Math C13-2 Final 6/6/95 C. Robinson

Closed book. You may use hand calculators.

1. (30 Points) Let $F(z) = z^3$.
   a) Describe the orbit of z_0 if $|z_0| > 1$.
   b) Describe the orbit of z_0 if $|z_0| < 1$.
   c) Determine the filled in Julia set and the Julia set.

2. (50 Points) Consider the map given in polar coordinates by

$$F\begin{pmatrix}\theta \\ r\end{pmatrix} = \begin{pmatrix} 8\,\theta - \dfrac{\pi}{8} \\ 1 + \dfrac{1}{16}\,r + \dfrac{2}{\pi}\,\theta \end{pmatrix}$$

for $0 \le \theta \le \dfrac{\pi}{2}$ and $1 \le r \le 2$. (Notice this definition is only for part of the plane.) Let

$$V_L = \{(\theta, r) : 0 \le \theta \le 3\,\pi/32,\ 1 \le r \le 2\}$$
$$V_R = \{(\theta, r) : \pi/4 \le \theta \le 11\,\pi/32,\ 1 \le r \le 2\}.$$

   a) Let $g(\mathbf{q}) = \dfrac{f(\mathbf{q}) - \mathbf{q}}{\|f(\mathbf{q}) - \mathbf{q}\|}$. Show that g is essential as a map from ∂V_L to C and ∂V_R to C.
   b) Show that $\{V_L, V_R\}$ is a Markov partition for the invariant set of all points whose complete orbit lies in the region $S = \{(\theta, r) : 0 \le \theta \le \pi/2,\ 1 \le r \le 2\}$.
   c) Let Λ be the invariant set in the region S. Let x_0 be a point in S and $\mathbf{v}_0 = \begin{pmatrix}1 \\ 0\end{pmatrix}$ be the vector point in the θ direction. Prove that the Lyapunov exponent for x_0 and $\mathbf{v}_0$ is greater than or equal to $\ln(8)$.
   d) Show that Λ has a chaotic orbit. Note: You do not need to prove all the claims, but discuss why various properties for the orbit are true.

3. (30 Points) Let $F\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}2\,xy + y \\ 3\,y - x\end{pmatrix}$. Find and classify the fixed points.

(over)

4. (30 Points) Consider $Q(x) = x^2 - \frac{43}{36}$. Notice that the fixed points are $p_\pm = \frac{1}{2} \pm \frac{1}{3}\sqrt{13} = 0.5 \pm 1.202$ and the orbit of period two is $\{q_\pm\} = \{\frac{-3 \pm 4}{6}\} = \{\frac{1}{6}, \frac{-7}{6}\}$. Also if x_0 is any point in the open interval $(-p_+, p_+)$ then either (i) there is an iterate $Q^{k_0}(x_0)$ which equals p_- or (ii) the forward orbit $Q^k(x_0)$ is asymptotic to the orbit $\{q_-, q_+\}$. How many different Lyapunov exponents are there for points in $[-p_+, p_+]$? What are they? (You may use the facts given as statements. You only need to answer the questions.)

5. (30 Points) Consider the iterated function system with no rotation, points $\mathbf{p}_0 = (0.5, 0)$, $\mathbf{p}_1 = (0, 0.5)$, $\mathbf{p}_2 = (0.5, 1)$, and $\mathbf{p}_3 = (1, 0.5)$. and contraction factor $\beta = 0.4$. Let S be the attractor determined by this iterated function system.
   a) Describe the set S.
   b) Determine the fractal dimension of S.
   c) Determine the topological dimension of S.
   d) How many boxes of size 0.16 are needed to cover S?

6. (30 Points) Write a short discussion of the question "What is a chaotic dynamical system?" Include a discussion of measurements of chaos and some examples (without showing that the examples are chaotic).