Math 303 **Final: June 2005** **C. Robinson**

1. (50 Points) Consider the system of differential equations

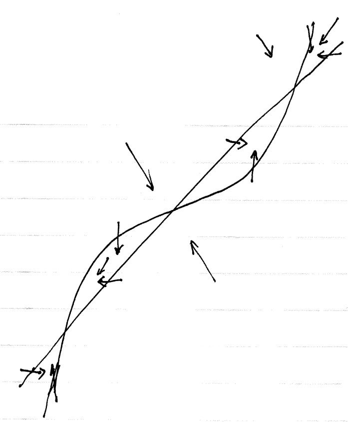
$$\dot{x} = y - x^3$$
$$\dot{y} = x - y.$$

a. Determine the nullclines and the signs of $\dot{x}$ and $\dot{y}$ in the various regions of the plane.
b. Determine the fixed points and the (linearized) stability type of each fixed point.
c. Using the information from parts (a) and (b), sketch by hand a rough phase portrait. Explain and justify your sketch.
d. Indicate what are the ω-limit sets of various points in the plane. In particular, pay attention to how the stable manifolds of saddle fixed points separates the plane into different regions.

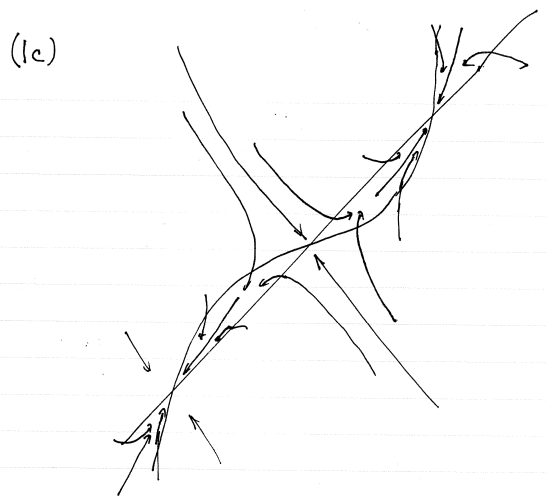
Ans:

(**a**) The nullclines are $y = x$ for $\dot{y} = 0$, and $y = x^3$ for $\dot{x} = 0$.
$\dot{x} < 0$ on $y = x$ and $x > 1$; $\dot{x} > 0$ on $y = x$ and $0 < x < 1$; $\dot{x} < 0$ on $y = x$ and $-1 < x < 0$; $\dot{x} > 0$ on $y = x$ and $x < -1$.
$\dot{y} < 0$ on $y = x^3$ and $x > 1$; $\dot{y} > 0$ on $y = x^3$ and $0 < x < 1$; $\dot{y} < 0$ on $y = x^3$ and $-1 < x < 0$; $\dot{y} > 0$ on $y = x^3$ and $x < -1$.
$\mathbf{R}_1 = \{x > y$ and $y < x^3\}$: $\dot{x} < 0$ and $\dot{y} > 0$.
$\mathbf{R}_2 = \{x < y$ and $y > x^3\}$: $\dot{x} > 0$ and $\dot{y} < 0$.
$\mathbf{R}_3 = \{x > y, y > x^3,$ and $x < -1\}$: $\dot{x} > 0$ and $\dot{y} > 0$.
$\mathbf{R}_4 = \{x < y, y < x^3,$ and $-1 < x < 0\}$: $\dot{x} < 0$ and $\dot{y} < 0$.
$\mathbf{R}_5 = \{x > y, y > x^3,$ and $0 < x < 1\}$: $\dot{x} > 0$ and $\dot{y} > 0$.
$\mathbf{R}_6 = \{x < y, y < x^3,$ and $1 < x\}$: $\dot{x} < 0$ and $\dot{y} < 0$.

(1a)



(b) The fixed points are $(0,0)$, $(1,1)$, and $(-1,-1)$. $DF_{(x,y)} = \begin{pmatrix} -3x^2 & 1 \\ 1 & -1 \end{pmatrix}$. At $(0,0)$, the determinant is $\Delta = -1$ so the origin is a saddle. At $\pm(1,1)$, the determinant is $\Delta = 2$, and the trace is $\tau = -4$, and the fixed points are stable nodes.
(c) Most orbits go to $\pm(1,1)$.



(d) The ω-limits of points on the stable manifold of the origin is the origin. To the left and below this stable manifold the ω-limit is the point $(-1,-1)$. To the right and above this stable manifold the ω-limit is the point $(1,1)$.

2. (50 Points) Consider the system of differential equations

$$\dot{x} = ax - y - x(x^2 + y^2)$$
$$\dot{y} = 6x + ay - y(x^2 + y^2).$$

The only fixed point is the origin $(0,0)$. (You do not need to prove this.)

a. Classify the fixed point at the origin, depending on the parameter a.

b. Consider the test function $L(x,y) = \dfrac{6x^2 + y^2}{2}$. Calculate the time derivative of L, $\dot{L}$.

c. For $a = -2$, show that $(0,0)$ is asymptotically stable with basin of attraction all of $\mathbb{R}^2$ (globally asymptotically stable).

d. For $a = 2$, show that there is a periodic orbit.

Ans:

(a) $DF_{(0,0)} = \begin{pmatrix} a & -1 \\ 6 & a \end{pmatrix}$. The characteristic equation is $0 = \lambda^2 - 2a\lambda + a^2 + 6$. The determinant is $\Delta = a^2 + 6 > 0$. The trace is $2a$. Thus the fixed point is attracting for $a < 0$, repelling for $a > 0$, and a center for $a = 0$.

(b)

$$\begin{aligned}
\dot{L} &= 6x\dot{x} + y\dot{y} \\
&= 6x(a - y - x(x^2 + y^2)) + y(6x + ay - y(x^2 + y^2)) \\
&= 6x^2(a - (x^2 + y^2)) + y^2(a - x^2 - y^2) \\
&= (6x^2 + y^2)(a - x^2 - y^2).
\end{aligned}$$

(c) For $a = -2$, $\dot{L} < 0$ for $(x, y) \neq (0, 0)$. Also, $L(x, y) > 0$ for $(x, y) \neq (0, 0)$ is positive definite with a unique minimum at the origina. Thus, it is a strict Lyapunov function. Therefore, the origin is globally asymptotically stable.
(d) For $a = 2$, the origin is repelling. In particular, on the set $2 = 2\,L(x, y) = 6x^2 + y^2 \geq x^2 + y^2$, $\dot{L} \geq 0$. Also, $\dot{L} \leq 0$ for $12 = 2\,L(x, y) = 6x^2 + y^2 \leq 6(x^2 + y^2)$. Thus, the set

$$\mathbf{D} = \{(x, y) : 1 \leq L(x, y) \leq 6\}$$

is positively invariant and contains no fixed points. Therefore, there is a periodic orbit by the Poincaré-Bendixson Theorem.

3. (60 Points) Let $V(x) = -2x^6 + 15x^4 - 24x^2$, for which $V'(x) = -12x^5 + 60x^3 - 48x$, $V'(0) = V'(\pm 1) = V'(\pm 2) = 0$, $V(0) = 0$, $V(\pm 1) = -11$, and $V(\pm 2) = 16$. Also, $V(x)$ goes to $-\infty$ as x goes to $\pm\infty$. Let $L(x, y) = V(x) + {}^{y^2}\!/_2$.

a. Plot the potential function $V(x)$ and sketch the phase portrait for the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = -V'(x).$$

b. Sketch the phase portrait for the system of differential equations

$$\dot{x} = y$$
$$\dot{y} = -V'(x) - y\,L(x, y).$$

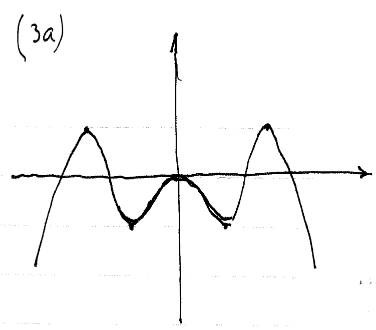
Pay special attention to the stable and unstable manifolds of saddle fixed points.

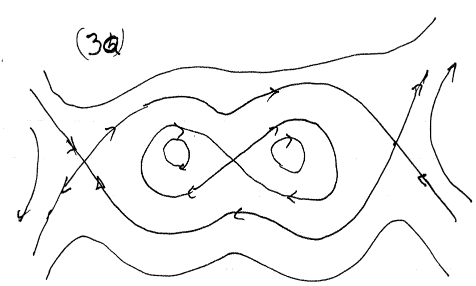
c. What are the possible ω-limit sets of points for the system of part (b)?

d. Let $\mathbf{A} = \{(x, y) \in L^{-1}(0) : -2 < x < 2\}$. Does the system of part (b) have sensitive dependence on initial conditions at points of the set $\mathbf{A}$? Does it have have sensitive dependence on initial conditions when restricted to $\mathbf{A}$?

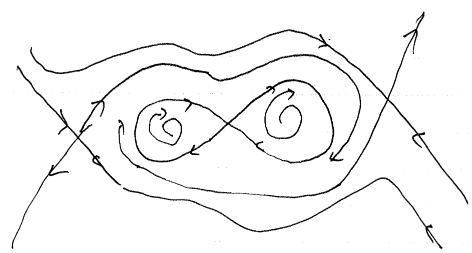
e. What are the attracting sets and attractors for the system of part (b)?

Ans:
(a) The graph has 3 peaks, with the outer two taller than the inner one.
The stable and unstable manifolds of $(0, 0)$ coincide: they are homoclinic. One branch of stable manifold of $(2, 0)$ goes to $(-2, 0)$, and one branch of the stable manifold of $(-2, 0)$ goes to $(2, 0)$.





(**b**) The figure eight containing $(0,0)$ persists. The orbits inside spiral out toward it. Those outside spiral in.



(**c**) **A**, $(0,0)$, $(-2,0)$, $(2,0)$, $(1,0)$, $(-1,0)$, the part of **A** with $x \geq 0$, and the part of **A** with $x \leq 0$.
(**d**) It does have sensitive dependence at all points of **A**, but not when restricted to **A**.
(**e**) The attracting sets are **A**, $\mathbf{A} \cup \{(x,y) : L(x,y) \leq 0,\ -2 < x < 2\}$, $\mathbf{A} \cup \{(x,y) : L(x,y) \leq 0,\ 0 < x < 2\}$, and $\mathbf{A} \cup \{(x,y) : L(x,y) \leq 0,\ -2 < x < 0\}$. Only **A** is an attractor.

4. (20 Points) Consider the system of differential equations

$$\dot{\tau} = 1 \qquad\qquad (\text{mod } 2\,\pi\,)$$
$$\dot{x} = (x - x^2)\,\big(2 + \cos(\tau)\big).$$

Notice that there are two periodic orbits: $\gamma_0 = \{\,(\tau, 0) : 0 \leq \tau \leq 2\pi\,\}$ and $\gamma_1 = \{\,(\tau, 1) : 0 \leq \tau \leq 2\pi\,\}$.

a. What is the divergence of the system of equations?

b. Find the derivative of the Poincaré map for the two periodic orbit γ_0 and γ_1. Is each periodic orbit orbitally asymptotically stable or unstable (repelling)?

Ans:

(**a**) The divergence is $(1 - 2x)((2 + \cos(\tau))$.
(**b**) At $x = 0$, the divergence is $2 + \cos(\tau)$, so

$$P'(0) = \exp\left(\int_0^{2\pi} 2 + \cos(\tau)\,d\tau\right) = e^{4\pi}.$$

Thus, γ_0 is orbitally asymptotically unstable.
At $x = 1$, divergence is $-2 - \cos(\tau)$,

$$P'(1) = \exp\left(\int_0^{2\pi} -2 - \cos(\tau)\,d\tau\right) = e^{-4\pi}.$$

Thus, γ_0 is orbitally asymptotically stable.

5. (20 Points) Consider the system of differential equations given by

$$\dot{\tau} = 1 + \frac{1}{2}\sin(\theta) \qquad (\bmod\ 2\pi\,)$$
$$\dot{\theta} = 0 \qquad (\bmod\ 2\pi\,).$$

Explain why the system has sensitive dependence on initial conditions at all points. Hint: The flow is a shear with the speed of τ depending on θ.

Ans:

For a point (τ_0, θ_0), a nearby point has a different value of theta (τ_0, θ_1). For most points, is is possible to choose θ_1 so that $\sin(\theta_1) > \sin(\tau_0)$. Then, the orbit through (τ_0, θ_1) goes faster and runs ahead. It can get about π units ahead. Certainly $r = {}^{\pi}\!/2$ will work to give sensitive dependence.