SUPPLEMENTAL PROBLEMS AND HINTS FOR MATH 368

Chapter 3: Linear Programming

SP3.1 For this problem, mark each statement True or False and justify each answer. The statements relate to a standard maximum linear program with the objective function $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ the constraint inequality $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ for an $m \times n$ coefficient matrix $\mathbf{A}$ and constant vector $\mathbf{b} \in \mathbb{R}^m_+$, and $\mathbf{0} \leq \mathbf{x}$.

a. If a standard maximization linear program does not have an optimal solution, then either the objective function is unbounded on the feasible set $\mathscr{F}$ or $\mathscr{F}$ is the empty set.

b. If $\bar{\mathbf{x}}$ is an optimal solution of a standard maximization linear program, then $\bar{\mathbf{x}}$ is an extreme point of the feasible set.

c. A slack variable is used to change an equality into an inequality.

d. A solution is called a basic solution if exactly m of the variables are nonzero.

e. The basic feasible solutions correspond to the extreme points of the feasible region.

f. The bottom entry in the right column of a simplex tableau gives the maximum value of the objective function.

g. For a tableau for a maximization linear program, if there is a column with all negative entries including the one in the row for the objective function, then the linear programming problem has no feasible solution.

h. The value of the objective function for a MLP at any basic feasible solution is always greater than the value at any non-basic feasible solution.

i. If a standard maximization linear program MLP has nonempty feasible set, then it has an optimal basic solution.

j. In the two-phase simplex method, if an artificial variable is positive for the optimal solution for the artificial objective function, then there is no feasible solution to the original linear program.

SP3.2 Show that the following problem permits the objective function to assume arbitrarily large values:

$$\begin{array}{ll} \text{Maximize} & 2x + y \\ \text{Subject to:} & x - y \leq 3 \\ & -3x + y \leq 1 \\ & 0 \leq x,\, 0 \leq y. \end{array}$$

3.3:3 *Hint:* The problem become degenerate in the sense that the constant on the right side become zero. That is OK. $0 \leq 0/3 < 1$ so that is a good place to pivot.

Do not pivot on a location with 0 or a negative entry in that location in the new pivot column.

The entries in the right hand column should keep ≥ 0 during the row reduction. These are the values of the basic variables, and we want all values ≥ 0.

3.4:5 *Hint:* This corresponds to Case 1 in the proof of Theorem 3.4.2.

3.4:6 *Hint:* This corresponds to Case 2 in the proof of Theorem 3.4.2. You need to find a vector $\mathbf{y}$ such that $\mathbf{A}\mathbf{y} = \mathbf{0}$ (i.e. in the null space), $\mathbf{c} \cdot \mathbf{y} > 0$, and all $y_i \geq 0$. For such a $\mathbf{y}$, $\mathbf{A}(\mathbf{x} + t\mathbf{y}) = \mathbf{A}\mathbf{x} = \mathbf{b}$, $x_i + t\, y_i \geq 0$ for $t \geq 0$ so it is feasible, and $f(\mathbf{x} + t\mathbf{y}) = f(\mathbf{x}) + t\,\mathbf{c} \cdot \mathbf{y}$ goes to infinity as t goes to infinity.

To find this $\mathbf{y}$, apply the simplex method to equations as given (ignoring the given $\mathbf{x}^T = (0, 12, 0, 16, 0, 2)$). You should come to a situation where an entry in the objective function row is negative and all the entries above it are negative (you only need nonpositive). This is the situation of Rule 4 on page 91 of Walker and my $3'$ on page 6 of my lecture notes. To find $\mathbf{y}$, keep all the free (nonbasic) variables except the one x_k for the column found. Write down the general solution of the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ taking all the terms involving this one free variable to the right side of the equation. The constants give the basic solution and the coefficients of this one free variable will be all nonnegative and will give the $\mathbf{y}$ desired.

Date: April 22, 2013.

Chapter 4: Linear Programming Duality

SP4.1 For this problem, mark each statement *true* or *false* and justify each answer. The statements relate to a standard maximum linear program MLP with the objective function $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ and the constraint inequality $\mathbf{Ax} \leq \mathbf{b}$ for an $m \times n$ coefficient matrix $\mathbf{A}$ and constant vector $\mathbf{b} \in \mathbb{R}^m_+$, and $\mathbf{x} \geq \mathbf{0}$.

- **a.** The dual mLP problem is to minimize $\mathbf{y}$ in $\mathbb{R}^m$ subject to $\mathbf{Ay} \geq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.
- **b.** If $\bar{\mathbf{x}}$ is an optimal solution to the primal MLP and $\hat{\mathbf{y}}$ is a feasible solution to the dual mLP, then $f(\bar{\mathbf{x}}) = g(\hat{\mathbf{y}})$.
- **c.** If a slack variable is $\bar{s}_i > 0$ in an optimal solution, then the addition to the objective function that would be realized by one more unit of the resource corresponding to its inequality is positive.
- **d.** If a maximization linear program MLP and its dual minimization linear problem mLP each have nonempty feasible sets (some feasible point), then each problem has an optimal solution.
- **e.** If the optimal solution of a standard MLP has a slack variable $s_i = 0$, then the i^{th} resource has zero marginal value, i.e., one unit of the i^{th} resource would add nothing to the value of the objective function.

Chapter 6: Unconstrained Extrema

SP6.1 Consider the sets

$$\mathbf{S}_1 = \{ (x, y) \in \mathbb{R}^2 : -1 < x < 1 \}$$
$$\mathbf{S}_2 = \{ (x, y) \in \mathbb{R}^2 : x \geq 1,\ y \geq 0 \}.$$

- **a.** For the sets $\mathbf{S}_1$ and $\mathbf{S}_2$, discuss which points are in the boundary and which points are not using the definition of the boundary.
- **b.** Discuss why $\mathbf{S}_1$ is open in two ways: (i) $\mathbf{S}_1 \cap \partial(\mathbf{S}_1) = \emptyset$ and (ii) for every point $\mathbf{p} \in \mathbf{S}_1$, there is an $r > 0$ such that $\mathbf{B}(\mathbf{p}, r) \subset \mathbf{S}_1$.
- **c.** Discuss why $\mathbf{S}_2$ is closed in two ways: (i) $\partial(\mathbf{S}_2) \subset \mathbf{S}_2$ and (ii) for every point $\mathbf{p} \in \mathbf{S}_2^c$, there is an $r > 0$ such that $\mathbf{B}(\mathbf{p}, r) \subset \mathbf{S}_2^c$.

SP6.2 Let $f : \mathbb{R}_+ \to \mathbb{R}$ be continuous, $f(0) = 1$, and $\lim_{x\to\infty} f(x) = 0$.

- **a.** Show that there is a $p > 0$ such that the maximum value of $f(x)$ on $[0, p]$ is larger than any value of $f(x)$ for $x > p$. *Hint:* Take p such that $f(x) < \frac{1}{2} f(0)$ for $x \geq p$.
- **b.** Show that $f(x)$ has a maximum on $\mathbb{R}_+$.
- **c.** Does $f(x)$ have to have a minimum on $\mathbb{R}_+$? Explain why or why not.

SP6.3 Let $\mathscr{D} = \{ (x, y) \in \mathbb{R}^2_+ : xy \geq 1 \}$ and $\mathscr{B} = \{ (x, y) \in \mathbb{R}^2_+ : x + y \leq 10 \}$. (Note that $\mathscr{D}$ is not compact.) Assume that $f : \mathbb{R}^2_+ \to \mathbb{R}$ is a continuous function with $f(x, y) > f(2, 3)$ for $x + y > 10$, i.e. for $(x, y) \in \mathbb{R}^2_+ \setminus \mathscr{B}$.

- **a.** Why must f attain a minimum on $\mathscr{D} \cap \mathscr{B}$?
- **b.** Using reasoning like for SP6.2, explain why f attains a minimum on all of $\mathscr{D}$.

SP6.4 Compute the second order Taylor polynomial (without explicit remainder) for $f(x, y) = e^x \cos(y)$ around $(x_0, y_0) = (0, 0)$.

SP6.5 Find all the critical points and classify them as local maximum, local minimum, or neither for the following functions.

a. $f(x, y, z) = x^4 + x^2 - 6xy + 3y^2 + z^2$

b. $f(x, y, z) = 3x - x^3 - 2y^2 + y^4 + z^3 - 3z$

SP6.6 Suppose $f : \mathscr{D} \subset \mathbb{R}^n \to \mathbb{R}$ is C^1 and has a maximum at a point $\mathbf{x}^*$ in the boundary of $\mathscr{D}$, $\partial(\mathscr{D})$. Does $Df(\mathbf{x}^*)$ have to equal $\mathbf{0}$ at $\mathbf{x}^*$? *Hint:* What sort of examples have we considered?

SP6.7 Find the points at which each of the following functions attains a maximum and minimum on the interval $0 \le x \le 3$. For parts (a) and (b), also find the maximum and minimum values. *Remember* to consider the end points of the interval $[0, 3]$.

a. $f(x) = x^2 - 2x + 2$.

b. $g(x) = -x^2 + 2x + 4$.

c. The function $h(x)$ satisfies $h'(x) > 0$ for all $0 \le x \le 3$.

d. The function $k(x)$ satisfies $k'(x) < 0$ for all $0 \le x \le 3$.

e. The function $u(x)$ satisfies $u'(x) = 0$ for all $0 \le x \le 3$.

Chapter 7: Constrained Extrema

SP7.1 (Implicit Function Theorem) A firm uses two inputs, q_1 and q_2 to produce a single output Q, given by the production function $Q = kq_1^{2/5}q_2^{1/5}$. Let P be the price of the output Q, p_1 be the price of q_1, and p_2 be the price of q_2. The profit is given by $\pi = Pkq_1^{2/5}q_2^{1/5} - p_1q_1 - p_2q_2$. The inputs that maximize profits satisfy

$$0 = \frac{2Pk}{5}\, q_1^{-3/5}\, q_2^{1/5} - p_1 \qquad \text{and}$$
$$0 = \frac{Pk}{5}\, q_1^{2/5}\, q_2^{-4/5} - p_2.$$

a. Show that this two equations can be used to determine the amounts of inputs q_1 and q_2 in terms of the prices p_1, p_2, and P. Show that the relevant matrix has nonzero determinant.

b. Write the matrix equation for the partial derivatives of q_1 and q_2 with respect to p_1, p_2 and P in terms of the variables.

c. Solve for the matrix of partial derivatives of q_1 and q_2 in terms of p_1, p_2 and P.

SP7.2 (Implicit Function Theorem) A nonlinear Keynesian IS-LM model for national income involves the following quantities:

Y	Gross domestic product (GDP)
G	Government spending
r	Interest rate
M	Money supply

In addition, there are three quantities which are functions of the other variables (intermediate variables). Investment expenditure I is a function of the interest rate given by $I(r) = \dfrac{I_0}{r+1}$. The consumption is a function of Y given by $C(Y) = C_0 + \frac{5}{6}\,Y + \frac{1}{6}\,e^{-Y}$ with C_0 a constant. The gross domestic product is the sum of consumption, investment expenditure, and government spending, $Y = C + I + G = C_0 + \frac{5}{6}\,Y + \frac{1}{6}\,e^{-Y} + \dfrac{I_0}{r+1} + G$. The money supply equals the liquidity function, $M = \dfrac{Y}{r+1}$. With these assumptions, the model yields the following two equations:

$$0 = C_0 - \frac{1}{6}\,Y + \frac{1}{6}\,e^{-Y} + \frac{I_0}{r+1} + G \qquad \text{and}$$
$$0 = \frac{Y}{r+1} - M.$$

a. Using the Implicit Function Theorem, show that these two equations define Y and r as dependent variables which are determined by the independent variables G and M, i.e., these equations define Y and r as functions of G and M.
b. Write the matrix equation that the partial derivatives of Y and r with respect to G and M must satisfy.
c. Solve for the matrix equation for the partial derivatives of Y and r with respect to G and M.

SP7.3 Find the points satisfying the first order conditions for a constrained extrema and then apply the second order test to determine whether they are local maximum or local minimum.
a. $f(x, y, z) = xyz$ and $g(x, y, z) = 2x + 3y + z = 6$.
b. $f(x, y, z) = 2x + y^2 - z^2$, $g_1(x, y, z) = x - 2y = 0$, and $g_2(x, y, z) = x + z = 0$.

SP7.4 For each of the following objective and constraint functions, find the maximizer and minimizers.
a. $f(x, y, z) = x^2 + y^2 + z^2$, subject to $g(x, y, z) = x + y + z = 12$ and $h(x, y, z) = x^2 + y^2 - z = 0$.
b. $f(x, y, z) = x + y + z$, subject to $g(x, y, z) = x^2 + y^2 = 2$ and $h(x, y, z) = x + z = 1$.
c. Minimize $f(x, y, z) = x^2+y^2+z^2$, subject to $g(x, y, z) = x+2y+3z = 6$ and $h(x, y, z) = y+z = 0$.

SP7.5 (cf 7.4:3) Consider the problem

Maximize: $f(x, y) = 2 - 2y$
Subject to $g_1(x, y) = y + (x - 1)^3 \leq 0$
$g_2(x, y) = -x \leq 0,$
$g_3(x, y) = -y \leq 0.$

Carry out the following steps to show that the maximizer is a point at which the constrain qualification fails.
a. By drawing a figure, show that the feasible set is a three sided (nonlinear) region with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$.
b. Plot several level curves $f^{-1}(C)$ of the objective function to your figure from part (a) and conclude geometrically that $(0, 1)$ is a maximizer and $(1, 0)$ is a minimizer of $f(x, y)$ on the feasible set.
c. Show that the constraint qualification fails at $(0, 1)$. Also, show that $Df(0, 1)$ cannot be written as a linear combination of the derivatives $Dg_i(0, 1)$ of the effective constraints.

SP7.6 Maximize the revenue

$$\pi = p_1 y_1 + p_2 y_2 = p_1 x_1^{1/2} + p_2 x_1^{1/2} x_2^{1/3}$$

subject to a wealth constraint on the inputs

$$w_1 x_1 + w_2 x_2 \leq C > 0, \qquad x_1 \geq 0 \qquad x_2 \geq 0.$$

a. Write down the constraint functions and the equations that must be satisfied for Theorem 7.4.1.
b. Take $w_1 = w_2 = 2$, $p_1 = p_2 = 1$, and $C = 8$, and find explicit values of x_1 and x_2 that attains the maximum.

SP7.7 A firm produces a single output q with two inputs x and y, with production function $q = xy$. The output must be at least q_0 units, $xy \geq q_0 > 0$. The firm is obligated to use at least one unit of x, $x \geq 1$. The prices of x and y are w and 1 respectively. Assume that the firm wants to minimize the cost of the inputs $f(x, y) = wx + y$.
a. Is the feasible set closed? Compact? Convex?
b. Write down the first order KKT conditions.
c. Find the minimizer by solving the KKT-1,2 equations.
Hints: (i) Note that one of the equations for KKT-1 implies that the multiplier for $0 \geq q_0 - xy$ is nonzero and so this constraint must be effective at a solution.
(ii) If $0 \geq 1 - x$ is tight, then $q \leq w$ because both multiplier must be less than or equal to zero.
(iii) If the multiplier for $0 \geq 1 - x$ is zero, then $q \geq w$ because $x \geq 1$.

SP7.8 Let $f : \mathbb{R}^n_{++} \to \mathbb{R}$ be defined by $f(x_1, \ldots, x_n) = \ln(x_1^{\alpha_1} \cdots x_n^{\alpha_n})$, where all the $\alpha_i > 0$. Is f convex or concave?

SP7.9 Assume that $g_i : \mathscr{D} \subset \mathbb{R}^n \to \mathbb{R}$ for $1 \leq i \leq k$ are convex and bounded on $\mathscr{D}$.
a. Show that $f(\mathbf{x}) = \max_{1 \leq i \leq k} g_i(\mathbf{x})$ is convex.
Hint: $\max_i\{ a_i + b_i \} \leq \max_i\{ a_i \} + \max_i\{ b_i \}$.
b. Is $g(x) = \min_{1 \leq i \leq k} g_i(\mathbf{x})$ convex? Why or why not?
Hint: $\min\{ a_i + b_i \} \geq \min\{ a_i \} + \min\{ b_i \}$.
c. If the g_i are concave, is $\min_{1 \leq i \leq k} g_i(\mathbf{x})$ concave?

SP7.10 Let I, $p_i > 0$ for $1 \leq i \leq n$. Show that

$$\mathscr{B}(\mathbf{p}, I) = \{\mathbf{x} \in \mathbb{R}^n_+ : p_1x_1 + \cdots + p_nx_n \leq I\}$$

satisfies Slater's condition. *Hint:* Split up $\frac{1}{2}I$ evenly among the amounts spent on the various commodities.

SP7.11 Assume that $f, g : \mathbb{R}^n \to \mathbb{R}$ are C^1, f is concave, and g is convex. Let $\mathscr{D}_g = \{\mathbf{x} \in \mathbb{R}^n_+ : g(\mathbf{x}) \leq b\}$, $\bar{\mathbf{x}} \in \mathscr{D}_g \cap \mathbb{R}^n_{++}$ with $g(\bar{\mathbf{x}}) < b$, and $\mathbf{p}^* \in \arg\max\{ f(\mathbf{x}) : \mathbf{x} \in \mathscr{D}_g \} \cap \mathbb{R}^n_{++}$ with $Df(\mathbf{p}^*) \neq \mathbf{0}$ and $Dg(\mathbf{p}^*) \neq \mathbf{0}$. *Explain why* $\mathbf{p}^*$ is a minimizer of $g(\mathbf{y})$ on $\mathscr{D}_f = \{ \mathbf{y} \in \mathbb{R}^n_+ : f(\mathbf{y}) \geq f(\mathbf{p}^*) \}$ with $g(\mathbf{p}^*) = b$.

SP7.12 Give an example of a concave function $f : \mathbb{R}_+ \to \mathbb{R}$ that is bounded above but does not attain a maximum.

SP7.13 Indicate which of the following statements are *true* and which are *false* and justify each answer. For a true statement explain why it is true and for a false statement either indicate how to make it true or indicate why the statement is false.

a. If $\mathscr{D} = \{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq b_i \text{ for } 1 \leq i \leq m \}$ is convex, then each of the g_i must be convex.

b. If $f : \mathscr{D} \subset \mathbb{R}^n$ is continuous and $f(\mathbf{x})$ attains a maximum on $\mathscr{D}$, then $\mathscr{D}$ is compact.

c. If $f, g_i : \mathbb{R}^n \to \mathbb{R}$ are C^1 for $1 \leq i \leq m$, $\mathscr{D} = \{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq b_i \text{ for } 1 \leq i \leq m \}$, $f(\mathbf{x})$ satisfies KKT-1,2,3 at $\mathbf{x}^* \in \mathscr{D}$, and the constraint qualification holds at $\mathbf{x}^*$, then $\mathbf{x}^*$ must be a maximizer of $f(\mathbf{x})$ on $\mathscr{D}$.

d. If $\mathscr{D} = \{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq b_i \text{ for } 1 \leq i \leq m \}$ is convex, $f : \mathbb{R}^n \to \mathbb{R}$ is C^1, $\mathbf{x}^*$ satisfies KKT-1,2,3 and is a maximizer of f, then f must be concave.

e. Assume that $f, g_i : \mathbb{R}^n \to \mathbb{R}$ are C^1 for $1 \leq i \leq m$, $\mathscr{D} = \{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq b_i \}$, the constraint qualification is satisfied at all points of $\mathscr{D}$, and $\mathbf{p}_1, \ldots, \mathbf{p}_k$ are the set of all the points in $\mathscr{D}$ that satisfy KKT-,1,2,3. Then, $f(\mathbf{x})$ attains a maximum on $\mathscr{D}$ and $\max\{f(\mathbf{x}) : \mathbf{x} \in \mathscr{D}\} = \max\{f(\mathbf{p}_j) : 1 \leq j \leq k\}$.

f. For a C^2 function $f : \mathbb{R}^n \to \mathbb{R}$ with a critical point $\mathbf{x}^*$ at which the second derivative (or Hessian matrix) $D^2 f(\mathbf{x}^*)$ is negative definite, then $\mathbf{x}^*$ is a maximizer of f on $\mathbb{R}^n$.

g. Let $f, g_j : \mathbb{R}^n_+ \to \mathbb{R}$ be C^1 for $1 \leq j \leq m$, $\mathscr{D} = \{ \mathbf{x} : g_j(\mathbf{x}) \leq b_j \text{ for } 1 \leq j \leq m \}$, and $\{x^*_k\}_{k=1}^K$ be the set of points where either (i) the KKT-1,2,3 conditions hold or (ii) the constraint qualification fails. Then f must have a maximum on $\mathscr{D}$ at one of the points $\{x^*_k\}_{k=1}^K$.

h. Assume that $g_j : \mathbb{R}^n_+ \to \mathbb{R}$ are continuous and convex for $1 \leq j \leq m$, $\mathscr{D} = \{ \mathbf{x} \in \mathbb{R}^n_+ : g_j(\mathbf{x}) \leq b_j \text{ for } 1 \leq j \leq m \}$, $f : \mathbb{R}^n \to \mathbb{R}$ is concave, and f has a local maximum on $\mathscr{D}$ at $\mathbf{x}^*$. Then $\mathbf{x}^*$ is a global maximizer of f on $\mathscr{D}$.

i. If $\mathscr{D} = \{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \leq b_i \text{ for } 1 \leq i \leq m \}$ is convex and f is concave on $\mathscr{D}$, then f must have a maximum on $\mathscr{D}$.

Correspondences & Parametric Maximization

SP.PM.1 Let $\mathbf{S} = [0, 1]$ and $\mathbf{S} = \mathbb{R}$. For each of the following correspondences $\mathscr{D} : \mathbf{S} \to \mathscr{P}(\mathbb{R})$, (i) draw its graph and (ii) determine whether it is uhc, and/or continuous. *Hint:* By Proposition 1, the correspondence is uhc iff it is closed-graph. (They satisfy the other assumptions of the proposition.)

a.
$$\mathscr{D}(s) = \begin{cases} [0, 2s] & \text{for } s \in \left[0, 1/2\right), \\ [0, 2-2s] & \text{for } s \in \left[1/2, 1\right]. \end{cases}$$

b.
$$\mathscr{D}(s) = \begin{cases} [0, 1-2s] & \text{for } s \in \left[0, 1/2\right], \\ [0, 2-2s] & \text{for } s \in \left(1/2, 1\right]. \end{cases}$$

c.
$$\mathscr{D}(s) = \begin{cases} [0, 1-2s] & \text{for } s \in \left[0, 1/2\right), \\ [0, 2-2s] & \text{for } s \in \left[1/2, 1\right]. \end{cases}$$

d.
$$\mathscr{D}(s) = \{0, s\} \qquad \text{for } s \in [0, 1] \qquad \text{(two points for each } s\text{)}.$$

e.
$$\mathscr{D}(s) = \begin{cases} \{0\} & \text{for } s < 0 & \text{(one point for each } s\text{)}, \\ \{-1, 1\} & \text{for } s \geq 0 & \text{(two points for each } s\text{)}. \end{cases}$$

SP.PM.2 Let $\mathbf{X} = [0, 1] = \mathbf{S}$, and $f : \mathbf{S} \times \mathbf{X} \to \mathbb{R}$ be defined by $f(s, x) = 3 + 2x - 3s - 5xs$. Here, $\mathscr{D}(s) = [0, 1]$ for all s. Find $f^*(s)$ and $\mathscr{D}^*(s)$ for each value of s. Using the f^* and $\mathscr{D}^*$ you have found, discuss why $f^*(s)$ is a continuous function and $\mathscr{D}^*(s)$ is a uhc correspondence. (Do not just quote a theorem.) *Hint*: If $f_x(s, x) > 0$ for all $x \in [0, 1]$, the then maximum occurs for $x = 1$. If $f_x(s, x) < 0$ for all $x \in [0, 1]$, then the maximum occurs for $x = 0$.

SP.PM.3 (Sundaram 9.5:12) Let $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be defined by

$$f(s, x) = (x - 1) - (x - s)^2.$$

Define the correspondence $\mathscr{D} : \mathbb{R}_+ \to \mathscr{P}(\mathbb{R}_+)$ by $\mathscr{D}(s) = [0, 1]$ for $s \geq 0$. Do the hypotheses of the Parametric Maximum Theorem 2 hold for this problem? Verify, through direct calculation whether the conclusions of the Parametric Maximum Theorem hold for $\mathscr{D}^*(s)$ and $f^*(s)$.

Hint: Find the critical point, x_s and verify that $\dfrac{\partial^2 f}{\partial x^2} < 0$. If $x_s \in \mathscr{D}(s)$ then it is the maximizer.

If $x_s \notin \mathscr{D}(s)$ is $\dfrac{\partial f}{\partial x}$ always positive or always negative on $[0, 1]$? Is the maximizer the right or left end point.

SP.PM.4 Let $f(s, x) = \sin(x) + sx$, for $s \in \mathbf{S} = [-1, 1]$ and $x \in \mathscr{D}(s) = [0, 3\pi]$.

a. Discuss why the Maximum Theorem applies.

b. Without finding explicit values, sketch the graph of f^* and $\mathscr{D}^*$. Discuss why these graphs look as they do and how they satisfy the conclusion of the Maximum Theorem.
Hint: Draw the graph of $f(s, x)$ as a function of x for three cases of s: (i) $s < 0$, (ii) $s = 0$, and (iii) $s > 0$.

SP.PM.5 Let $\mathbf{S} = [0, 2]$, $\mathbf{S} = [0, 1]$, $f : \mathbf{S} \times \mathbf{\Theta} \to \mathbb{R}$ be defined by $f(s, x) = -(x + s - 1)^2$, and $\mathscr{D}(s) = [0, s]$. Find $f^*(s)$ and $\mathscr{D}^*(s)$ for each value of s. Draw the graphs of f^* and $\mathscr{D}^*$

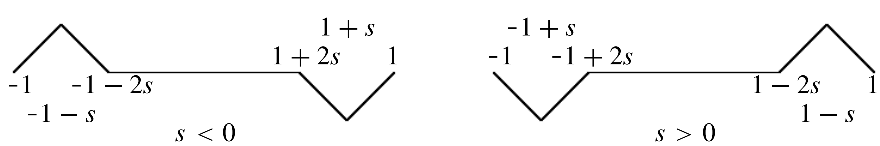
SP.PM.6 Let $\mathbf{S} = \left[-{}^1\!/\!_2, {}^1\!/\!_2\right]$ and $\mathbf{X} = [-1, 1]$. Let the correspondence $\mathscr{D} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$ and function $f : \mathbf{S} \times \mathbf{X} \to \mathbb{R}$ be defined by

$$\mathscr{D}(s) = \begin{cases} [-1, 1 + 4s] & \text{for } s < 0, \\ [-1, 1] & \text{for } s = 0, \\ [-1 + 4s, 1] & \text{for } s > 0, \end{cases}$$

$$\text{for } s < 0, \qquad f(s, x) = \begin{cases} x + 1 & \text{for } -1 \leq x \leq -1 - s, \\ -x - 1 - 2s & \text{for } -1 - s \leq x \leq -1 - 2s, \\ 0 & \text{for } -1 - 2s \leq x \leq 1 + 2s \\ -x + 1 + 2s & \text{for } 1 + 2s \leq x \leq 1 + s, \\ x - 1 & \text{for } 1 + s \leq x \leq 1. \end{cases}$$

$$f(0, x) = 0,$$

$$\text{for } s > 0, \qquad f(s, x) = \begin{cases} -x - 1 & \text{for } -1 \leq x \leq -1 + s, \\ x + 1 - 2s & \text{for } -1 + s \leq x \leq -1 + 2s, \\ 0 & \text{for } -1 + 2s \leq x \leq 1 - 2s, \\ x - 1 + 2s & \text{for } 1 - 2s \leq x \leq 1 - s, \\ -x + 1 & \text{for } 1 - s \leq x \leq 1. \end{cases}$$



a. Sketch the graph of $\mathscr{D}$. Do f and $\mathscr{D}$ meet all the conditions of the Maximum Theorem? If yes, justify your claim. If no, list all the conditions you believe are violated and explain why you believe each of them is violated.

- **b**. For each s, determine the value of $f^*(s)$ and the set $\mathscr{D}^*(s)$, and sketch the graphs of f^* and $\mathscr{D}^*$.
  Hint: Consider $s > 0$, $s = 0$, and $s < 0$ separately. Also, you may have to split up $\mathbf{\Theta}$ into subintervals where $\mathscr{D}(s)$ contains the point that maximizes $f(s, x)$ on $[-1, 1]$ and where it does not.
- **c**. Is f^* continuous? Is $\mathscr{D}^*(s) \neq \emptyset$ for each s? If so, determine whether $\mathscr{D}^*$ is uhc and/or continuous on $\mathbf{S}$.

SP.PM.7 (cf Sundaram 9.5:13.) Let $\mathbf{S} = [0, 1]$ and $\mathbf{X} = [0, 2]$. Let the correspondence $\mathscr{D} : \mathbf{S} \to \mathscr{P}(\mathbf{X})$ be defined by

$$\mathscr{D}(s) = \begin{cases} [0, 1 - 2s] & \text{for } s \in \left[0, 1/2\right), \\ [0, 2 - 2s] & \text{for } s \in \left[1/2, 1\right]. \end{cases}.$$

Let the function $f : \mathbf{S} \times \mathbf{X} \to \mathbb{R}$ be defined by

$$f(s, x) = \begin{cases} 0 & \text{if } s = 0, \quad x \in [0, 2], \\ \dfrac{x}{s} & \text{if } s > 0, \quad x \in [0, s), \\ 2 - \left(\dfrac{x}{s}\right) & \text{if } s > 0, \quad x \in [s, 2s], \\ 0 & \text{if } s > 0, \quad x \in (2s, 2]. \end{cases}.$$

- **a**. Sketch the graph of f for $s > 0$. Sketch the graph of $\mathscr{D}$. Do f and $\mathscr{D}$ meet all the conditions of the Maximum Theorem? If yes, justify your claim. If no, list all the conditions you believe are violated and explain why you believe each of them is violated. (Is f continuous at $s = 0$?)
- **b**. For each s, determine the value of $f^*(s)$ and the set $\mathscr{D}^*(s)$, and sketch the graphs of f^* and $\mathscr{D}^*$.
  Hint: You may have to split up $\mathbf{S}$ into subintervals where $\mathscr{D}(s)$ contains the point that maximizes $f(s, x)$ on $[0, 2]$ and where it does not.
- **c**. Is f^* continuous? Is $\mathscr{D}^*(s) \neq \emptyset$ for each s? If so, determine whether $\mathscr{D}^*$ is uhc and/or continuous on $\mathbf{S}$.

Finite Horizon Dynamic Programming, FHDP

SP.FH.1 Consider the Consumption-Savings FHDP with $\delta = 1$, $r_t(w, c) = c^{\frac{1}{3}}$, transition function $f_t(c, w) = (w - c)$, $\Phi_t(w_t) = [0.w_t]$, and $T = 2$. Find the value functions and optimal strategy for each stage.

SP.FH.2 Consider the Consumption-Savings FHDP with $T > 0$, $r_t(w, c) = \ln(c)$ ($\delta = 1$), transition function $f_t(w, c) = w - c$, and $\Phi_t(w) = [0, w]$ for all periods. Find the value functions and optimal strategy for each stage. *Remark*: The reward function equals minus infinity for $c = 0$, but this just means that it is very undesirable.
Hint: Compute, $V_T(w_T)$ and $V_{T-1}(w_{T-1})$. Then guess the form of V_{T-t}, and prove it is valid by induction.

SP.FH.3 Consider the Consumption-Savings FHDP with $r(w, c)) = 1 - e^{-c}$, transition function $f_t(w_t, c) = w_t - c$, and $\Phi_t(w_t) = [0, w_t]$. Find the value functions and optimal strategy for each stage.

SP.FH.4 Consider the FHDP with $\delta = 1$, $r_t(s, c) = 1 - \dfrac{1}{1 + c}$, transition function $f_t(s, c) = (s - c)$, $\Phi_t(s_t) = [0.s_t]$, and $T \geq 2$.

- **a**. Find the value function and optimal strategy for $t = T$ and $T - 1$.
- **b**. Using backward induction, verify that $V_t(s) = 1 + t - \dfrac{(1 + t)^2}{1 + t + s}$. Also, determine the optimal strategy for each t.

Stationary Dynamic Programming, SDP

SP.SD.1 Consider the SDP problem with reward function $r(s, a) = u(a) = a^{2/3}$, transition function $f(s, a) = k(s - a)$ with $k \geq 1$, $\Phi(s) = [0, s]$, and $0 < \delta < 1$.

- **a**. Using the guess that $V(s) = M\, s^{2/3}$, find the action $a = \sigma(s)$ in terms of M that maximizes the right hand side of the Bellman equation.
- **b**. Substitute the solution of part (b) in the Bellman equation to determine the constant M and $V(s)$.
- **c**. What is the optimal strategy?

SP.SD.2 Consider the SDP problem with reward function $r(s, a) = u(a) = \ln(a)$, transition function $f(s, a) = (s - a)^\beta$ with $0 < \beta \leq 1$ $\Phi(s) = [0, s]$, and $0 < \delta < 1$. (Note that $u(a) = \ln(a)$ is unbounded below at 0, and the choices are open at 0, but it turns out that the Bellman equation does have a solution. This SDP is a model of Brock-Mirman for an amount s_t of capital at period-t, consumption a_t at period-t. The production function f determines the capital at the next period, $s_{t+1} = f(s_t, a_t)$.)
Start with the guess that this SDP has a value function of the form $V(s) = A + B\ \ln(s)$.

- **a**. Find the action $a = \sigma(s)$ in terms of A and B that maximizes the right hand side of the Bellman equation.
- **b**. Substitute the solution of part (b) in the Bellman equation to determine the constants A and B. *Hint*: In the Bellman equation, the coefficients of $\ln(y)$ on the two sides of the equation must be equal and the constants must be equal. Solve for B first and then A.
- **c**. What is the optimal strategy?

SP.SD.3 Consider the SDP problem with $\mathbf{S} = [0, \infty)$, $\mathbf{A}$, $\Phi(s) = [0, s]$, $f(s, a) = 2s - 2a$, $r(s, a) = 2 - 2e^{-a}$, and $\delta = 1/2$. Start with the guess that this SDP has a value function of the form $V(s) = A - B\,e^{-bs}$.

- **a**. Find the action $\bar{a} = \sigma(s)$ that maximizes the right hand side of the Bellman equation. (The answer can contain the unspecified parameters A, B, and b.)
- **b**. What equations must A, B, and b satisfy to be a solution of the Bellman equation?
- **c**. Solve for A, B, and b.
- **d**. Give the value function and the optimal strategy in terms of the original data for the problem.

SP.SD.4 Consider the SDP problem with discount $0 < \delta < 1$, bounded reward function $r(s, a) = \dfrac{a}{1 + a} = 1 - \dfrac{1}{1 + a}$, and transition function $f(s, a) = k(s - a)$ with $k > 1$ and $k\delta = 1$. Start with the guess that this SDP has a value function of the form $V(s) = \dfrac{s}{1 + Bs} = \frac{1}{B}\left[1 - \dfrac{1}{1 + Bs}\right]$.

- **a**. What is the Bellman equation for this problem?
- **b**. Find the action $a = \sigma(s)$ that maximizes the right hand side of the Bellman equation.
- **c**. Substitute the solution of part (b) in the Bellman equation to determine the constant B.
- **d**. What is the optimal strategy?

SP.SD.5 Consider the Consumption-Savings FHDP with $T > 0$, $r_t(w, c) = \delta^t \ln(c)$ with $0 < \delta \leq 1$, transition function $f_t(w, c) = A\,w^\beta - c$ with $A > 0$ and $\beta > 0$, and $\Phi_t(w) = [0, A\,w^\beta]$ for all periods. Verify that $V_{T-t}(w) = \delta^{T-t} \ln(w)\,\beta(1 + \beta\delta + \cdots + \beta^t\delta^t) + v_{T-t}$ is the value function for correctly chosen constants v_{T-t}. Also find the optimal strategy for each stage.
Remark: The reward function equals minus infinity for $c = 0$, but this just means that it is very undesirable.

SP.SD.6 *Indicate* which of the following statements are always *true* and which are *false*. Also give a short *reason* for you answer.

- **a**. For a finite horizon dynamic programming problem with C^1 reward functions $r_t(s, a)$ and C^1 transition functions $f_t(s, a)$ and continuous feasible action correspondences $\Phi_t(s)$, the optimal strategy *must be* a continuous function.
- **b**. The feasible set $\mathscr{F}$ for a linear program is always a convex set.
- **c**. If $f : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous, then the point $c^* = \sigma^*(s)$ that maximizes $f(s, c)$ for $c \in [0, s]$ is a continuous function of s.