

Melnikov Method for Autonomous Hamiltonians

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ABSTRACT. This paper presents the method of applying the Melnikov method to autonomous Hamiltonian systems in dimension four. Besides giving an application to Celestial Mechanics, it discusses the problem of convergence of the Melnikov function and the derivative of the Melnikov function.

1. Introduction

The original Melnikov method applies to periodically forced Hamiltonian systems in dimension two (one degree of freedom). In this paper, our main emphasis is to show how a modification of this method applies directly to autonomous Hamiltonian systems in dimension four (two degrees of freedom). These original unperturbed system needs to be completely integrable. By restricting to the energy surface, it is possible to convert the autonomous system in dimension four into a periodically forced Hamiltonian systems in dimension two and then apply the usual Melnikov method. It is our contention that this is not the natural method to treat this type of system. In fact the calculations are much more direct and natural when done directly with the autonomous system. Thus in this paper we advocate applying the Melnikov method directly to the autonomous Hamiltonian system. These ideas go back at least to the papers of P. Holmes and J. Marsden, [3] and [4]. Also see [9] and [6]. In Section 3 we discuss the main theorems in the theory of autonomous systems and their proofs. In Section 4 we indicate how these theorems can be applied to the circular restricted three body problem.

Because we want to connect our treatment with the material for periodically forced systems, we start by describing the periodically forced situation using our approach in Section 2.

This paper also points out the connection between the convergence of the improper integral giving the Melnikov function and the smoothness and dependence on parameters of the stable manifold of a periodic orbit. (See the discussion of Lemma 3.8 and the remarks which follow it.) This connection has also been made recently in the context of Celestial Mechanics by J. Casasayas, E. Fontich, and A. Nunes, [2].

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A final point made in this paper is the possibility of differentiating the Melnikov function by differentiating the integrand of its representation as a improper integral. See Theorem 3.5 in Section 3.

2. Periodically force systems

Consider a Hamiltonian $H : \mathbb{R}^2 \rightarrow \mathbb{R}$, with associated Hamiltonian vector field $X = X_H$. Next, consider the periodically force perturbation,

$$\begin{aligned}\dot{\mathbf{z}} &= X(\mathbf{z}) + \epsilon Y(\mathbf{z}, \tau) \\ \dot{\tau} &= 1\end{aligned}$$

where τ is considered as a periodic variable of some period T . We assume that X has a homoclinic orbit to a hyperbolic fixed point (in the \mathbf{z} -space). Let γ_0 be the associated periodic orbit in the (\mathbf{z}, τ) -space, \tilde{S} , and

$$\Gamma_0 = \{(\mathbf{z}_0, \tau_0) \mid \mathbf{z}_0 \text{ is homoclinic to } \gamma\}$$

be the set of homoclinic orbits in the extended phase space \tilde{S} .

Since γ_0 is hyperbolic, it can be continued for $\epsilon \neq 0$ to a periodic orbit γ_ϵ in the extended phase space \tilde{S} . For σ equal s and u , let $W^\sigma(\gamma_\epsilon, X + \epsilon Y)$ be the stable and unstable manifolds of γ_ϵ respectively. In order to determine the orbits on Γ_0 to a particular orbit on $W^\sigma(\gamma_\epsilon, X + \epsilon Y)$ for $\epsilon \neq 0$, we let $N_{(\mathbf{z}_0, \tau_0)}$ be a (one dimensional) transversal to Γ_0 at a point $(\mathbf{z}_0, \tau_0) \in \Gamma_0$. Then for $\sigma = s, u$, set

$$\zeta^\sigma(\mathbf{z}_0, \tau_0, \epsilon) = W^\sigma(\gamma_\epsilon, X + \epsilon Y) \cap N_{(\mathbf{z}_0, \tau_0)}.$$

In our development, we use strongly the fact that these stable and unstable manifold depend differentiably on the parameter and position on the manifold, so are smooth in the space \tilde{S} , i.e., for $\sigma = s$ and u , $\zeta^\sigma(\mathbf{z}_0, \tau_0, \epsilon)$ is a smooth function of all its variables.

The Hamiltonian function is constant on Γ_0 and has nonzero gradient, so it is a good measure of the displacement perpendicular to Γ_0 . With this motivation, we define the *Melnikov function* by

$$M(\mathbf{z}_0, \tau_0) = \frac{\partial}{\partial \epsilon} [H \circ \zeta^u(\mathbf{z}_0, \tau_0, \epsilon) - H \circ \zeta^s(\mathbf{z}_0, \tau_0, \epsilon)] \Big|_{\epsilon=0}.$$

Thus, M is a measurement of the infinitesimal separation of the stable and unstable manifolds as measured by the Hamiltonian function H . By parameterizing the solutions of the equations and restricting to a transverse cross section, M can be considered a function of a real variable (as it usually is). Notice that we use the fact that the stable and unstable manifolds depend smoothly on the parameter to define M .

There are two theorems related to M . The first states that a nondegenerate zero of M corresponds to a transverse homoclinic orbit of γ_ϵ for $\epsilon \neq 0$ but small enough. We refer to [9], [10], or other standard references for this theorem. (Also see Theorem 3.3 in the next section.) The second theorem relates M to an improper integral. We state this second theorem in order to emphasize the fact that with the proper statement, it can be calculated in any coordinate system (not just symplectic coordinates).

THEOREM 2.1.

$$\begin{aligned}
 (*) \quad M(\mathbf{z}_0, \tau_0) &= \int_{-\infty}^{\infty} \mathbf{k} \cdot (X \times Y)_{(\phi_0^t(\mathbf{z}_0), t+\tau_0)} dt \\
 (**) \quad &= \int_{-\infty}^{\infty} (DH \cdot Y)_{(\phi_0^t(\mathbf{z}_0), t+\tau_0)} dt.
 \end{aligned}$$

We remark that the integral in equality (*) is the usual one derived for the Melnikov function: the integrand is the scalar cross product (component in the \mathbf{k} direction) or wedge product of the two vector fields. This integrand must be calculated in symplectic coordinates. The the integral in equality (**) is valid in any set of coordinates, even those which are not symplectic. See [9] or [10] for a proof of the second form of the integral. Also in the discussion below of the autonomous case, we indicate a sketch of the proof why the above theorem is true. The proof in [9] uses the smoothness of the stable and unstable manifolds and not the uniform estimate for $0 \leq t < \infty$ on the difference of solutions for $\epsilon \neq 0$ and $\epsilon = 0$ as many proofs do. (See e.g. [1] and [2].)

3. Autonomous Hamiltonians in four dimensions

In this section, the phase space is some four dimensional space which we label as S^4 : in Celestial Mechanics applications S^4 is usually \mathbb{R}^4 or some product of copies of the circle (periodic variables) and copies of the reals. The autonomous Hamiltonian $H^\epsilon : S^4 \rightarrow \mathbb{R}$ is given as

$$H^\epsilon(\mathbf{z}) = H_0(\mathbf{z}) + \epsilon H_1(\mathbf{z}) + O(\epsilon^2).$$

The corresponding Hamiltonian vector field is given as

$$X^\epsilon(\mathbf{z}) = X_0(\mathbf{z}) + \epsilon Y(\mathbf{z}) + O(\epsilon^2).$$

The basic assumptions are as follows.

- (A1) We assume that X_0 is completely integrable, with a second independent integral F . (The function F is mostly likely not an integral of X^ϵ for $\epsilon \neq 0$.)
- (A2) We assume that X_0 has a hyperbolic periodic orbit γ_0 with set of homoclinic orbits

$$\begin{aligned}
 \Gamma_0 &= \text{comp}[W^u(\gamma_0, X_0) \setminus \gamma_0] \\
 &= \text{comp}[W^s(\gamma_0, X_0) \setminus \gamma_0].
 \end{aligned}$$

Here Γ_0 is one of the components of the stable and unstable manifolds indicated and has dimension two because of the second integral.

EXAMPLE 3.1. The easiest example of the above type of system is one which decouples for $\epsilon = 0$ into a subsystem with a homoclinic orbit and another That is oscillator: e.g., assume $H_0(\mathbf{z}, \mathbf{w}) = F(\mathbf{z}) + G(\mathbf{w})$ where

$$F(z_1, z_2) = \frac{z_2^2}{2} - \frac{z_1^2}{2} + \frac{z_1^3}{3}$$

has a homoclinic orbit to the fixed point at $\mathbf{0}$, and

$$G(w_1, w_2) = \frac{\alpha}{2}(w_1^2 + w_2^2)$$

is an oscillator. A simple coupling is of the type $H_1(\mathbf{z}, \mathbf{w}) = z_1 w_1$.

EXAMPLE 3.2. P. Holmes and J. Marsden [4] gave an example of a system which does not decouple. (Compare this example with the closely related one from Celestial Mechanics given in [1].) Let

$$H^\epsilon(p, q, \theta, I) = \frac{p^2}{2} + U(q) + \frac{I^2}{2q^2} + \epsilon \sin(\theta),$$

so

$$\dot{I} = -\epsilon \cos(\theta)$$

and $F = I$ is a second integral for $\epsilon = 0$. The potential function $U(q)$ must be chosen so that there is a homoclinic orbit in the (p, q) -subsystem for $\epsilon = 0$ and I fixed.

We set up the definition of the Melnikov function using transversals much as we did above for the periodically forced case. For each point $\mathbf{z}_0 \in \Gamma_0$, let $N_{\mathbf{z}_0}$ be a two dimensional plane perpendicular to Γ_0 at \mathbf{z}_0 . Let γ_ϵ be the continuation of the periodic orbit γ_0 for $\epsilon \neq 0$ with $\gamma_\epsilon \subset (H^\epsilon)^{-1}(h_0)$ where $h_0 = H_0(\gamma_0)$ is the energy of the unperturbed orbit. Then for $\sigma = s$ and u and $\epsilon \neq 0$, let

$$\zeta^\sigma(\mathbf{z}_0, h_0, \epsilon) = N_{\mathbf{z}_0} \cap W^\sigma(\gamma_\epsilon, X^\epsilon)$$

be the continuation of \mathbf{z}_0 . Notice that both the stable and the unstable manifolds continue to lie in the same energy surface $(H^\epsilon)^{-1}(h_0)$, so H_0 is not a good measure of the separation; on the other hand, the second integral F is constant on Γ_0 and is independent of H_0 so it is a good measure of the distance perpendicular to Γ_0 within this energy surface. The *Melnikov function*, $M : \Gamma_0 \rightarrow \mathbb{R}$ is then defined as the infinitesimal displacement of the stable and unstable manifolds as measured by the second integral F :

$$M(\mathbf{z}_0, h_0) = \frac{\partial}{\partial \epsilon} [F \circ \zeta^u(\mathbf{z}_0, h_0, \epsilon) - F \circ \zeta^s(\mathbf{z}_0, h_0, \epsilon)] \Big|_{\epsilon=0}.$$

By expressing M in terms of specific coordinates and restricting to a transverse section, it is possible to give it as a function of a single variable (plus the parameter h_0). See the example in the next section.

Again, there are two theorems about the Melnikov function. The first states that a nondegenerate zero of M corresponds to a transverse homoclinic point. The second theorem indicates how to calculate M in terms of an improper integral. In general for this case, the integral representation of M is only conditionally convergent, so the times must be chosen correctly in the improper integral. The third theorem below justifies differentiating under the integral sign to calculate the derivative of the Melnikov function.

THEOREM 3.3. *Make the basic assumptions A1-A2). If $M(\cdot, h_0) : \Gamma_0 \rightarrow \mathbb{R}$ has a nondegenerate zero at $\mathbf{z}_0 \in \Gamma_0$, then X^ϵ has a transverse homoclinic orbit \mathbf{z}_ϵ nearby for small enough nonzero ϵ .*

THEOREM 3.4. *Make the basic assumptions (A1-A2). Assume that T_j and $-T_j^*$ are chosen so that the points $\phi_0(T_j, \mathbf{z}_0)$ and $\phi_0(-T_j^*, \mathbf{z}_0)$ converge to the same point $\mathbf{z}_\infty \in \gamma_0$. Then*

$$M(\mathbf{z}_0, h_0) = \lim_{j \rightarrow \infty} \int_{-T_j^*}^{T_j} (DF \cdot Y)_{\phi_0(t, \mathbf{z}_0)} dt.$$

THEOREM 3.5. *Make the basic assumptions (A1-A2) as in Theorem 3.4 but assume that the integral converges absolutely (not conditional convergence). (See remarks following Lemma 3.8 for conditions which imply this assumption.) Then*

$$D_{\mathbf{z}}M_{(\mathbf{z}, h_0)}\mathbf{v} = \int_{-\infty}^{\infty} D_{\mathbf{z}}((DF \cdot Y)_{\phi_0(t, \mathbf{z})})\mathbf{v} dt.$$

In the above equation, $D_{\mathbf{z}}$ is differentiation with respect to the initial spatial coordinate \mathbf{z} , and is calculated in the direction of the vector \mathbf{v} .

REMARK 3.6. The integral in Theorems 3.4 and 3.5 can be calculated in any coordinates, even non-symplectic coordinates. This is relevant to applications in Celestial Mechanics because McGehee coordinates are non-symplectic coordinates. See Section 4.

REMARK 3.7. Although the convergence is only conditional in general, there are several circumstances in which the convergence is absolute. We make this more explicit below.

The proof of Theorem 3.3 follows as usual from the Implicit Function Theorem, so we do not give any details. See [10].

IDEA OF THE PROOF OF THEOREM 3.4. As in the usual proof,

$$M(\mathbf{z}_0, h_0) = \int_{-T_j^*}^{T_j} (DF \cdot Y)_{\phi_0(t, \mathbf{z}_0)} dt + R(-T_j^*, T_j)$$

where the remainder term

$$\begin{aligned} R(-T_j^*, T_j) &= \frac{\partial}{\partial \epsilon} [F \circ \phi(-T_j^*, \zeta^u(\mathbf{z}_0, \epsilon), \epsilon)] \Big|_{\epsilon=0} \\ &\quad - \frac{\partial}{\partial \epsilon} [F \circ \phi(T_j, \zeta^s(\mathbf{z}_0, \epsilon), \epsilon)] \Big|_{\epsilon=0}. \end{aligned}$$

In [9], we prove a lemma about the convergence of the remainder. The statement of Lemma 2.1 in [9] only includes the final statement of the following lemma but the proof in that paper implies the following extended statement. Clearly Lemma 3.8 implies the Theorem 3.4. \square

LEMMA 3.8. *Assume that T_j and $-T_j^*$ are chosen so that $\phi_0(-T_j^*, \mathbf{z}_0)$ converges to \mathbf{z}_{∞}^- and $\phi_0(T_j, \mathbf{z}_0)$ converges to \mathbf{z}_{∞}^+ as $j \rightarrow \infty$. Then*

$$\frac{\partial}{\partial \epsilon} [F \circ \phi(-T_j^*, \zeta^u(\mathbf{z}_0, \epsilon), \epsilon)] \Big|_{\epsilon=0}$$

converges to

$$\frac{\partial}{\partial \epsilon} [F \circ \gamma(\mathbf{z}_{\infty}^-, \epsilon)] \Big|_{\epsilon=0}$$

and

$$\frac{\partial}{\partial \epsilon} [F \circ \phi(T_j, \zeta^s(\mathbf{z}_0, \epsilon), \epsilon)] \Big|_{\epsilon=0}$$

converges to

$$\frac{\partial}{\partial \epsilon} [F \circ \gamma(\mathbf{z}_{\infty}^+, \epsilon)] \Big|_{\epsilon=0}.$$

Here $\gamma(\mathbf{z}, \epsilon)$ is the function which gives the perturbed orbit γ_{ϵ} for $\epsilon \neq 0$ as a graph over the unperturbed orbit. In particular, if $\mathbf{z}_{\infty}^- = \mathbf{z}_{\infty}^+$ then $R(-T_j^, T_j)$ goes to zero as j goes to infinity.*

REMARK 3.9. By Lemma 3.8, the improper integral in Theorem 3.4 always gives $M(\mathbf{z}_0, h_0)$ provided T_j and $-T_j^*$ are chosen so that $\mathbf{z}_\infty^- = \mathbf{z}_\infty^+$.

REMARK 3.10. If $DF_{(\mathbf{z})} \equiv 0$ for all $\mathbf{z} \in \gamma_0$, then the convergence is not conditional and the integral is just the improper integral from $-\infty$ to ∞ . This feature is essentially what is true in the proof for the usual Melnikov function when using the function H which vanishes at the fixed point.

REMARK 3.11. If the periodic orbit does not move with ϵ , $\gamma_\epsilon \equiv \gamma_0$, then the convergence is also a usual improper integral (and not conditional) since the limiting values are zero and do not need to cancel each other. When applying this theorem to parabolic orbits which are asymptotic to periodic orbits at infinity, this is often the case since the orbits at infinity are not affected by the small coupling terms. See the example of the circular restricted three body problem in Section 4.

REMARK 3.12. What makes the convergence work in the proof of Lemma 3.8 is the stable manifold theory. Casasayas, Fontich, and Nunes proved in [2] that this type of result is often true even when the periodic orbit is weakly hyperbolic (from higher order terms). Notice that they emphasize that it is possible to prove a stable manifold theorem in the parabolic case with a particular smoothness. Then then prove that this smoothness of the stable manifold is sufficient to imply a convergence of the above type to give the Melnikov function as an improper integral. They treat the usual time periodic perturbations but their results go over to the above context of autonomous systems as well.

PROOF OF THEOREM 3.5. The fact that we can differentiate under the integral sign follows from a theorem in analysis about differentiating improper integrals. What is needed to differentiate under the integral sign of an improper integral is that the integrals for $M(\mathbf{z}, h_0)$ and $D_{\mathbf{z}}M(\mathbf{z}, h_0)\mathbf{v}$ converge uniformly in \mathbf{z} . See [5] or [7]. In our situation the convergence of both integrals is uniform for the same reason that the integral converges in the proof of Lemma 3.8 as given in [9]. \square

4. Planar Restricted Three Body Problem Example

Both J. Llibre and C. Simo in [8] and Z. Xia in [11] applied the usual Melnikov method to the circular planar three body problem. In this section, we indicate how the same results can be obtain (possibly) more directly by using the method for the autonomous systems indicated in the last section. Our remarks about the applicability in non-symplectic coordinates also justifies making the calculations in McGehee coordinates.

Below we sketch the introduction of the coordinates into the problem. See the above references for more detailed explanations and treatment. Our treatment most closely follows that in [11].

In this situation, the third mass $m_3 \equiv 0$, $m_1 = \mu$ is a small parameter, and $m_2 = 1 - \mu$. Jacobi coordinates are used as a starting place. The problem is circular so the variable

$$\mathbf{Q} = (\cos(t), \sin(t))$$

gives the motion of the vector from m_2 to m_1 . The position of m_3 relative to m_1 and m_2 is given by \mathbf{q} .

Next, McGehee coordinates are introduced. The variable x measures the distance from infinity,

$$x^{-2} = |\mathbf{q}|,$$

so x goes to zero as $|\mathbf{q}|$ goes to infinity. The direction to m_3 is given by θ ,

$$\frac{\mathbf{q}}{|\mathbf{q}|} = (\cos \theta, \sin \theta).$$

This angle is measured in rotating coordinates by introducing the 2π -periodic variable $s = t - \theta$. The momentum variable for m_3 , \mathbf{p} is broken down into radial and angular components by introducing the variables y and ρ respectively by the equation

$$\mathbf{p} = [y + x^2 \rho i][\cos \theta + i \sin \theta].$$

(This last equations uses the notation of complex variables for points in \mathbb{R}^2 .) The energy equation in these variables becomes

$$H^\mu = \frac{1}{2}y^2 + \frac{1}{4}x^4\rho^2 - U^\mu$$

where $U^0 = x^2$.

When $\mu = 0$, $\dot{\rho} = 0$, so ρ , which is the angular momentum of m_3 (the Jacobi constant), is a second integral. After a change of the time variable, the equations become

$$\begin{aligned} x' &= -\frac{1}{2}x^3y \\ y' &= -x^4 + x^2\rho^2 + \mu g_1(x, s) + O(\mu^2) \\ \rho' &= \mu g_2(x, s) + O(\mu^2) \\ s' &= 1 - x^4\rho. \end{aligned}$$

Here s is considered a 2π periodic variable.

When $\mu = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a fixed point in $\begin{pmatrix} x \\ y \end{pmatrix}$ -space. This fixed point corresponds to a periodic orbit in the total space (with s the periodic variable). This periodic orbit is weakly hyperbolic but it has smooth stable and unstable manifolds which correspond to the parabolic orbits of the original problem. We parameterize the homoclinic orbits

$$\begin{aligned} x &= \xi_0(t, \rho_0, s_0) \\ y &= \eta(t, \rho_0, s_0) \\ s &= s_0 + \sigma(t, \rho_0) \end{aligned}$$

where we take the initial conditions so that $\eta(0, \rho_0, s_0) = 0$, i.e. the orbits cross the transverse section $y = 0$ at $t = 0$. Fixing ρ_0 at $t = 0$ is like fixing the energy of the system.

For $\mu = 0$, ρ is constant and so is a second integral of the system. Thus the Melnikov function is

$$\begin{aligned} M(s_0, \rho_0) &= \int_{-\infty}^{\infty} (D\rho) \cdot Y_{\phi_0(t, \rho_0, s_0)} dt \\ &= \int_{-\infty}^{\infty} g_2 \circ \phi_0(t, \rho_0, s_0) dt \\ &= \int_{-\infty}^{\infty} \xi_0^4(t) \sin(s_0 + \sigma(t)) S(t, \rho_0, s_0) dt, \end{aligned}$$

where

$$S(t, \rho_0, s_0) = \left[1 - \frac{1}{[1 + 2\xi_0^2(t) \cos(s_0 + \sigma(t)) + \xi_0^4(t)]^{3/2}} \right].$$

The above form of the Melnikov function is derived in [11] by starting with the form arising from a periodically forced system and then integrating by parts. This type of derivation is not surprising since the equivalence of the usual Melnikov integral and that obtained for autonomous systems is proved in [9] using integration by parts. The point that is being made here is the above treatment is very direct and seems less ad hoc than what is found in [11]. (I am certain that Z. Xia was very aware of the representation of the Melnikov integral by means of a second integral and that he just preferred using the more familiar form.) Notice that the orbit at infinity does not vary with μ so the integral converges absolutely.

Once the above form is obtained, symmetry considerations imply that there is a zero of M for $s_0 = 0$ and π , $M(0, \rho_0) = M(\pi, \rho_0) = 0$.

The orbit at infinity does not vary with μ , so the above integral representation of M is not conditional. Therefore by Theorem 3.5, $\frac{\partial M}{\partial s_0}(s_0, \rho_0)$ can be calculated by differentiating under the integral sign with respect to s_0 which appears in a very simple way. (The integral of this derivative with respect to s_0 also converges uniformly in s_0 .) Special arguments are needed to get that the improper integral representing $\frac{\partial M}{\partial s_0}(s_0, \rho_0)$ for $s_0 = 0$ or π is nonzero, and so the corresponding zero of $M(\cdot, \rho_0)$ is a nondegenerate zero as a function of s_0 . These arguments which verify this fact done very differently in [11] and [8].

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