Math 313-1 Test 2: February 20, 2002 C. Robinson

No books, no notes. You may use hand calculators

1. (30 Points) Let $g(x) = \frac{2}{5}\,x^3 - \frac{7}{5}\,x$. The fixed points are 0, and $\pm\sqrt{6}$. There is a period-2 orbit of 1 and -1. The critical points are $\pm\sqrt{\frac{7}{6}}$. Notice that

$$g\left(\left(-\sqrt{6},\sqrt{6}\right)\right) \subset \left(-\sqrt{6},\sqrt{6}\right).$$

You may use the fact that every point in $[-\sqrt{6},\sqrt{6}]$ is in the basin of either the fixed points or points of period two. See back of test for plot of $g(x)$ and $g^2(x)$.

   a. Calculate the Schwarzian derivative,

$$S_g(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2.$$

   b. Determine the Lyapunov exponents $h(x_0)$ for all the points x_0 in $[-\sqrt{6},\sqrt{6}]$. Explain why your answer is correct.

   c. Let $x_0 = g\left(\sqrt{\frac{7}{6}}\right)$. What is the Lyapunov exponent of x_0.

2. (25 Points) Let f be a continuous function defined on the interval $[1,6]$ with $f(1) = 5$, $f(2) = 6$, $f(3) = 4$, $f(4) = 1$, $f(5) = 2$, and $f(6) = 3$. Assume the the function is linear between these integers.

   (a) Label the intervals between the integers and give the transition graph.

   (b) For which n is there a period-n orbit? Give the Symbol sequence in terms of the intervals which will give each period that exists.

3. (20 Points) Let $f(x) = x^3$ and $g(y) = \frac{1}{4}\,y^3 + \frac{3}{2}\,y^2 + 3\,y$. Verify that $y = C(x) = 2\,x - 2$ is a conjugacy between f and g.

4. (25 Points) Let

$$T(x) = \begin{cases} 5\,x & x \le 0.5 \\ 5\,(1-x) & x \ge 0.5. \end{cases}$$

   a. Sketch the graph of T.

   b. Describe the set of points x such that $x, T(x), T^2(x) \in [0,1]$,

$$\{x : T^j(x) \in [0,1] \text{ for } 0 \le j \le 2\}.$$

   It is made up of how many intervals of what length? What is its total length?

   c. Let

$$K = \{x : T^j(x) \in [0,1] \text{ for all } j \ge 0\}.$$

   Explain which numbers in $[0,1]$ belong to K in terms of the numbers expansion base 5, i.e.,

$$x = \sum_{j=1}^{\infty} \frac{a_j}{5^j}.$$

   d. Give a number in K that is not an end point of one of the intervals in the finite process defining K.

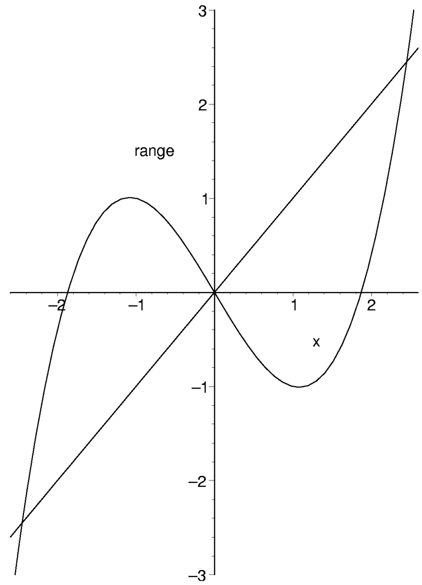

FIGURE 1. Plot of graph of $g(x)$ for problem 1.

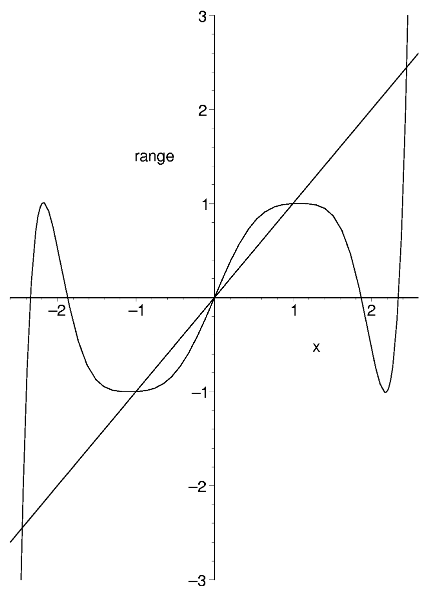

FIGURE 2. Plot of graph of $g^2(x)$ for problem 1.