mu}.$$

$$f'(x_\mu^\pm) = 1 \pm \sqrt{1+4\mu}.$$

(a) $1 = 1 \pm \sqrt{1+4\mu}$ need $1 + 4\mu = 0$

or $\mu_0 = -\frac{1}{4}$. Fixed point $x_0 = \frac{1}{-1/2} = -2.$

$$f_{-\frac{1}{4}}(-2) = -\tfrac{1}{4}(2)^2 - 1 = -1 - 1 = -2 \quad \text{fixed Point}$$

$$f_{-\frac{1}{4}}'(-2) = 2(-\tfrac{1}{4})(-2) = 1.$$

This is a potential tangential bifurcation

(b) $-1 \stackrel{?}{=} 1 \pm \sqrt{1+4\mu}.$

$\mp\sqrt{1+4\mu} = 2.$ need $\sqrt{1+4\mu} = 2$ so $x_\mu = \frac{1-\sqrt{1+4\mu}}{2\mu}.$

Square

$1 + 4\mu = 4$ $\quad 4\mu = 3 \quad \mu_0 = 3/4.$

$$x_0 = \frac{1 - \sqrt{1 + 4\cdot 3/4}}{3/2} = \frac{1-\sqrt{4}}{3/2} = \frac{1-2}{3/2} = -2/3.$$

$$f_{3/4}(-2/3) = \tfrac{3}{4}(-\tfrac{2}{3})^2 - 1 = \tfrac{1}{3} - 1 = -2/3 \quad \text{Fixed Pt.}$$

$$f_{3/4}'(-2/3) = 2(\tfrac{3}{4})(-\tfrac{2}{3}) = -1.$$

This is a potential period doubling bifurcation.