1. (20 Points) Consider the equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x + y \\ 4\,x \end{pmatrix}.$$

Using the Heun (Improved Euler) Method, $h = 0.1$, and initial condition $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, find the approximate solution at time $t = 0.2$.

Ans:

t_n	$\mathbf{x}_n$	$\mathbf{k}_1$	$\mathbf{z}$	$\mathbf{k}_2$	$\frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)$
0.0	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 1.3 \\ 2.4 \end{pmatrix}$	$\begin{pmatrix} 3.7 \\ 5.2 \end{pmatrix}$	$\begin{pmatrix} 3.35 \\ 4.6 \end{pmatrix}$
0.1	$\begin{pmatrix} 1.335 \\ 2.46 \end{pmatrix}$	$\begin{pmatrix} 3.795 \\ 5.34 \end{pmatrix}$	$\begin{pmatrix} 1.7145 \\ 2.994 \end{pmatrix}$	$\begin{pmatrix} 4.7085 \\ 6.858 \end{pmatrix}$	$\begin{pmatrix} 4.25175 \\ 6.099 \end{pmatrix}$
0.2	$\begin{pmatrix} 1.760175 \\ 3.0699 \end{pmatrix}$				

2. (20 Points) Consider the following system of linear differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -6 & 3 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with eigenvalues -3 and -5 and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

a. Give the general real solution.

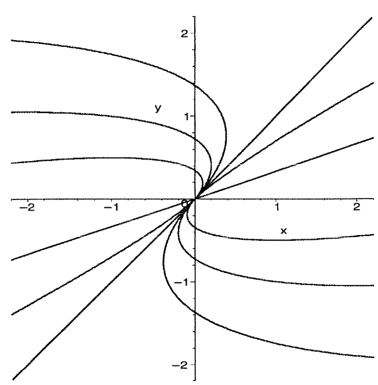
b. Draw the phase portrait. Be sure to indicate the relative strengths of the eigendirections.

Ans:

(a)

$$C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-5t} \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

(b) Most trajectories come in asymptotic to the direction of the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.



3. (25 Points) Consider the equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - x^2 \\ x - y \end{pmatrix}.$$

a. Find the two fixed points. Classify each of them as stable node, stable focus, saddle, unstable node, etc.

b. Draw the phase portraits using the nullclines and the answer to part (a).

Ans:

(**a**) The fixed points are $(0,0)$ and $(1,1)$. The matrix of partial derivatives is

$$\begin{pmatrix} -2x & 1 \\ 1 & -1 \end{pmatrix}.$$

At $(0,0)$, it is

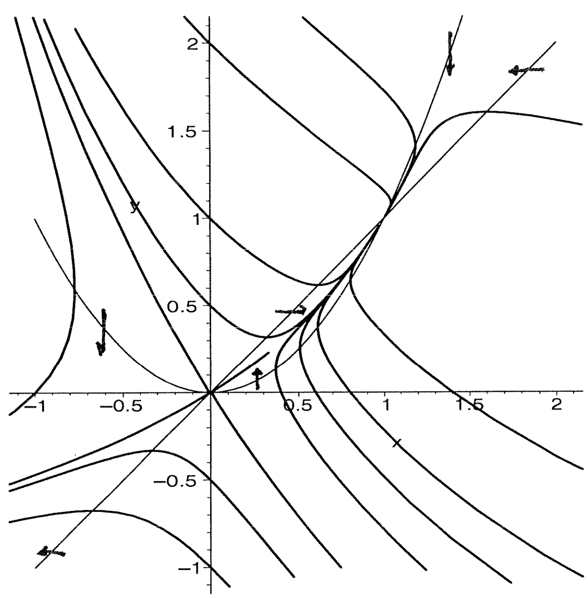
$$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

which has determinant -1 and is a saddle. At $(1,1)$, The matrix of partial derivatives is

$$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix},$$

which has determinant 1, trace -3, characteristic equation $\lambda^2 + 3\lambda + 1$, eigenvalues $\lambda = (-3 \pm \sqrt{5})/2$, and is a stable node.

(**b**) The nullclines are $\{\dot{y} = 0\} = \{y = x\}$, where the vector field is vertical, and $\{\dot{x} = 0\} = \{y = x^2\}$, where the vector field is horizontal. The phase portrait should show the saddle at $(0,0)$ and the stable node at $(1,1)$.



4. (35 Points) Let $V(x) = -\dfrac{x^4}{4} - \dfrac{x^3}{3} + x^2$. Notice that $V'(x) = -x^3 - x^2 + 2x$, $V'(1) = V'(0) = V'(-2) = 0$, $V(1) = 5/12$, and $V(0) = 0$, $V(-2) = 8/3 = 2.66\ldots.$

a. Sketch the graph of $V(x)$.

b. Draw the phase portrait for the system

$$\dot{x} = y$$
$$\dot{y} = x^3 + x^2 - 2x.$$

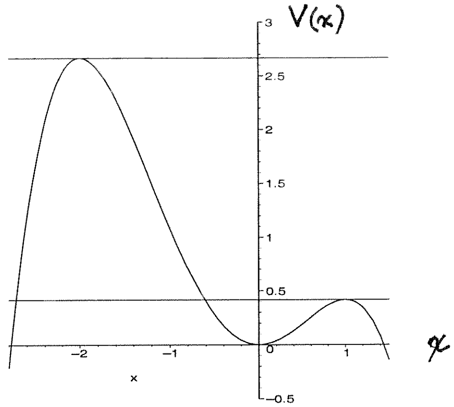
c. Find a Lyapunov function for

$$\dot{x} = y$$
$$\dot{y} = x^3 + x^2 - 2x - y.$$

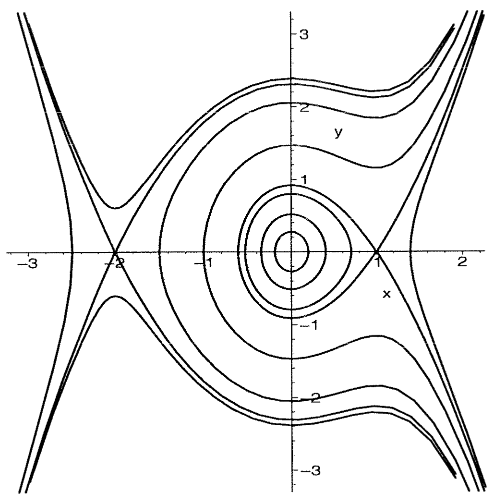
Verify that it is a Lyapunov function. Using this Lyapunov function, what is the largest region which can be (easily) shown to be in the basin of the attracting fixed point.

Ans:

(**a**) The graph of $V(x)$ has local maxima at $x = -2$ and 1 and a local minimum at $x = 0$. The height at -2 is higher than at 1.



(**b**) The phase portrait should have two saddles at $(x, y) = (-2, 0)$ and $(1, 0)$, and a center at $(0, 0)$. The stable manifold of $(1, 0)$ to the left loops around $(0, 0)$ and is homoclinic back to $(1, 0)$. To the right, the stable and unstable manifolds of $(1, 0)$ extend off to infinity.
The stable and unstable manifolds of $(-2, 0)$ to the left of $(-2, 0)$ extend off to (minus) infinity. The stable and unstable manifolds of $(-2, 0)$ to the right go below and above the fixed points $(0, 0)$ and $(1, 0)$ and extend off to infinity.



(**c**) Let $L(x, y) = V(x) + {}^{y^2}\!/_2$. Then, $\dot{L} = (-x^3 - x^2 + 2x)(y) + y\,(x^3 + x^2 - 2x - y) = -y^2$. This is a weak Lyapunov function. The center for the undamped equation become attracting. The region cannot include the other fixed points $(-2, 0)$ and $(1, 0)$. Therefore, we need to take a value less than $V(1) = {}^5\!/_{12}$ and $V(-2) = {}^{14}\!/_3$, i.e., less than ${}^5\!/_{12}$. There is a point $-2 < x_1 < 0$ where $V(x_1) = {}^5\!/_{12}$. Let

$$\mathbf{U} = \{(x, y) : L(x, y) < {}^5\!/_{12},\; x_1 < x < 1\}.$$

Then, L is a weak Lyapunov function in $\mathbf{U}$. $L(x, y) > L(0, 0)$ at points in $\mathbf{U}$ other than $(0, 0)$. The set $\mathbf{Z_U} = \{(x, 0) : x_1 < x < 1\}$. The maximal invariant set in $\mathbf{Z_U}$ is $(0, 0)$. Therefore, by the theorem in the book, all of $\mathbf{U}$ is in the basin of attraction of $(0, 0)$.