Math 303 Final: December 14, 2001 C. Robinson

No books, no notes, start each problem on a new page of your blue book

1. (50 Points) Consider the system of differential equations

$$\dot{x} = (x-1)\,(y-1)$$
$$\dot{y} = 3 - xy$$

which has fixed points at $(1, 3)$ and $(3, 1)$.

a. Determine the type of the linearized equations at each fixed point (saddle, stable node, etc.).

b. Determine the nullclines and the signs of $\dot{x}$ and $\dot{y}$ in the regions determined by the nullclines.

c. Draw the phase portrait for the system using the information from parts (a) and (b). Explain your sketch of the phase portrait.

2. (45 Points) Consider the system of differential equations

$$\dot{x} = y + 2\,\mu\,x - x^3,$$
$$\dot{y} = -x.$$

a. Check whether the origin is weakly attracting or repelling for $\mu = 0$ by using the test function $L(x, y) = \dfrac{x^2 + y^2}{2}$.

b. Show that there is a Hopf bifurcation as μ varies.

c. Is the bifurcation subcritical or supercritical? Is the periodic orbit attracting or repelling? Does the periodic orbit appear for $\mu < 0$ or $\mu > 0$?

3. (30 Points) Show that the system of differential equations

$$\dot{x} = y + x - x\,(x^2 + 4\,y^2)$$
$$\dot{y} = -x$$

has a periodic orbit. (Over for problems 4 and 5)

4. (45 Points) Consider the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = x - 2\,x^3 + y\,(x^2 - x^4 - y^2).$$

The test function

$$L(x,y) = \frac{-x^2 + x^4 + y^2}{2},$$

has $\dot{L} = -2\,y^2\,L(x,y)$.

a. Draw the phase portrait. Hint: What do the level curves of L look like?
b. What are the attracting sets and attractors for this system of differential equations.
c. For points (x_0, y_0) with $L(x_0, y_0) = 0$, what are $\alpha(x_0, y_0)$ and $\omega(x_0, y_0)$?

5, (30 Points) Consider the forced damped pendulum given by

$$\begin{aligned} \dot{x} &= y && \text{modulo } 2\,\pi \\ \dot{y} &= -\sin(x) - y + \cos(\tau) && \\ \dot{\tau} &= 1 && \text{modulo } 2\,\pi. \end{aligned}$$

Here, the x variable is considered an angle variable and is taken modulo 2π. The forcing variable τ is also taken modulo 2π.

a. What is the divergence of the system of equations?
b. If V_0 is the volume of a region D, what is the volume of the region $D(t) = \phi(t; D)$? (The region $D(t)$ is the region formed by following the trajectories of points starting in D at time 0 and following them to time t.)
c. Show that the region $R = \{(x, y, \tau) : |y| \leq 2$ is a trapping region. and $\bigcap_{t \geq 0} \phi(t; R)$ is an attracting set.