WHAT IS A CHAOTIC ATTRACTOR?

CLARK ROBINSON

Abstract. Devaney gave a mathematical definition of the term chaos, which had earlier been introduced by Yorke. We discuss issues involved in choosing the properties that characterize chaos. We also discuss how this term can be combined with the definition of an attractor.

1. Introduction

J. Yorke coined the word 'chaos' as applied to deterministic systems. R. Devaney gave the first mathematical definition for a map to be chaotic on the whole space where a map is defined. Since that time, there have been several different definitions of chaos which emphasize different aspects of the map. Some of these are more computable and others are more mathematical. See [9] a comparison of many of these definitions.

There is probably no one best or correct definition of chaos. In this paper, we discuss what we feel is one of better mathematical definition. (It may not be as computable as some of the other definitions, e.g., the one by Alligood, Sauer, and Yorke.) Our definition is very similar to the one given by Martelli in [8] and [9]. We also combine the concepts of chaos and attractors and discuss chaotic attractors.

2. Basic definitions

We start by giving the basic definitions needed to define a chaotic attractor. We give the definitions for a diffeomorphism (or map), but those for a system of differential equations are similar.

The orbit of a point $\mathbf{x}^*$ by $\mathbf{F}$ is the set $O(\mathbf{x}^*, \mathbf{F}) = \{ \mathbf{F}^i(\mathbf{x}^*) : i \in \mathbb{Z} \}$.

An *invariant set* for a diffeomorphism $\mathbf{F}$ is an set $\mathbf{A}$ in the domain such that $\mathbf{F}(\mathbf{A}) = \mathbf{A}$. Therefore, for every $\mathbf{x}$ in $\mathbf{A}$, the orbit $O(\mathbf{x}, \mathbf{F})$ is entirely contained in $\mathbf{A}$.

An invariant set $\mathbf{A}$ is *topologically transitive* provided that there is a point $\mathbf{x}^*$ in $\mathbf{A}$ such that the orbit or $\mathbf{x}^*$ is dense in $\mathbf{A}$, i.e., $\mathrm{cl}\,(O(\mathbf{x}^*, \mathbf{F})) = \mathbf{A}$.

A diffeomorphism $\mathbf{F}$ has *sensitive dependence on initial conditions at all points of* $\mathbf{A}$ provided that for each point $\mathbf{x} \in \mathbf{A}$ there is an $r > 0$ such that for all $\delta > 0$, there are $\mathbf{y}$ and $n \geq 1$ with $\|\mathbf{y} - \mathbf{x}\| \leq \delta$ and $\|\mathbf{F}^n(\mathbf{y}) - \mathbf{F}^n(\mathbf{x})\| > r$.

A diffeomorphism $\mathbf{F}$ has *sensitive dependence on initial conditions when restricted to* $\mathbf{A}$ provided that for each point $\mathbf{x} \in \mathbf{A}$ there is an $r > 0$ such that for all $\delta > 0$, there are $\mathbf{y} \in \mathbf{A}$ and $n \geq 1$ with $\|\mathbf{y} - \mathbf{x}\| \leq \delta$ and $\|\mathbf{F}^n(\mathbf{y}) - \mathbf{F}^n(\mathbf{x})\| > r$. This condition means that the nearby point $\mathbf{y}$ whose orbit move away from the orbit of $\mathbf{x}$ can be chosen in the set $\mathbf{A}$ and not just in the ambient space.

1991 *Mathematics Subject Classification.* 34C, 58F.
Key words and phrases. chaos, attractor.

Martelli [9] defines a concept which means that an orbit is not Lyapunov stable. An orbit $O(\mathbf{x}^*, \mathbf{F})$ in an invariant set $\mathbf{A}$ is called *unstable with respect to* $\mathbf{A}$ for $\mathbf{F}$ provided that there is an $r > 0$ such that for every $\delta > 0$, there is an $\mathbf{y} \in \mathbf{A}$ and $n \geq 1$ such that $\|\mathbf{y} - \mathbf{x}^*\| \leq \delta$ and $\|\mathbf{F}^n(\mathbf{y}) - \mathbf{F}^n(\mathbf{x}^*)\| > r$.

Let $\mathbf{A}$ be a compact topologically transitive invariant set for a diffeomorphism $\mathbf{F}$. In [9], they remark that $\mathbf{F}$ has sensitive dependence on initial conditions when restricted to $\mathbf{A}$ if and only if there is a dense orbit $O(\mathbf{x}^*)$ in $\mathbf{A}$ that is unstable with respect to $\mathbf{A}$.

A point $\mathbf{x}_0$ is *chain recurrent* provided that for each $\epsilon > 0$ there are a set of points $\{ \mathbf{x}_i : 1 \leq i \leq n+1 \}$ such that $\mathbf{x}_{n+1} = \mathbf{x}_0$ and

$$\|\mathbf{F}(\mathbf{x}_i) - \mathbf{x}_{i+1}\| < \epsilon \qquad \text{for } 0 \leq i \leq n.$$

The *chain recurrent set* $\mathcal{R}(\mathbf{F})$ is the set of all points which are chain recurrent for $\mathbf{F}$. Chain recurrence is a weak type of recurrence.

3. Attractor, chaos, and chaotic attractor

Our definition of an attractor (as in [10] and [11]) is similar to asymptotic stability of a fixed point, which requires that the point is Lyapunov stable in addition to assuming that nearby point converge to the fixed point under iteration. Therefore, we give the definition in terms of a trapping region.

Definition 3.1. A set $\mathbf{U}$ is called a *trapping region* $\mathbf{U}$ for map $\mathbf{F}$ provided its closure is compact and

$$\mathrm{cl}(\mathbf{F}(\mathbf{U})) \subset \mathrm{int}(\mathbf{U}).$$

Since the closure in mapped into the interior, the set is mapped well inside itself.

A set $\mathbf{A}$ is called an *attracting set* if there exists trapping region $\mathbf{U}$ such that

$$\mathbf{A} = \bigcap_{j \geq 0} \mathbf{F}^j(\mathbf{U}).$$

Because a trapping region has compact closure, an attracting set is necessarily compact.

A set $\mathbf{A}$ is called an *attractor* for $\mathbf{F}$ if it is an attracting set such that there are no nontrivial subattracting sets, i.e., no attracting set $\mathbf{A}' \subset \mathbf{A}$ such that $\emptyset \neq \mathbf{A}' \neq \mathbf{A}$.

The following theorem, which is a consequence of Conley's Fundamental Theorem of Dynamical Systems (see [10]), relates the "minimality" of the attracting set to the chain recurrent set.

Theorem 3.2.

(a) *An attractor is contained in the chain recurrent set,* $\mathbf{A} \subset \mathcal{R}(\mathbf{F})$.
(b) *An attracting set* $\mathbf{A}$ *that is contained in the chain recurrent set,* $\mathbf{A} \subset \mathcal{R}(\mathbf{F})$, *is an attractor.*
(c) *An attractor is an isolated chain transitive component of the chain recurrent set.*

There are other definitions of an attractor. The most common other definition was given by Milnor. The basin of attraction of an invariant set $\mathbf{A}$ is the set

$$W^s(\mathbf{A}) \equiv \{ \mathbf{x} : \omega(\mathbf{x}) \subset \mathbf{A} \},$$

where $\omega(\mathbf{x})$ is the ω-limit set of the point. An invariant set $\mathbf{A}$ is a *Milnor attractor* provided that the Lebesgue measure of $W^s(\mathbf{A})$ is positive. The ideas is that there is a set of positive measure whose points tend to the invariant set under future iteration. This means that there is a positive probability of observing the invariant set by choosing an initial condition. A Milnor attractor neither has to attract all the point in a neighborhood nor does it have to be "Lyapunov stable", i.e., there can be points near $\mathbf{A}$ whose orbit must go a long distance from $\mathbf{A}$ before finally converging to $\mathbf{A}$.

J. Yorke coined the word of chaos for a deterministic system in [7]. Several different properties of a map that he called chaotic were discussed, but no formal mathematical definition was given.

R. Devaney was the first to give a precise mathematical definition in [4]. He defined a map $\mathbf{F} : \mathbf{X} \to \mathbf{X}$ to be *chaotic on* $\mathbf{X}$ provided that the following conditions are satisfied:

1. $\mathbf{F}$ has sensitive dependence on $\mathbf{X}$.
2. $\mathbf{F}$ is topologically transitive, i.e., there exists an $\mathbf{x}^*$ such that $\operatorname{cl}(O(\mathbf{x}^*, \mathbf{F})) = \mathbf{X}$.
3. The periodic points are dense in $\mathbf{X}$.

Notice that his definition only applies to a map on its whole domain. Banks et al showed in [2] that conditions (2) and (3) imply condition (1). There has been a continuing discussion about the "correct" or "best" definition of chaos. Some definitions are more mathematical while other definitions are more computable. See [9] for a discussion of various definitions given by different people.

Besides the question of the "correct" definition of chaos, there is also the issue of how to combine the definition of chaos with that of an attractor. The rest of this paper addresses this latter question together with our perspective on a mathematical definition of chaos.

The third condition on the density of the periodic points does not seem as central to the idea of chaos. It is also true C^1 generically on attractors. See Section 6. In Section 4, we also give examples that show that we need to assume that $\mathbf{F}$ restricted to the attractor $\mathbf{A}$ has sensitive dependence on initial conditions, not just sensitive dependence on initial conditions in $\mathbf{M}$ at all points of $\mathbf{A}$. Because of the existence of examples of the type given in Section 4, we keep Devaney's second condition of topological transitivity on the attractor, even though an attractor is automatically chain transitive. (This condition is not required in [11], but we now feel should be added to the definition.)

Definition 3.3. A diffeomorphism $\mathbf{F}$ is said to be *chaotic* on an invariant set $\mathbf{A}$ provided that the following conditions are satisfied:

(a) The diffeomorphism $\mathbf{F}$ has sensitive dependence on initial conditions when restricted to the attractor $\mathbf{A}$.

(b) The diffeomorphism $\mathbf{F}$ is is topological transitivity on $\mathbf{A}$, i.e., there exists an $\mathbf{x}^*$ such that $\operatorname{cl}(O(\mathbf{x}^*, \mathbf{F})) = \mathbf{A}$.

As set $\mathbf{A}$ is called a *chaotic attractor* for a diffeomorphism $\mathbf{F}$ provided that the set $\mathbf{A}$ is an attractor for $\mathbf{F}$ and $\mathbf{F}$ is chaotic on $\mathbf{A}$.

We have added condition (b) that the map is topologically transitive to the definition given in [11] in order to avoid some the the pathology of the examples given in Section 4. Also, a saddle periodic orbit is not chaotic because it has sensitive dependence in the ambient space but not when restricted to the periodic orbit.

This definition is very similar to the one of Martelli. He calls an invariant set $\mathbf{A}$ for $\mathbf{F} : \mathbf{M} \to \mathbf{M}$ *chaotic* provided that there exits an $\mathbf{x}^* \in \mathbf{A}$ that satisfied the following properties.

(i) the orbit $O(\mathbf{x}^*)$ is unstable with respect to $\mathbf{A}$, i.e., there is an $r > 0$ such that for every $\delta > 0$, there is an $\mathbf{y} \in \mathbf{A}$ and $n \geq 1$ such that $\|\mathbf{y} - \mathbf{x}^*\| \leq \delta$ and $\|\mathbf{F}^n(\mathbf{y}) - \mathbf{F}^n(\mathbf{x}^*)\| > r$. (Therefore, $\mathbf{F}$ has sensitive dependence on initial conditions when restricted to $\mathbf{A}$.)

(ii) $\mathrm{cl}\,(O(\mathbf{x}^*, \mathbf{F})) = \mathbf{A}$. (So, $\mathbf{F}$ is topologically transitive on $\mathbf{A}$.)

Alligood, Sauer, and Yorke give another definition in [1]. They define a set set $\mathbf{A}$ to be a *chaotic attractor* provided that the following conditions are satisfied.

(a) The set $\mathbf{A}$ is a Milnor attractor, the Lebesgue measure of the basin of attraction $(W^s(\mathbf{A}))$ is positive.

(b) There exists a point $\mathbf{p}_0 \in \mathbf{A}$ such that the following conditions are satisfied:
   (i) There is at least one positive Lyapunov exponents $h_1(\mathbf{p}_0) > 0$, and all of the Lyapunov exponents are nonzero, $h_j(\mathbf{p}_0) \neq 0$ for all j.
   (ii) $\omega(\mathbf{p}_0, \mathbf{F}) = \mathbf{A}$. (So, $\mathbf{F}$ is topologically transitive on $\mathbf{A}$.)
   (iii) $\omega(\mathbf{p}_0, \mathbf{F})$ is not a periodic orbit. In some contexts, they also require that the attractor is not a collection of fixed points and saddle connections. (See Example 4.1.)

Since Lyapunov exponents are calculate by computer simulation and for experimental data, this definition seems like a good test for an attractor, but we do not think it makes the best mathematical definition. Since they require that the point $\mathbf{p}_0$ is in the attractor, pathology of examples like 4.1 is avoided. However, this condition is not easily verified in examples, and so it makes the conditions less computable.

As mentioned in the introduction, several other definitions of chaos are given in the paper [9].

As a final remark, many people discuss chaos for systems without show that they have a transitive attractor. For example, some papers on the Lorenz system show that it contains a horseshoe system, but not that it has a transitive attractor. The only proof that we know that shows the the Lorenz system for the usual parameter values has a chaotic attractor is the one by Warwick Tucker, which is computer assisted. See [12].

4. Examples motivating the definition of chaos

We give several examples which illustrate the need for the conditions in our definition of a chaotic attractor.

Example 4.1 (Saddle connection)**.** The following example is given in [11]. Consider the system of differential equations

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2\,x^3 + y\,(x^2 - x^4 - y^2). \end{aligned}$$

The test function

$$L(x, y) = \frac{-x^2 + x^4 + y^2}{2}$$

satisfies the following time derivative:

$$\dot{L} = -2\,y^2\,L \begin{cases} \geq 0 & \text{when } L < 0 \\ = 0 & \text{when } L = 0 \\ \leq 0 & \text{when } L > 0. \end{cases}$$

Therefore, solutions with initial conditions in a neighborhood tend to the level set $\mathbf{A}^* = L^{-1}(0)$, which is the fixed point together with the two homoclinic orbits. Therefore, this set is an attracting set. See Figure 1. All the points on $\mathbf{A}^*$ are chain recurrent, so it is also an attractor.

The flow on this set does not appear to be chaotic. It also does not satisfy two of our conditions to be called a chaotic attractor.

The flow in not topologically transitive on the attractor, although there are points outside the attractor whose ω-limit set is the entire attractor.

It does have sensitive dependence on initial conditions (in ambient space) at all points of $\mathbf{A}^*$. However, it does not have sensitive dependence on initial conditions when restricted to $\mathbf{A}^*$. Therefore, it fails to satisfy our definition of a chaotic attractor on two accounts.

For a point $\mathbf{p}_0$ in $\mathbf{A}^*$, $\omega(\mathbf{p}_0, \mathbf{F}) = \{\mathbf{0}\}$, so $h_1(\mathbf{p}_0, \mathbf{F}) > 0$ and $h_2(\mathbf{p}_0, \mathbf{F}) < 0$. Since $\omega(\mathbf{p}_0, \mathbf{F})$ cannot equal $\mathbf{A}^*$ for $\mathbf{p}_0$ in $\mathbf{A}^*$, $\mathbf{A}^*$ does not satisfy the Alligood, Sauer, and Yorke definition for a chaotic attractor. For points $\mathbf{p}_0$ outside $\mathbf{A}^*$, $\omega(\mathbf{p}_0, \mathbf{F}) = \mathbf{A}^*$, but numerical simulation indicates that the Lyapunov exponents are probably zero.

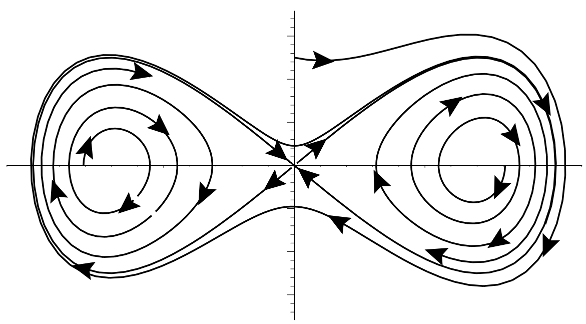

FIGURE 1. Phase portrait

By taking the time one map of the flow, we can make an example for a diffeomorphism.

Example 4.2. By adding the angle variables

$$\dot{\theta}_1 = 2 + \sin(\theta_2) \qquad \pmod{2\pi}$$
$$\dot{\theta}_2 = 0 \qquad \pmod{2\pi},$$

to Example 4.1, we obtain an example which has sensitive dependence on initial conditions when restricted to the attractor. The sensitive dependence comes from the shear in the θ-variables. However, this example is not topologically transitive on the attractor, so it still does not satisfy our definition of a chaotic attractor.

Example 4.3. Let $\mathbf{F}_1$ be the time one map of Example 4.1 and let $\mathbf{F}_2$ have an attractor $\mathbf{A}_2$ that has sensitive dependence when restricted to $\mathbf{A}_2$. Then, $\mathbf{F}_1 \times \mathbf{F}_2$ has sensitive dependence on $\mathbf{A}^* \times \mathbf{A}_2$, but is not topologically transitive on $\mathbf{A}^* \times \mathbf{A}_2$.

Examples 4.1–4.3 do not seem to have dynamics that should be called chaotic. Examples 4.2–4.3 do satisfy condition (a) of our definition but not (b), i.e., they are not topologically transitive. The possibility of such examples is the reason we decided to add topological transitivity to our definition of a chaotic attractor given in [11].

Example 4.4. The above examples are not generic. If we add a time dependent perturbation to Example 4.1, then the homoclinic connection is broken. The time one map of

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= x - 2\,x^3 + y\,(x^2 - x^4 - y^2) + 0.01\,\cos(2\pi\tau)\\ \dot{\tau} &= 1 \pmod 1.\end{aligned}$$

has the phase portrait given in Figure 2. This map probably has a chaotic attractor by our definition, but we do not attempt to verify this fact.

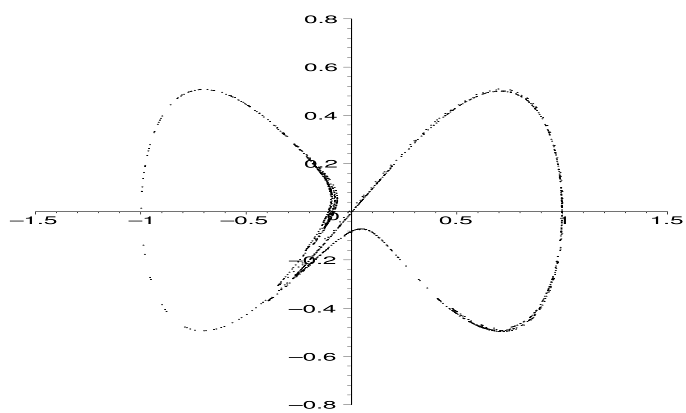

FIGURE 2. Phase portrait

Example 4.5. Martelli, Dang, and Seph in [9] give an example like the following which they say should not be called chaotic. Consider the system of differential equations (in polar coordinated)

$$\dot{r} = \begin{cases} e^{-(1-r)^{-2}} & \text{if } 1/2 \le r < 1\\ 0 & \text{if } 1 \le r \le 2\\ -e^{-(r-2)^{-2}} & \text{if } 2 < r.\end{cases}$$

$$\dot{\theta} = r.$$

The set $\{(r,\theta) : 1 \le r \le 2\ \}$ is an attractor (chain recurrent). It has sensitive dependence on initial conditions when restricted to the attractor. It should not be called chaotic.

5. A chaotic attractor with zero entropy

The example given in this section is different than the ones of the last section: it does satisfy our definition of a chaotic attractor, but it has zero topological entropy.

We have decided to give a definition that does call this map chaotic, but other might argue that it should not be so designated.

Since the definition of topological entropy is rather technical, we leave it to the references. See for example [10] or [6]. The idea is it measures the complexity of the system. If a system has positive topological entropy, then the number of "different orbits" grows exponentially as the length of the orbits considered grows. In the example below, the number only grows linearly. Any system with a transverse homoclinic orbit has positive topological entropy.

Example 5.1. We give an example which is transitive on the whole two torus. It could be made into an attractor in a large dimensional system by adding contracting directions. Let α be an irrational number. Define the map

$$\mathbf{F}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+\alpha \\ x+y \end{pmatrix} \qquad (\text{mod } 1).$$

(Both variables are taken modulo one.) The map is a skew product on the two torus. It is an irrational rotation in the x-variable. The rotation in the y-variable depends on the point x.

To see that the map has zero entropy, consider several points $(x_0, y_1), \dots (x_0, y_k)$ in the same fiber above a single x_0. The amount these points are rotated depends on the iterate, but all are rotated by the same amount on each iterate. Therefore, this map has zero entropy for points starting in all the same fiber. Also, the map of the x-variable is just an irrational rotation and has zero entropy. By a theorem of Bowen (see Remark 9.1.11 in [10]), the total map has zero entropy. The basic idea is that orbits for this map diverge at a linear rate and not an exponential rate, so the map has zero entropy.

The map has sensitive dependence on initial conditions because of the shear factor. If two points start at (x_0, y_0) and $(x_0 + \delta, y_0)$, then they will move apart in the y-variable under iteration.

We do not show directly that the map is topologically transitive. By the Birkhoff Transitivity Theorem (Theorem 8.2.1 in [10]), it is enough to show that for any two open sets $\mathbf{U}$ and $\mathbf{V}$, there is an $n > 0$ such that $\mathbf{F}^n(\mathbf{U}) \cap \mathbf{V} \neq \emptyset$. Inside a pair of such open sets, we can find squares; we can take an integer k large enough so that we can find (x_0, y_0) and (x_0', y_0') such that

$$[x_0, x_0 + 1/k] \times [y_0, y_0 + 1/k] \subset \mathbf{U} \qquad \text{and}$$
$$[x_0', x_0' + 2/k] \times [y_0', y_0' + 2/k] \subset \mathbf{V}.$$

Therefore, it is enough to use such squares for the two sets. Let $(x_j, y_j) = \mathbf{F}^j(x_0, y_0)$, so $x_j = x_0 + j\alpha$. Then, the j^{th}-iterate of the interval $[x_0, x_0 + 1/k] \times \{y_0\}$ is a line in the covering space (before taking modulo one) from (x_j, y_j) to
$(x_j + 1/k, y_j + j/k)$. Therefore, if $j \geq k$, then this line goes at least one complete time around the y-direction while the x-variable increases by $1/k$. Next, take a $j_0 \geq k$, such that $x_0' \leq x_{j_0} \leq x_0' + 1/k$. Then, $x_{j_0} + 1/k \leq x_0' + 2/k$, and all the x values on the line segment (x_{j_0}, y_{j_0}) to $(x_{j_0} + 1/k, y_{j_0} + j_0/k)$ lie between x_0' and $x_0' + 2/k$, and some y-value modulo one must equal y_0'; thus, the image of the line segment must intersect $\mathbf{V}$. This proves that the j_0-iterate of $\mathbf{U}$ must intersect $\mathbf{V}$.

Since this this intersection property is true for any pair of open sets, $\mathbf{F}$ is topologically transitive by the Birkhoff Transitivity Theorem.

The Lyapunov exponents of this system are both zero. It is clear that the length of a vector purely in the y-direction is preserved under iteration, so such a vector give rise to a zero Lyapunov exponent. The system preserves area, so the other Lyapunov exponent must also be zero. Therefore, this map does not satisfy the Alligood, Sauer, and Yorke conditions to be called chaotic. Their definition requires the orbits to separate at an exponential rate, which is closer to requiring that the system has positive topological entropy.

6. Generic Properties

The Closing Lemma of Pugh implies that for a generic C^1 diffeomorphism, the periodic points are dense in the nonwandering set. Recently, Hayashi proved a very strong generalization of the Closing Lemma called the Connecting Lemma, which has been used to prove other generic properties. In particular, Bonatti and Crovisier [3] used Hayashi's Connecting Lemma to show that for a C^1 generic diffeomorphism $\mathbf{F}$, it is topologically transitive on any of its attractors. If the attractor is topologically transitive, then all the points in the attractor are nonwandering. Therefore, for a C^1 generic diffeomorphism $\mathbf{F}$, it automatically satisfies our condition (b) for a chaotic attractor and also Devaney's condition (3) about the density of the periodic points.

References

[1] Alligood, K., T. Sauer, and J. Yorke, *Chaos: An Introduction to Dynamical Systems*, Springer–Verlag, New York–Berlin–Heidelberg, 1997.

[2] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, "On Devaney's definition of chaos", *Amer. Math. Monthly*, 99 (1992), pp. 332-334.

[3] C. Bonatti and S. Crovisier, "Récurrence et génricité", *Invent. Math.*, 158 (2004), pp. 33-104.

[4] Devaney, R., *An Introduction to Chaotic Dynamical Systems*, Addison–Wesley Publ. Co., New York & Reading, MA, 1989.

[5] Guckenheimer, J. and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer–Verlag, New York–Berlin–Heidelberg, 1983.

[6] Katok, H. and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge UK and New York, 1995.

[7] Li, T. and J. Yorke, "Period three implies chaos", *Amer. Math. Monthly* **82** (1975), 985 - 992.

[8] M. Martelli, *Introduction to Discrete Dynamical Systems and Chaos*, John Wiley & Sons, Inc., New York, 1999.

[9] M. Martelli, M. Dang, and T. Seph, "Defining Chaos", *Math. Mag.*, 71 (1998), pp. 112-122.

[10] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos* 2nd Edition, CRC Press, Boca Raton Florida, 1999.

[11] R.C. Robinson, *An Introduction to Dynamical Systems: Continuous and Discrete*, Pearson Prentice Hall, Upper Saddle River New Jersey, 2004.

[12] W. Tucker, "Lorenz system", *C R Acad. Sci. Paris Math*, 1999.

Department of Mathematics, Northwestern University, Evanston IL 60208
E-mail address: `clark@math.northwestern.edu`