Math C13-2 Take Home Final due June 9, 1999 C. Robinson

1. Consider the differential equations
$$\dot{x} = -2x + y$$
$$\dot{y} = 4 - 2xy.$$
   (a) Find the fixed points and classify them as asymptotically stable, stable, or unstable.
   (b) Solve the linearized equation at each fixed point.
   (c) Sketch the phase portrait for the nonlinear system. Include the isoclines where $\dot{x} = 0$ and $\dot{y} = 0$, and a small copy of the phase portrait of the linearized equation centered at each fixed point.

2. Consider the differential equations
$$\dot{x} = y$$
$$\dot{y} = -x + x^3 - y.$$
   (a) Find a (weak) Lyapunov function for these equations near $(0, 0)$.
   (b) Use the Lyapunov function to discuss the basis of attraction of $(0, 0)$. How big is it?

3. Consider the differential equations
$$\dot{x} = y$$
$$\dot{y} = -x - y \ln(0.1 + x^2 + 4y^2).$$
   Let $E(x, y) = \dfrac{x^2 + y^2}{2}$.
   (a) Calculate $\dot{E}$.
   (b) Show $\dot{E}(x, y) \geq 0$ for $\sqrt{x^2 + y^2} = 0.25$.
   (c) Show $\dot{E}(x, y) \leq 0$ for $\sqrt{x^2 + y^2} = 1$
   (d) Let $A = \{(x, y) \,:\, 0.25 \leq \sqrt{x^2 + y^2} \leq 1\,\}$. Using the Poincaré Bendixson Theorem, show for $(x_0, y_0) \in A$ that $\omega(x_0, y_0)$ is a periodic orbit. Why can't $\omega(x_0, y_0)$ contain a fixed point?

4. Consider the map given in polar coordinates by
$$F\begin{pmatrix}\theta\\ r\end{pmatrix} = \begin{pmatrix} 8\,\theta - \dfrac{\pi}{8} \\ 1 + \dfrac{1}{16}r + \dfrac{2}{\pi}\,\theta \end{pmatrix}$$
   for $0 \leq \theta \leq \dfrac{\pi}{2}$ and $1 \leq r \leq 2$. (Notice this definition is only for part of the plane.)
   (a) Let $\mathbf{p}_0$ be a point and $\mathbf{v}_0 = \begin{pmatrix}1\\0\end{pmatrix}$ be the vector point in the θ direction. Prove that the Lyapunov exponent for the vector $\mathbf{v}_0$ along the orbit of $\mathbf{p}_0$ is equal to $\ln(8)$.
   (b) Using the determinant of the matrix of partial derivatives, determine the second Lyapunov exponent of the orbit.
   (c) Explain why there can be a chaotic orbits. Note: You do not need to prove all the claims, but discuss why various properties for the orbit might be true.

5. Consider the figure with corners at the four points $\mathbf{p}_0 = (0.5, 0)$, $\mathbf{p}_1 = (0, 0.5)$, $\mathbf{p}_2 = (0.5, 1)$, and $\mathbf{p}_3 = (1, 0.5)$. At the next step, take four copies inside which are smaller by a factor of 0.4 with each copy touching a different corner of the original figure. Repeat with each remaining quadrilateral. Call the limiting set S.
   (a) Describe the set S.
   (b) Determine the box dimension of S.

6. Write a short discussion of the question "What is a chaotic dynamical system?" Include a discussion of measurements of chaos and some examples (without showing that the examples are chaotic).