9.3.1 An Invariant Measure for an Expanding Markov Map

A C^2 piecewise expanding map of the interval is a relatively simple example with an absolutely continuous ergodic invariant measure. This case was proved by Lasota and Yorke (1973). We consider an even simpler situation which has the Markov Property.

Definition. Let $I = [0, 1]$ be the closed unit interval. A map $f : I \to I$ is called a C^2 *piecewise expanding Markov map* provided there is a partition into subintervals $I_i = [x_i, x_{i+1}]$ for $i = 1, \dots, k$ with endpoints $0 = x_1 < x_2 < \cdots < x_{k+1} = 1$ satisfying the following properties.

(i) The map f is C^2 on $\bigcup_i \operatorname{int}(I_i)$ with a uniform bound on the function and its first two derivatives.

(ii) (Onto) The image of f covers the union of the open subintervals,

$$f(\bigcup_i \operatorname{int}(I_i)) \supset \bigcup_i \operatorname{int}(I_i).$$

(iii) (Piecewise expanding) There exist constants $1 < \alpha < \beta$ such that $\alpha \le |f'(x)| \le \beta$ for all $x \in \bigcup_i \operatorname{int}(I_i)$.

(iv) (Markov Property) If $f(\operatorname{int}(I_i)) \cap \operatorname{int}(I_j) \ne \emptyset$, then $f(\operatorname{int}(I_i)) \supset \operatorname{int}(I_j)$.

In the simplest case, f induces a full k-shift, i.e., $f(\operatorname{int}(I_i)) = \operatorname{int}(I)$ for all $1 \le i \le k$. An example of such a map is $f(x) = k\,x \bmod 1$.

The main theorem below states that a C^2 piecewise expanding Markov map has an invariant measure μ which is absolutely continuous with respect to Lebesgue measure, λ, i.e., $\mu = \rho\lambda$ where $\rho : I \to \mathbb{R}$ is an L^1 function. In fact the density function is bounded and bounded away from 0, so $\log(\rho)$ is bounded. This condition on ρ implies that the measure μ is equivalent to Lebesgue, in the sense that it has the same sets of zero measure as Lebesgue.

Theorem. *Assume $f : I \to I$ is a C^2 piecewise expanding Markov map. Then there exists an invariant measure μ with the following properties.*

(a) *The measure μ is absolutely continuous with respect to Lebesgue measure, $\mu = \rho^*\,\lambda$. Moreover, the density function ρ^* is continuous on each $\operatorname{int}(I_i)$ and $\log(\rho^*)$ is bounded so the measure is equivalent to Lebesgue.*

(b) *Assume that the subshift of finite type induced by f is irreducible. Then the map f is ergodic with respect to μ.*

To the complexity of the proof, we assume that f induces a full k-shift. It turns out in this case that the density function ρ^* is continuous on all of I. The book Pollicott and Yuri (1998) has a proof of the general case, which leads to a density function with possibly discontinuities at the end points of the partition intervals.

When f induces a full k-shift, f restricted to each $\operatorname{int}(I_i)$ has an inverse $\psi_i : \operatorname{int}(I) \to \operatorname{int}(I_i)$ which extends to a C^1 function ψ_i from all of I to I_i. (If f merely has the Markov Property, then the different ψ_i gave different domains which must be taken care of in the proof.) (If f does not have the Markov Property, then Lasota and Yorke consider the total variation of the function to get a similar convergence.)

Before we start the proof, we want to derive an operator whose fixed point correspond to invariant measures which are absolutely continuous with respect to Lebesgue. If μ is of the form $\rho\,\lambda$, then

$$\int_I h(y)\,d\mu(y) = \int_I h(y)\rho(y)\,d\lambda(y)$$

for any continuous function h. For the measure μ to be invariant, we need

$$\int_I h(y)\,d\mu(y) = \int_I h\circ f(x)\,d\mu(x)$$

for any continuous function h, i.e.,

$$\begin{aligned}\int_I h(y)\rho(y)\,d\lambda(y) &= \int_I h\circ f(x)\rho(x)\,d\lambda(x)\\ &= \sum_{i=1}^{k}\int_{I_i} h\circ f(x)\rho(x)\,d\lambda(x)\\ &= \sum_{i=1}^{k}\int_I h(y)\rho(\psi_i(y))|\psi_i'(y)|\,d\lambda(y)\\ &= \int_I h(y)\,\big[\sum_{i=1}^{k}\rho(\psi_i(y))|\psi_i'(y)|\big]\,d\lambda(y),\end{aligned}$$

where the third equality uses the change of variable formula for the substitution $x = \psi_i(y)$. If we define

$$\begin{aligned}\mathcal{L}(\rho)(y) &= \sum_{i=1}^{k}\rho(\psi_i(y))|\psi_i'(y)|\\ &= \sum_{x\in f^{-1}(y)}\frac{\rho(x)}{|f'(x)|},\end{aligned}$$

then we need

$$\mathcal{L}(\rho) = \rho.$$

Conversely, if $\mathcal{L}(\rho) = \rho$, then $\mu = \rho\lambda$ is an invariant measure for f which is absolutely continuous with respect to Lebesgue. The operator $\mathcal{L}$ is called the *Perron-Frobenius operator*. It maps densities for which $\log(\rho)$ is bounded to densities with the same property. Also, by the change of variables formula,

$$\int_I \mathcal{L}(\rho)(y)\,d\lambda(y) = \int_I \rho(y)\,d\lambda(y),$$

so $\mathcal{L}$ preserves densities of integral one. Finally, $\mathcal{L}$ is easily seen to by a bounded linear operator on the space of continuous functions on all of I. (This fact uses the extension of ψ_i to all of I. If f induces a subshift of finite type, then it is necessary to allow discontinuities at the x_j which can be the end points of the domain of definition of one of the ψ_i.) Notice that we do not prove that $\mathcal{L}$ is a contraction.

Proof of Part (a) of the Theorem:.

The idea of the proof is to show that the sequence of densities 1, $\mathcal{L}(1)$, $\mathcal{L}^2(1)$, $\cdots$ is bounded and equicontinuous on the intervals $\mathrm{int}(I_i)$ separately. It follows that $P_n = \dfrac{1}{n}[1 + \mathcal{L}(1) + \cdots + \mathcal{L}^{n-1}(1)]$ is bounded and equicontinuous on the intervals $\mathrm{int}(I_i)$ separately. By the Ascoli-Arzela Theorem, there is a subsequence P_{n_j} which converge to a function ρ^* which is continuous on $\bigcup_{i=1}^{k} \mathrm{int}(I_i)$. Also ρ^* is fixed by $\mathcal{L}$ because

$$\begin{aligned}\mathcal{L}(\rho^*) \leftarrow \mathcal{L}(P_{n_j}) &= \mathcal{L}(\frac{1}{n_j}[1 + \mathcal{L}(1) + \cdots + \mathcal{L}^{n_j - 1}(1)]) \\ &= \frac{1}{n_j}[\mathcal{L}(1) + \cdots + \mathcal{L}^{n_j}(1)] \\ &= P_{n_j} + \frac{1}{n_j}[\mathcal{L}^{n_j}(1) - 1] \to \rho^*.\end{aligned}$$

To show that the family of densities in equicontinuous, we show that the logarithm of the densities $\mathcal{L}^n(1)$ is uniformly Lipschitz on each $\mathrm{int}(I_i)$, i.e., there is a constant C independent of n such that for $y, y' \in I$,

$$\log(\mathcal{L}^n(1)(y)) - \log(\mathcal{L}^n(1)(y'))| \leq C\,|y - y'|,$$

or

$$\frac{\mathcal{L}^n(1)(y)}{\mathcal{L}^n(1)(y')} \leq e^{C\,|y-y'|}.$$

As a first step in the proof of the of the uniform Lipschitz constant, we prove the following lemma about the ratios of derivatives of powers of f are uniformly Lipschitz. For $\mathbf{i} = (i_1, \ldots, i_n)$, let $\psi_{\mathbf{i}}$ be the inverse of f^n restricted to $\bigcap_{j=0}^{n-1} f^j(I_{i_{j+1}})$. Notice for $x \in \bigcap_{j=0}^{n-1} f^j(I_{i_{j+1}})$, $x \in I_{i_1}$, $f(x) \in I_{i_2}, \ldots,$ $f^{n-1}(x) \in I_{i_n}$. Therefore, for any $y, y' \in I$, $\psi_{i_n}(y), \psi_{i_n}(y') \in I_{i_n}$, $\psi_{(i_{n-j},\ldots,i_n)}(y), \psi_{(i_{n-j},\ldots,i_n)}(y') \in I_{i_{n-j}}$, and $\psi_{\mathbf{i}}(y), \psi_{\mathbf{i}}(y') \in I_{i_1}$. Let $I_{\mathbf{i}} = \psi_{\mathbf{i}}(I)$.

Lemma 3.1. *Let f and $\psi_{\mathbf{i}}$ be as above. There is a constant C_1 independent of n and $\mathbf{i}$, such that the following are true.*

(1) *For $x, x' \in \bigcap_{j=0}^{n-1} f^j(I_{i_{j+1}})$.*

$$\frac{|(f^n)'(x)|}{|(f^n)'(x')|} \leq e^{C_1\,|f^n(x) - f^n(x')|}.$$

(2) *For $y, y' \in \mathrm{int}(I)$ and for $\mathbf{i} = (i_1, \ldots, i_n)$,*

$$\frac{|\psi_{\mathbf{i}}'(y')|}{|\psi_{\mathbf{i}}'(y)|} \leq e^{C_1\,|y-y'|}.$$

PROOF. Because the derivative of the inverse is one over the derivative of the function, the two parts are equivalent.

Considering the second part,

$$\begin{aligned}
\log\left|\frac{\psi'_{(i_1,\ldots,i_n}(y')}{\psi'_{(i_1,\ldots,i_n}(y)}\right| &= \sum_{j=n}^{n}\log\left|\frac{\psi_{i_j}(\psi_{(i_{j+1},\ldots,i_n}(y'))}{\psi_{i_j}(\psi_{(i_{j+1},\ldots,i_n}(y))}\right| \\
&= \sum_{j=n}^{n}\log\left|\frac{f'(\psi_{(i_j,\ldots,i_n}(y))}{f'(\psi_{(i_j,\ldots,i_n}(y'))}\right. \\
&\leq \sum_{j=n}^{n}\log(1 + C_2\,|\psi_{(i_j,\ldots,i_n}(y) - \psi_{(i_j,\ldots,i_n}(y')|) \\
&\leq \sum_{j=n}^{n}\log\big(1 + C_2\,\frac{|y-y'|}{\alpha^{n+1-j}}\big) \\
&\leq \log\big(1 + C_2\,|y-y'|\sum_{m=1}^{\infty}\alpha^{-m}\big) \\
&= \log\big(1 + \frac{C_2\,|y-y'|}{\alpha-1}\big) \\
&\leq \frac{C_2\,|y-y'|}{\alpha-1}.
\end{aligned}$$

where C_2 bounds $\dfrac{\sup_t|f''(t)|}{\inf_x|f'(x)|}$. Letting $C_1 = C_2/(\alpha-1)$, we get the result. □

Notice that $\mathcal{L}^n$ is the $\mathcal{L}$-function for f^n. Using the last lemma,

$$\begin{aligned}
\mathcal{L}^n(1)(y) &= \sum_{(i_i,\ldots,i_n)}|\psi'_{(i_i,\ldots,i_n)}(y)| \\
&\leq \sum_{(i_i,\ldots,i_n)}|\psi'_{(i_i,\ldots,i_n)}(y')|e^{C_1\,|y-y'|} \\
&= \mathcal{L}^n(1)(y')e^{C_1\,|y-y'|}.
\end{aligned}$$

Taking logarithms, we get that

$$\log(\mathcal{L}^n(1)(y)) - \log(\mathcal{L}^n(1)(y')) \leq C_1\,|y-y'|,$$

where C_1 is independent of n. Because $\int_I \mathcal{L}^n(1)(y)\,d\lambda(y) = 1$, $\mathcal{L}^n(1)(y)$ and $\log(\mathcal{L}^n(1)(y))$ are uniformly bounded. It follows that $\mathcal{L}^n(1)$ and so P_n are a bounded and equicontinuous family of functions. By the argument given earlier, a subsequence converge to a fixed point ρ^* for $\mathcal{L}$. Because $\log(\rho^*(y))$ is uniformly bounded, the measure induced by ρ^* is equivalent to Lebesgue. □

Proof of Part (b) of the Theorem:.

In this part of the proof, we again assume that the subshift induced by f is the full k-shift. See Pollicott and Yuri (1998) for a proof of the general case of f, and for a stronger conclusion than ergodicity (strongly mixing).

Let A be an invariant set in the Borel σ-algebra with $\mu(A) > 0$, and so $\lambda(A) > 0$; by invariant by f we mean that $A = f^{-1}(A)$. To show that f is ergodic, we need to show that $\mu(A) = 1$ or $\mu(A^c) = \mu(I \setminus A) = 0$. Given any $\delta > 0$, we want to show the Lebesgue measure of A^c is small relative to δ so $\lambda(A^c) = 0$ and so $\mu(A^c) = 0$.

We first want to see in what sense the cylinder sets form a generating partition partition for the Borel σ-algebra. (cf. Walters, Theorem 5.25.) Fix a set B in the Borel σ-algebra. For any

$\epsilon > 0$, there exists an open set Q such that $B \subset Q$ and $\lambda(Q \setminus B) < \epsilon/2$. (See Royden Proposition 15 page 63.) For any $\mathbf{i} = (i_1, \dots, i_m)$, the length of $I_{\mathbf{i}}$ is less than α^{-m}. Because the sets $I_{\mathbf{i}}$ get smaller with the length of $\mathbf{i}$, there exists a countable cover of Q and so of B by cylinder sets with $\lambda(\bigcup_{\mathbf{i}\in J} I_{\mathbf{i}} \setminus Q) < \epsilon/2$ and $\lambda(\bigcup_{\mathbf{i}\in J} I_{\mathbf{i}} \setminus B) < \epsilon$. (The open set Q is the countable union of open intervals, each of with can be covered within an arbitrarily small Lebesgue measure by a countable number of the cylinder sets.) Therefore, for each set B in the Borel σ-algebra, there is a countable number of cylinder sets $\{I_{\mathbf{i}}\}_{\mathbf{i}\in J}$ such that $B \subset \bigcup_{\mathbf{i}\in J} I_{\mathbf{i}}$ and $\lambda(\bigcup_{\mathbf{i}\in J} I_{\mathbf{i}} \setminus B) < \epsilon$.

Next we show that given any $\delta > 0$, there exits at least one cylinder $I_{\mathbf{j}}$ with $\mathbf{j} = (j_1, \dots j_n)$ for which

$$\begin{aligned} \lambda(A \cap I_{\mathbf{j}}) &\geq (1-\delta)\,\lambda(I_{\mathbf{j}}), \qquad \text{or equivalently} \\ \lambda(A^c \cap I_{\mathbf{j}}) &\leq \delta\,\lambda(I_{\mathbf{j}}). \end{aligned}$$

If this is false for all $\mathbf{i}$, then $\lambda(A \cap I_{\mathbf{i}}) \leq (1-\delta)\,\lambda(I_{\mathbf{i}})$ for all $\mathbf{i}$, and so $\lambda(A \cap \bigcup_{\mathbf{i}\in J} I_{\mathbf{i}}) \leq (1-\delta)\,\lambda(\bigcup_{\mathbf{i}\in J} I_{\mathbf{i}})$ for any countable union of cylinder sets. Because the cylinder sets form a generating partition, any set B in the Borel σ-algebra can be approximated by a countable union of disjoint cylinders, and so $\lambda(A \cap B) \leq (1-\delta)\,\lambda(B)$. By taking $B = A$, we get that $\lambda(A) \leq (1-\delta)\,\lambda(A)$, which contradicts the fact that $\delta > 0$ and both $\mu(A) > 0$ and $\lambda(A) > 0$.

By Lemma 3.1,

$$\frac{\sup_{x\in I_{\mathbf{j}}} |(f^n)'(x)|}{\inf_{x\in I_{\mathbf{j}}} |(f^n)'(x)|} \leq C_0 = e^{C_1}$$

for $\mathbf{j} = (j_1, \dots, j_n)$ as above. Since $f^n(I_{\mathbf{j}}) = I$ and $f^n(A)^c = A^c$, $A^c = f^n(A)^c \cap f^n(I_{\mathbf{j}}) = f^n(A^c \cap I_{\mathbf{j}})$. Applying the change of variables formula for integration,

$$\begin{aligned} \lambda(A^c) &= \lambda(f^n(A^c \cap I_{\mathbf{j}})) \\ &= \int_{I_{\mathbf{j}}\cap A^c} |(f^n)'(x)|\, d\lambda(x) \\ &\leq \sup_{x\in I_{\mathbf{j}}}\{|(f^n)'(x)|\}\, \lambda(I_{\mathbf{j}} \cap A^c) \\ &\leq C_0 \inf_{x\in I_{\mathbf{j}}}\{|(f^n)'(x)|\}\, \lambda(I_{\mathbf{j}} \cap A^c) \\ &\leq C_0\, \frac{\lambda(I_{\mathbf{j}} \cap A^c)}{\lambda(I_{\mathbf{j}})} \qquad \left(\text{since } \inf_{x\in I_{\mathbf{j}}}\{|(f^n)'(x)|\}\, \lambda(I_{\mathbf{j}}) \leq \lambda(f^n(I_{\mathbf{j}})) = 1\right) \\ &\leq C_0\, \delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary and C_0 is independent of n and so of $\mathbf{j}$ and δ, $\lambda(A^c) = 0$ and so $\mu(A^c) = 0$. This completes the proof that f is ergodic. □

References

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