OBSTRUCTION ARGUMENT FOR TRANSITION CHAINS OF TORI INTERSPERSED WITH GAPS

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Abstract. We consider a dynamical system that exhibits a two-dimensional normally hyperbolic invariant manifold diffeomorphic to an annulus. We assume that in the annulus there exist transition chains of invariant tori interspersed with Birkhoff zones of instability. We prove the existence of orbits that follow the transition chains and cross the Birkhoff zones of instability.

1. Introduction

This paper is a continuation to [21], in which we describe a topological method for proving the existence of orbits that shadow transition chains of invariant tori interspersed with Birkhoff zones of instability. In [21] the boundaries of the Birkhoff zones of instabilities were assumed to be smooth. In this paper we consider the general case when all the invariant tori in the transition chains and at the boundaries of the Birkhoff zones of instabilities are only Lipschitz. Here by a primary torus we mean a 1-dimensional invariant torus that cannot be homotopically deformed into a point in the annulus. Also, by a Birkhoff zone of instability in an annulus, we mean a region between two primary invariant tori that does not contain any other primary invariant torus in between.

We consider a discrete dynamical system whose phase space contains a hyperbolic invariant manifold diffeomorphic to an annulus. The dynamics restricted to the annulus is assumed to be a monotone twist map. If the twist map is close to integrable, the KAM theorem yields many invariant tori in the annulus, close to the integrable ones. Besides the KAM tori, there also exist other primary invariant tori. Due to normal hyperbolicity, all primary invariant tori possess stable and unstable manifolds. Under some generic non-degeneracy conditions on the dynamics, the stable and unstable manifold of nearby tori intersect transversally. One can link together nearby tori through their heteroclinic connections and form transition chains of such tori. However, gaps are also formed between the primary invariant tori. Some of the gaps can be large, in the sense that one may not be able to extend the transition chains across those gaps using standard analytical arguments. Thus, we obtain transition chains of tori alternating with gaps. In this paper we assume that these gaps are Birkhoff zones of instability with Lipschitz boundaries, and that the transition chains can be extended all the way to the boundaries of these gaps. We use topological arguments to prove that there exist orbits that follow infinitely many transition chains and also cross the Birkhoff zones of instability in between successive transition chains.

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The motivation of this work resides within the Arnold diffusion problem. In 1964, Arnold [1] proposed a model of a Hamiltonian system consisting of a rotator and a pendulum with a small periodic perturbation of special type. He proved the existence of orbits that travel arbitrary far relative to the phase space of the rotor, for all sufficiently small perturbations. Arnold conjectured that this phenomenon is generic in the whole of Hamiltonian systems.

In Arnold's example, in the absence of the perturbation, the phase space of the rotator is a normally hyperbolic invariant manifold filled with primary invariant tori. The stable and unstable manifolds of each torus coincide. When the perturbation is added to the system, all invariant tori for the unperturbed system survive the perturbation. (This is not the case for general perturbations.) Also, the perturbation makes the stable and unstable split, so heteroclinic connections between nearby tori are formed. Thus, one can construct transition chains of tori that travel arbitrarily far in the phase space of the rotator. Then Arnold uses the "obstruction property" to show that there exist orbits that follow the transition chains (see also [2]). A torus T is said to satisfy the obstruction property if for every invariant manifold V intersecting transversely the stable manifold of T, the unstable manifold of T is contained in the closure of V. Often, one applies the obstruction property by taking an open neighborhood B of a point on the stable manifold, and inferring that the closure of the set $\{\phi_t(B)\,|\,t \geq 0\}$ contains the unstable manifold of T, where ϕ_t denotes the Hamiltonian flow. The obstruction property is used in combination with simple point-set topology to provide an argument for the existence of orbits shadowing a transition chain. We briefly describe this argument here. Suppose that we have a sequence of invariant tori $\{T_1, T_2, \dots, T_n, \dots\}$ such that each successive pair of tori in the sequence is linked by a heteroclinic connection. We choose a closed ball B_1 centered at a point on the stable manifold of T_1 and contained in some small neighborhood of T_1. Since $W^u(T_1) \pitchfork W^s(T_2) \neq \emptyset$, the obstruction argument implies that the stable manifold of T_2 intersects B_1. Then there exits a small closed ball $B_2 \subseteq B_1$, centered at a point on the stable manifold of T_2, that is taken by the flow ϕ_t into some small neighborhood of T_2. Applying again the obstruction argument we infer that the stable manifold of T_3 intersects the image of B_2 through the flow ϕ_t. Thus, there is a smaller closed ball $B_3 \subseteq B_2$ that is taken by the flow into some small neighborhood of T_3. This construction can be repeated inductively for an arbitrarily large number of steps, resulting in a sequence of closed balls $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$. Since the balls are compact, their intersection is non-empty. Any point in the intersection will shadow the prescribed sequence of tori.

The emphasis of this paper is on how to extend the obstruction argument in the case of transition chains of tori interspersed with gaps, modeled here as Birkhoff zones of instability. The obstruction mechanism as described above breaks down at the zones of instability. However, there exist connecting orbits that go from near one boundary of a zone of instability to near the other boundary of the zone of instability. The difficulty becomes how to link orbits that shadow the transition chains with orbits that shadow the connecting orbits, where the standard obstruction argument does not apply. To overcome this, we use a topological approach inspired from Easton's method of correctly aligned windows. Instead of closed balls as in the above argument, we use closed rectangular boxes (windows) that, under the dynamics, cross one another along some unstable-like directions. The unstable like-directions correspond to the hyperbolic unstable directions plus one preferred direction in the annulus which shears under the twist map on the annulus. One particular feature of windows is that they are robust objects, so we are able to make adjustments to the geometry of these windows to compensate for the lack of control on the dynamics within the zones of instability.

Topological methods were previously applied to the Arnold diffusion problem in [31, 26, 19, 21, 22]. Transition chains of tori formed with topologically crossing heteroclinic connections were considered in [20]. Ideas of producing diffusing orbits by combining the dynamics along heteroclinic connections with the dynamics across zones of instability appeared in [10, 29] through geometric methods, and in [33, 15, 16, 4, 25, 3] through variational methods.

The novelty of the approach in this paper consists of the following: we consider transition chains of tori that are not necessarily smooth; we consider that the stable and unstable manifolds of the consecutive tori in the chain have topologically crossing intersections; we consider a non-perturbative setting, in which the dynamics on Λ is not assumed to be nearly integrable.

2. Main Result

We describe a class of dynamical systems satisfying certain properties.

We consider a C^l-diffeomorphism $F : M \to M$ of a smooth $(2n+2)$-dimensional manifold M, where $l \geq 1$. We assume that there exists a compact invariant submanifold $\Lambda \subseteq M$ that is diffeomorphic to an annulus $[0,1] \times \mathbb{T}^1$. We assume that Λ is normally hyperbolic, with n-dimensional stable and unstable manifolds $W^s(p)$ and $W^u(p)$ at every point $p \in \Lambda$. Moreover, we assume that the restriction $F_{|\Lambda}$ of F to Λ is an area preserving monotone twist map.

For any invariant set in Λ, and in particular for any invariant primary torus $T \subseteq \Lambda$, we can define its stable and unstable manifolds:

$$W^s(T) = \bigcup_{x\in T} W^s(x), \quad W^u(T) = \bigcup_{x\in T} W^u(x).$$

These manifolds are not in general smooth, but their fibers $W^s(x), W^u(x)$ are as smooth at the map. The map F applied to $W^s(T)$ and $W^u(T)$ takes fibers into corresponding fibers, i.e.

$$F(W^s(x)) = W^s(F(x)), \quad F(W^u(x)) = W^u(F(x)).$$

We now describe a map, called the scattering map, acting on the normally hyperbolic invariant manifold Λ by following the heteroclinic excursions. A detailed treatment of the scattering map can be found in [13]. Assume that $W^u(\Lambda)$ and $W^s(\Lambda)$ have a differentiably transverse intersection along a homoclinic 2-dimensional manifold Γ that is C^l-smooth. This means that $\Gamma \subseteq W^u(\Lambda) \cap W^s(\Lambda)$ and, for each $x \in \Gamma$, we have

$$\begin{aligned} T_xM &= T_xW^u(\Lambda) + T_xW^s(\Lambda),\\ T_x\Gamma &= T_xW^u(\Lambda) \cap T_xW^s(\Lambda). \end{aligned}$$

By the normal hyperbolicity of Λ, for each $x \in W^s(\Lambda)$ there is a unique point $x^+ \in \Lambda$ such that $x \in W^s(x^+)$; also for each $x \in W^u(\Lambda)$ there is a unique point $x^- \in \Lambda$ such that $x \in W^u(x^-)$.

Let us assume the additional condition that for each $x \in \Gamma$ we have

$$T_xW^s(\Lambda) = T_xW^s(x^+) \oplus T_x(\Gamma),$$
$$T_xW^u(\Lambda) = T_xW^s(x^-) \oplus T_x(\Gamma),$$

where x^-, x^+ are the uniquely defined points in Λ corresponding to x.

Then we can define the wave operators $\Omega^+ : W^s(\Lambda) \to \Lambda$ by $\Omega^+(x) = x^+$, and $\Omega^- : W^u(\Lambda) \to \Lambda$ by $\Omega^-(x) = x^-$. The maps Ω^+ and Ω^- are C^{l-1}-smooth. The restrictions of the wave operators Ω^+, Ω^- to Γ are in general only continuous. By restricting Γ if necessary, we can ensure that Ω^+, Ω^- are homeomorphisms. A homoclinic manifold Γ for which the corresponding wave operators are homeomorphisms will be referred as a homoclinic channel. Thus, we can define the homeomorphism $S = \Omega^+ \circ (\Omega^-)^{-1}$ from an open subset $D^- \subseteq \Lambda$ to an open subset D^+ in Λ. We will refer to S as the scattering map associated to the homoclinic channel Γ. In the sequel we will regard S as a partially defined map, so the image of a torus A by S means the set $S(A \cap D^-)$. The scattering map has the following properties: If T and T' are two invariant smooth tori in Λ and $S(T)$ has a transverse intersection point with T', then $W^u(T)$ has a transverse intersection point with $W^s(T')$. If T and T' are two invariant Lipschitz tori in Λ and $S(T)$ has a topologically crossing intersection point with T', then $W^u(T)$ has a topologically crossing intersection point with $W^s(T')$. (The precise definition of a topological crossing intersection point will be given in Section 3.)

We now recall some facts on twist maps. We describe Λ through a system of action-angle coordinate (I, ϕ), with $I \in [0, 1]$ and $\phi \in \mathbb{T}^1$. The fact that the restriction $F_{|\Lambda}$ is a monotone twist map means that $\partial(\mathrm{pr}_\phi \circ F_{|\Lambda})/\partial I > 0$, where pr_ϕ is the projection onto the ϕ-coordinate. By an invariant primary torus (or, equivalently, an essential invariant circle) we mean a 1-dimensional torus invariant under F in Λ that cannot be homotopically deformed into a point inside Λ. Since $F_{|\Lambda}$ is a monotone twist map, each invariant primary torus T is the graph of some Lipschitz function $\tau(\phi)$ (see [5, 6]). A region in Λ between two primary invariant tori is called a Birkhoff zone of instability provided that there is no invariant primary torus in the interior of the region. It is known for a Birkhoff zone of instability that, if the boundary tori are topologically transitive, then there exist Birkhoff connecting orbits that go from any neighborhood of an arbitrary point on one boundary torus to any neighborhood of an arbitrary point on the other boundary torus (see [5, 6]).

We now describe the assumptions for the main theorem of this paper. We consider a bi-finite sequence of invariant primary tori $(T_i)_{i\in\mathbb{Z}}$ in Λ. These tori are Lipschitz tori. Below we will assume that the sequence of tori $(T_i)_{i\in\mathbb{Z}}$ can be partitioned into finite sequences with special properties; we will describe this partition by considering a certain increasing bi-infinite subsequence of indices $(i_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}$.

We assume that the tori in the sequence satisfy the following properties.

- (A1) The manifolds $W^u(\Lambda)$ and $W^s(\Lambda)$ have a topologically transverse intersection along a dynamical channel Γ, and each torus T_i in the given sequence intersects the domain D^- of the scattering map S associated to Γ (as described above).
- (A2) The restriction of F to each torus T_i is topologically transitive.
- (A3) Each subsequence of tori $(T_i)_{i=i_k+1,\dots,i_{k+1}}$, with $k \in \mathbb{Z}$, is a topological transition chain in the sense following sense: there exists a curve segment $\mathcal{T}_i \subseteq T_i$ in the domain of the scattering map S such that the image $S(\mathcal{T}_i)$ of

$\mathcal{T}_i$ under S intersects T_{i+1} at exactly one point, in a topologically crossing manner, for $i = i_k + 2, \ldots, i_{k+1} - 2$, and for $i = i_k + 1, i_{k+1} - 1$, the image $S(\mathcal{T}_i)$ intersects T_{i+1} at exactly three points, in a topologically crossing manner. (Depending on the case, this implies that $W^u(T_i)$ and $W^s(T_{i+1})$ have at least one or at least three topologically crossing points.)

(A4) The region in Λ between T_{i_k} and T_{i_k+1}, with $k \in \mathbb{Z}$, is a Birkhoff zone of instability.

(A5) Each torus T_i which is not at the boundary of one of the Birkhoff zones of instability specified by (A4), can be C^0-approximated from both sides with invariant tori from Λ, i.e., there exists two sequences of invariant tori $(T_{j_l^-(i)})_{l\geq 1} \subseteq \Lambda$ and $(T_{j_l^+(i)})_{l\geq 1} \subseteq \Lambda$ that approach T_i in the C^0-topology, such that the annulus bounded by $T_{j_l^-(i)}$ and $T_{j_l^+(i)}$ contains T_i in its interior, for all l.

The motivation for considering the above structures is given in [21]. We plan to expand on this motivation in a future paper [22]. The main result of this paper is the following:

Theorem 2.1. *We consider a discrete dynamical system $F : M \to M$ as above. Given a sequence of invariant tori $(T_i)_{i\in\mathbb{Z}}$ in Λ satisfying the properties (A1) – (A5) from above, for each sequence $(\epsilon_i)_{i\in\mathbb{Z}}$ of positive real numbers, there exist an orbit $(z_i)_{i\in\mathbb{Z}}$ and positive integers $(n_i)_{i\in\mathbb{Z}}$ such that*

$$\begin{aligned} z_{i+1} &= F^{n_i}(z_i), \quad \text{for all } i \in \mathbb{Z}, \\ d(z_i, T_i) &< \epsilon_i, \quad \text{for all } i \in \mathbb{Z}. \end{aligned}$$

Remark 2.2. One context in which the situation described by (A1)–(A5) can be occur is that of a priori unstable nearly integrable Hamiltonian systems. These are perturbed Hamiltonian systems for which the unperturbed integrable part possesses separatrices (following [8]). In this context, the map F in Theorem 2.1 represents a time discretization of the Hamiltonian flow.

In many examples of a priori unstable nearly integrable Hamiltonian systems (see [12, 13, 14]), the phase space of the unperturbed system contains a normally hyperbolic invariant manifold Λ_0, which is diffeomorphic to an annulus, and whose stable and unstable manifolds coincide. The restriction of F to the annulus is an integrable twist map. When the perturbation is added to the system, Λ_0 is survived by a normally hyperbolic invariant manifold Λ. Under some non-degeneracy conditions on the perturbation, the stable and unstable manifolds of Λ intersect transversally. These non-degeneracy condition can be verified through a Melnikov function or potential associated to the perturbation. Melnikov theory can also be used to verify the existence of a homoclinic channel Γ as in (A1), and to compute explicitly the scattering map S. See Example 2.3 below.

If the dynamics on Λ_0 satisfies the conditions required by the KAM theorem, then there exist many invariant primary tori surviving the perturbation. The KAM primary tori are topologically transitive, as in (A2). Furthermore, one can use the scattering map S associated to the homoclinic channel Γ to verify that the stable and unstable manifolds of sufficiently close invariant primary tori intersect. More precisely, if the image of of a curve segment of the torus T_i under the scattering map intersects transversally (topologically crossing) another torus T_{i+1}, then the unstable manifold of T_i intersects transversally (topologically crossing) the stable manifold of T_{i+1}. If the scattering map can be computed in terms of some Melnikov potential, this condition can be verified explicitly by studying the change of sign of some scalar function associated to the Melnikov potential. See Example 2.3 below. Condition (A3) also requires the existence of not only one but three such intersection points for the tori at both ends of a transition chain. The meaning of this condition is that when the image of a curve segment of the torus T_i under the scattering map S intersects another torus T_{i+1} three times in a topologically crossing manner, it determines two open regions bounded by $S(T_i)$ and T_{i+1} in Λ, one region on one side and the other region on the other side of T_{i+1}. See The existence of these regions is being used for applying the existence of Birkhoff connecting orbits property to cross the Birkhoff zones of instability described by condition (A4). Figure 4. (Here we note that the condition (A3) in the case $i = i_k + 1$, saying that there is a curve segment $\mathcal{T}_{i_k+1} \subset T_{i_k+1}$ in the domain of S such that $S(\mathcal{T}_{i_k+1})$ intersects T_{i_k+2} at exactly three points in a topologically crossing manner, implies that there is a curve segment $\mathcal{T}_{i_k+2} \subset T_{i_k+2}$ in the domain of S^{-1} such that $S^{-1}(\mathcal{T}_{i_k+2})$ intersects T_{i_k+1} at exactly three points in a topologically crossing manner.)

The KAM theorem leaves between invariant primary tori some 'large gaps' of an order of size larger than the order of size of the splitting of the stable and unstable manifolds of Λ. One can form transition chains of primary KAM tori by joining successive heteroclinic connections, and extend these transition chains to the boundary of those 'large gaps', as in (A3). The tori at the boundary of the large gaps are not in general KAM tori; they are only Lipschitz tori, they are not smooth. Since the KAM tori form a Cantor set, we can select the tori in the transition chain (except for the tori at the ends of the chain) so that they can be C^0-approximable from both sides by other KAM tori. Therefore we can ensure condition (A5).

In Theorem 2.1 we assume that the 'large gaps' are Birkhoff zones of instability, described by condition (A4). There exits various methods to verify the existence of Birkhoff zones of instability; some of them can be found in [23, 24, 27, 28].

For Theorem 2.1 we would also need to know that the tori at the boundary of the Birkhoff zones of instability are topologically transitive as in (A2). A sufficient condition for this is that these boundary tori can be obtained as C^0-limits of KAM tori.

We point out that in this remark we do not describe a specific class of Hamiltonian systems for which the conditions (A1) – (A5) are automatically satisfied, but rather we outline methods through which these conditions can be verified individually in examples.

Example 2.3. We consider a mechanical system consisting of one pendulum and one rotator with a weak, periodic coupling. This example have been considered in many papers, but the computation below follows [12]. This system is described by the following time-dependent Hamiltonian:

$$H_\varepsilon(p, q, I, \phi, t) \quad = \quad \frac{1}{2}p^2 + (1 - \cos(q)) + \frac{1}{2}I^2 + \varepsilon h(p, q, I, \phi, t; \varepsilon),$$

where $(p, q, I, \phi, t) \in \mathbb{R} \times \mathbb{T}^1 \times \mathbb{R} \times \mathbb{T}^1 \times \mathbb{T}^1$. The pendulum has a homoclinic orbit to $(0, 0)$. Let $(p^0(\sigma), q^0(\sigma))$ be a parametrization of such a homoclinic orbit, where $\sigma \in \mathbb{R}$ represents the time for the motion of the pendulum. The Melnikov potential for this homoclinic orbit is defined by

$$\mathcal{M}(\tau, I, \phi, t) = \; -\int_{-\infty}^{\infty} \big[h(p^0(\sigma), q^0(\sigma), I, \phi + I\sigma, t+\sigma; 0) \\ -h(0,0,I,\phi+I\sigma, t+\sigma;0)\big]\, d\sigma.$$

Assume the following non-degeneracy condition on the Melnikov potential $\mathcal{M}$:

(i) For each I in some interval (I^-, I^+), and each (ϕ, t) in some open set in $\mathbb{T}\times\mathbb{T}$, the map

$$\tau \in \mathbb{R} \to \mathcal{M}(\tau, I, \phi, t) \in \mathbb{R}$$

has a non-degenerate critical point τ^*, which can be parameterized as

$$\tau^* = \tau^*(I, \phi, t).$$

This condition implies that the unstable and stable manifolds of the annulus $\tilde{\Lambda} = \{(I, \phi, t) \,|\, I \in (I^-, I^+),\, \phi \in \mathbb{T}^1,\, t \in \mathbb{T}^1\}$ intersect transversally along a homoclinic 3-dimensional manifold $\tilde{\Gamma}$ that is described by the implicit equation $\tau^* = \tau^*(I, \phi, t)$, for (I, ϕ, t) in some open domain in $(I^-, I^+)\times\mathbb{T}^1\times\mathbb{T}^1$. When we we discretize the Hamiltonian flow by the time-1 map F, we obtain that $\Lambda = \{(I, \phi) \,|\, I \in (I^-, I^+),\, \phi \in \mathbb{T}^1, \}$ is a normally hyperbolic invariant manifold. Its stable and unstable manifolds intersect transversally along some 2-dimensional homoclinic manifold Γ corresponding to $\tilde{\Gamma}$. When we restrict Γ to some appropriate domain of (I, ϕ), we obtain a homoclinic channel Γ as in condition (A1). We can associate a scattering map S associated to this homoclinic channel Γ.

Assume the following non-degeneracy condition on the Melnikov potential $\mathcal{M}$:

(ii) For each (I, ϕ, t) as above, the function

$$(I, \phi, t) \to \frac{\partial \mathcal{M}}{\partial \phi}(\tau^*(I, \phi, t), I, \phi, t)$$

change its sign once (three times).

The existence of a zero of the above function means that the image of each torus T under the scattering map S has an intersection point with a torus T' which is $O(\varepsilon)$-close to T. If the zero of the function is non-degenerate, then the corresponding intersection of the tori is transverse. If the function changes its sign at a zero, then the corresponding intersection of the tori is topologically crossing. Of course, if the above function changes its sign three times, the corresponding three zeroes determine that $S(T)$ intersects T' three times in a topologically crossing manner, as described by condition (A3). Moreover, it follows that $S(T)$ and T' define two open regions in Λ bounded, with one region on one side and the other region on the other side of T'.

3. Topological shadowing

In this section we present a simple topological method to detect orbits with prescribed itineraries in a discrete dynamical system. This method is inspired from the works of C. Conley, R. Easton and R. McGehee [9, 17, 18], and some of its subsequent developments [7, 20, 32]. The proofs of the statements in this section can be found or follow immediately from similar statements in [21, 32].

Definition 3.1. Two immersed C^0-manifolds N_1 and N_2 in M, with complementary dimensions in M i.e., $\dim(N_1) + \dim(N_2) = \dim(M)$, are said to have a topologically crossing provided that there exist an orientable open neighborhood U

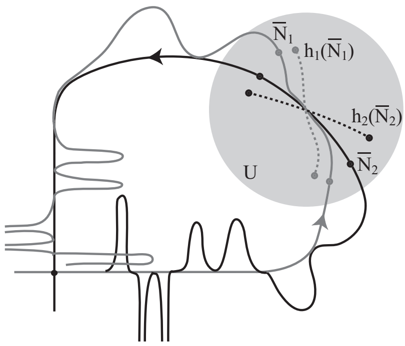

FIGURE 1. Topologically crossing manifolds.

of p in M, and compact embedded C^0-submanifolds with boundary $\bar{N}_1 \subseteq N_1 \cap U$ and $\bar{N}_2 \subseteq N_2 \cap U$ such that the following conditions hold

(i) $\dim(\bar{N}_1) = \dim(N_1)$ and $\dim(\bar{N}_2) = \dim(N_2)$,
(ii) $\partial\bar{N}_1 \cap \bar{N}_2 = \emptyset$ and $\partial\bar{N}_2 \cap \bar{N}_1 = \emptyset$,
(iii) $\bar{N}_1 \cap \bar{N}_2 = N_1 \cap N_2 \cap U$,
(iv) there exists a homotopy $h : [0,1] \times M \to M$ such that:
  (iv.a) $h_0(\bar{N}_1) = \bar{N}_1$ and $h_0(\bar{N}_2) = \bar{N}_2$,
  (iv.b) the homotopy h_t moves points by less than $\varepsilon/2$, where
$$\varepsilon = \min(\mathrm{dist}(\partial\bar{N}_1, \bar{N}_2), \mathrm{dist}(\partial\bar{N}_2, \bar{N}_1)),$$
  (iv.c) $h_1(\bar{N}_1)$ and $h_1(\bar{N}_2)$ are smooth manifolds in M,
  (iv.d) there is a choice of orientation on $h_1(\bar{N}_1), h_1(\bar{N}_2)$ and U such that the oriented intersection number relative to U is non-zero, i.e.,
$$\#_U(h_1(\bar{N}_1), h_1(\bar{N}_2)) \neq 0.$$

At the intuitive level, the above definition says that two manifolds are topologically crossing if they can be made differentiably transverse with non-zero oriented intersection number by the means of a sufficiently small homotopy. Since the embedded manifolds in the above definition are manifolds with boundaries, one has to require that the homotopy does not let the boundary of one manifold cross the other manifold. See Figure 1. From the above definition it follows that the compact embedded C^0-submanifolds with boundary $\bar{N}_1 \subseteq N_1$ and $\bar{N}_2 \subseteq N_2$ can be chosen to be closed disks. Also, the homotopy h_t can be chosen to be arbitrarily small. Since the oriented intersection number is a homotopy invariant, topological transversality is stable under small C^0-perturbations.

Definition 3.2. A window W in M is a homeomorphism $w : B^{n_u} \times B^{n_s} \to M$, together with its image $w(B^{n_u} \times B^{n_s})$ in M, where B^{n_u} and B^{n_s} are the closed unit balls in $\mathbb{R}^{n_u}$ and $\mathbb{R}^{n_s}$ respectively, with $n_u + n_s = \dim(M)$.

In the sequel, we will refer to any topological disk in W of the type $w(B^{n_u} \times \{y_0\})$ as an unstable-like leaf, and to any topological disk of the type $w(\{x_0\} \times B^{n_s})$ as a stable-like leaf, respectively.

Definition 3.3. Let W_1 and W_2 be two windows in M, and $w_1 : B^{n_u} \times B^{n_s} \to M$, $w_2 : B^{n_u} \times B^{n_s} \to M$ be their corresponding homeomorphisms. We say that W_1 is

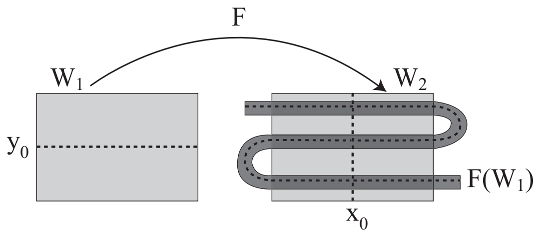

FIGURE 2. Correctly aligned windows.

correctly aligned with W_2 if for each $x_0 \in B^{n_u}$ and $y_0 \in B^{n_s}$, the unstable-like leaf $w_1(B^{n_u}, y_0)$ topologically crosses the stable-like leaf $w_2(x_0, B^{n_s})$, with the same non-zero oriented intersection number for all pairs of leaves.

See Figure 2. The above definition is equivalent to Definition 6 in [32]. In that version of correct alignment, the union of the boundaries of all unstable-like leaves of a windows is referred as the exit set, and the union of the boundaries of all stable-like leaves is referred as the entry set. Definition 6 in [32] requires that the exit set of W_1 is disjoint from W_2, W_1 is disjoint from the entry set of W_2, and there exits a homotopy, which does not alter the above conditions on the exit and entry sets, that deforms W_1 into a n_u-dimensional curve that projects onto the unstable-like direction of W_2 with non-zero Brouwer degree. Here, for convenience, we opted for a version of this definition that is expressed in terms of the topological transversality of leaves. (This version of the definition is also closer in spirit to the original version of correct alignment formulated in [17].)

Given two windows W_1 and W_2 and a homeomorphism F on M, if $F(W_1)$ is correctly aligned with W_2, we will say that W_1 is correctly aligned with W_2 under F. Note that the correct alignment of windows is robust, in the sense that if two windows are correctly aligned under a map, then they remain correctly aligned under a sufficiently small perturbation of that map.

The following result is a topological version of the Shadowing Lemma for hyperbolic dynamical systems.

Theorem 3.4. *Let $(W_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence of windows in M, with n_u-dimensional unstable-like leaves and n_s-dimensional stable-like leaves, where $n_u + n_s = dim(M)$. Let F_i be a collection of homeomorphisms on M. If W_i is correctly aligned with W_{i+1} under F_i for all i, then there exists a point $p \in W_0$ such that*

$$F_i \circ \ldots \circ F_0(p) \in W_{i+1}, \text{ for all } i.$$

In the context of this paper, the maps F_i will represent different powers of some map F.

We now discuss certain subsets of a window that they are themselves windows.

Definition 3.5. Let W be a window in M, and let $w : B^{n_u} \times B^{n_s} \to M$ be the associated homeomorphism. A subset $\hat{W}$ of W is said to be a horizontal sub-window of W if

$$\hat{W} = \bigcup_{x \in B^{n_u}} w(x, B^{n_s}_x),$$

where $\{B^{n_s}_x\}_x$ is a family of topological disks in B^{n_s} that depends continuously with $x \in B^{n_u}$.

A subset $\tilde{W}$ of W is said to be vertical sub-window of W if

$$\tilde{W} = \bigcup_{y \in B^{n_s}} w(B_y^{n_u}, y),$$

where $\{B_y^{n_u}\}_y$ is a family of topological disks in B^{n_u} that depends continuously with $y \in B^{n_s}$.

Note that by restricting w to the $\bigcup_{x \in B^{n_u}} w(x, B_x^{n_s})$ we obtain a homeomorphism from a topological rectangle to $\hat{W}$, thus $\hat{W}$ together with this restriction of w is itself a window. Similarly, $\tilde{W}$ together with the restriction of w to $\bigcup_{y \in B^{n_s}} w(B_y^{n_u}, y)$ is also a window.

We have the following straightforward result.

Lemma 3.6. *If the window W_1 is correctly aligned with the window W_2, and $\tilde{W}_2$ is a vertical sub-window of W_2, then W_1 is also correctly aligned with $\tilde{W}_2$. If W_1 is correctly aligned with W_2, and $\hat{W}_1$ is a horizontal sub-window of W_1, then $\hat{W}_1$ is also correctly aligned with W_2.*

It is however not true that if W_1 is correctly aligned with W_2, and $\hat{W}_2$ is a horizontal sub-window of W_2, then W_1 is correctly aligned with $\hat{W}_2$. It is also not true that if W_1 is correctly aligned with W_2, and $\tilde{W}_1$ is a vertical sub-window of W_1, then $\tilde{W}_1$ is correctly aligned with W_2.

The following statement provides a method of construction of correctly aligned windows about the topologically crossing intersection of two manifolds.

Proposition 3.7. *Suppose that N_1 and N_2 are two C^0-manifolds in M, of complementary dimensions, that are topologically crossing at a point p. Then, for every neighborhood V of p, there exists a pair of windows W_1 and W_2 contained in V, with distinguished homeomorphism $w_1 : B^{\dim N_1} \times B^{\dim N_2} \to M$ and $w_2 : B^{\dim N_1} \times B^{\dim N_2} \to M$ respectively, such that the following hold true:*

(i) *W_1 is correctly aligned with W_2,*

(ii) *the unstable-like leaf $w_1(B^{\dim N_1}, 0)$ is contained in N_1, and each unstable-like leaf $w_1(B^{\dim N_1}, y_0)$ topologically crosses N_2, for all $y_0 \in B^{\dim N_2}$,*

(iii) *the stable-like leaf $w_2(0, B^{\dim N_2})$ is contained in N_2, and each stable-like leaf $w_2(x_0, B^{\dim N_2})$ topologically crossing N_1, for all $x_0 \in B^{\dim N_1}$.*

Proof. The idea is to thicken the manifolds N_1 and N_2 into correctly aligned windows. Choose the embedded submanifolds $\bar{N}_1$ and $\bar{N}_2$ given by Definition 3.1 to be disks contained in V. Define two homeomorphisms $w_1, w_2 : B^{\dim N_1} \times B^{\dim N_2} \to V$ such that $w_1(B^{\dim N_1}, 0) \subseteq \bar{N}_1$ and $w_2(0, B^{\dim N_2}) \subseteq \bar{N}_2$. By choosing w_1 and w_2 so that $w_1(x_0, \cdot)$ and $w_2(\cdot, y_0)$ are sufficiently small for all $x_0 \in B^{\dim N_1}$ and all $y_0 \in B^{\dim N_2}$, the stability of topological transversality under small perturbations implies that the image of $w_1(\cdot, y_0)$ topologically crosses N_2 for each $y_0 \in B^{\dim N_2}$, and the image of $w_2(x_0, \cdot)$ topologically crosses N_1 for each $x_0 \in B^{\dim N_1}$. □

4. Proof of the main theorem

We are under the assumptions of Theorem 2.1. We are given a bi-infinite sequence $(T_i)_{i \in \mathbb{Z}}$ of invariant primary tori in Λ. We would like to show that there is an orbit (z_i) that (ε_i)-shadows this sequence of tori. For this purpose, we construct a sequence of windows along $(T_i)_{i \in \mathbb{Z}}$ that are correctly aligned.

Since Λ is normally hyperbolic, the map F is conjugate to its linearization near Λ (see [30]). More precisely, there exists a homeomorphism h from a neighborhood of the zero section in $T_\Lambda M \subset \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ to a neighborhood of Λ in M such that

$$F(h(x,v)) = h(F(x), (DF)_x(v)),$$

for all $x \in \Lambda$ and all $v \in T_x M$ sufficiently small. The map h induces a system of linearized coordinates in a neighborhood of Λ in M. Such a coordinate system is in general not smooth. The stable manifolds $W^s(x)$ will corresponds through this linearized coordinate system to the stable fibers E^s_x, and the unstable manifolds $W^u(x)$ will corresponds to the stable fibers E^u_x, where E^s and E^u denote the stable and unstable bundles associated to the normally hyperbolic manifold Λ, respectively. The map h can be used to define windows in a neighborhood of Λ in M. In our constructions below, the unstable-like leaves of the windows will correspond to the hyperbolic unstable directions plus one extra direction from the center directions, and the stable-like leaves will correspond to the hyperbolic stable directions plus one extra direction from the center directions.

4.1. **Construction of windows along a heteroclinic orbit.** We consider two invariant tori T_{i-1} and T_i in the sequence $(T_i)_{i\in\mathbb{Z}}$, such that $W^u(T_{i-1})$ topologically crosses $W^s(T_i)$ at a point $x_{i-1,i}$ in the homoclinic channel Γ. Due to normal hyperbolicity, there exist $x^-_{i-1} \in T_{i-1}$ and $x^+_i \in T_i$ such that $x_{i-1,i} \in W^u(x^-_{i-1}) \cap W^s(x^+_i)$. From the definition of the homoclinic channel Γ, the restriction to Γ of the wave operators $\Omega^\pm : \Gamma \to \Lambda$ are homeomorphisms onto their images. There exists a curve $\gamma^-_{i-1,i}$ in Γ corresponding through Ω^- to T_{i-1}, and there exists a curve $\gamma^+_{i-1,i}$ in Γ corresponding through Ω^- to T_i. These curves are topologically crossing at $x_{i-1,i}$ in Γ since $W^u(T_{i-1})$ and $W^s(T_i)$ are topologically crossing at $x_{i-1,i}$ in M. By the definition of the scattering map, we have that $\Omega^+(\gamma^-_{i-1,i}) = S(T_{i-1})$ and $\Omega^-(\gamma^+_{i-1,i}) = S^{-1}(T_i)$. (Note that $\gamma^-_{i-1,i}$ and $\gamma^+_{i-1,i}$ are not homeomorphic to the tori T_{i-1} and T_i, but only to some curve segments of these tori, since the maps $(\Omega^\pm)^{-1}$ are not defined on the whole of Λ.)

We will now construct a window $W_{i-1,i}$ about $x_{i-1,i}$; we will propagate $W_{i-1,i}$ backwards in time to $F^{-m^-_{i-1}}(W_{i-1,i})$ about a point $F^{-m^-_{i-1}}(x_{i-1,i})$ that is $(\varepsilon_{i-1}/2)$-close to $F^{-m^-_{i-1}}(x^-_{i-1}) \in T_{i-1}$; also, we will propagate $W_{i-1,i}$ forward in time to $F^{m^+_i}(W_{i-1,i})$ about a point $F^{m^+_i}(x_{i-1,i})$ that is $(\varepsilon_i/2)$-close to $F^{m^+_i}(x^+_i) \in T_i$.

The distance between $F^m(x_{i-1,i})$ and $F^m(x^+_{i-1,i})$, measured relative to $W^s(T_i)$, tends to 0 as $m \to \infty$. Also, the curve $F^m(\gamma^+_{i-1,i})$ approaches T_i in the C^0-topology, and the curve $F^m(\gamma^-_{i-1,i})$ approaches $F^m(S(T_{i-1}))$ in the C^0-topology, as $m \to \infty$. There exists m^+_i sufficiently large such that $F^{m^+_i}(\gamma^+_{i-1,i})$ is within a distance of $(\varepsilon^+_i/2)$ from T_i, and $F^{m^+_i}(\gamma^-_{i-1,i})$ is within a distance of $(\varepsilon^+_i/2)$ from $F^{m^+_i}(S(T_{i-1}))$. Consequently, we have that $F^{m^+_i}(x_{i-1,i})$ is within a distance of $(\varepsilon^+_i/2)$ from $F^{m^+_i}(x^+_i)$.

The iterate $F^{m^+_i}(W^u(T_{i-1}))$ of $W^u(T_{i-1})$ is topologically crossing $W^s(T_i)$ at $F^{m^+_i}(x_{i-1,i})$. We choose a topological disk D_i in $W^s(T_i)$, centered at $F^{m^+_i}(x^+_i)$ and contained in an $(\varepsilon^+_i/2)$-neighborhood of $F^{m^+_i}(x^+_i)$, such that $F^{m^+_i}(x_i)$ is an interior point to D_i. By replacing $W^u(T_{i-1})$ with some small topological disk centered at $x_{i-1,i}$, we can assume that $F^{m^+_i}(W^u(T_{i-1}))$ is itself a topological disk contained in an $(\varepsilon_i^+/2)$-neighborhood of $F^{m_i^+}(x_i^+) \in T_i$, and is topologically crossing $D_i \subseteq W^s(T_i)$ at $F^{m_i^+}(x_{i-1,i})$. We define a homoeomorphism $w_{i-1,i}$ on $B^{n+1}\times B^{n+1}$ such that its image under $F^{m_i^+}$ is contained in an (ε_i^+)-neighborhood of $F^{m_i^+}(x_i^+) \in T_i$. Moreover, we require that $w_{i-1,i}(B^1 \times \{0\}, 0) \subseteq \gamma_{i-1,i}^-$, $w_{i-1,i}(B^{n+1}, 0) \subseteq W^u(T_{i-1})$, $w_{i-1,i}(0, B^1\times\{0\}) \subseteq \gamma_{i-1,i}^+$, and $w_{i-1,i}(0, B^{n+1}) \subseteq W^s(T_i)$. By choosing $w_{i-1,i}$ so that its leaves are sufficiently small, we can can ensure, by the stability of topological crossing under small perturbations, that the image of each unstable-like leaf $w_{i-1,i}(B^{n+1}, y_0)$ under $F^{m_i^+}$ is a topological disk topologically crossing D_i, at a point interior to both disks, for all $y_0 \in B^{n+1}$.

Near Λ we have a conjugacy h between F and DF. We define a homeomorphism $w_i^+ : B^{n+1} \times B^{n+1} \to M$ as a rescaled restriction of h to $B^{n+1} \times B^{n+1}$. We require that the image of w_i^+ is contained in an (ε_i)-neighborhood of $F^{m_i^+}(x_i^+) \in T_i$. We can choose h, and implicitly w_i^+, so that $w_i^+(0, B^{n+1}) = D_i \subseteq W^s(T_i)$. By choosing the rescaling of h so that the leaves of w_i^+ are sufficiently small, we can ensure that each leaf $w_i^+(x_0, B^{n+1})$ is topologically crossing the image of each unstable-like leaf $w_{i-1,i}(B^{n+1}, y_0)$ under $F^{m_i^+}$, for all $x_0 \in B^{n+1}$ and all $y_0 \in B^{n+1}$. The image of w_i^+ is a window contained in an (ε_i^+)-neighborhood of $F^{m_i^+}(x_{i-1,i})$.

We require some additional condition on the construction of W_i^+. The intersection between the image of w_i^+ and Λ is a 2-dimensional topological rectangle R_i^+ that contains a segment of T_i; we require that two of the sides of this rectangle lie on on some pair of invariant tori nearby T_i, on opposite sides of T_i. We now make this requirement precise. Since T_i can be C^0-approximated from both sides by invariant tori, there exist a pairs of invariant tori $T_{j_l^-}(i)$ and $T_{j_l^+}(i)$, both within a distance of $(\varepsilon_i^+/2)$ from T_i in Λ, such that T_i is in the interior of the annulus bounded by $T_{j_l^-(i)}$ and $T_{j_l^+(i)}$. (We will use these tori to keep under control the dynamics induced by the restriction of F to Λ.) Every stable-like leaf $w_i^+(x_0, B^{n+1})$ of W_i^+ is topologically crossing at a point every unstable-like leaf $w_i^+(B^{n+1}, y_0)$ of W_i^+. The intersection between the stable-like leaf $w_i^+(0, B^{n+1})$ and Λ is a curve segment of T_i contained in R_i^+. The intersections between the stable-like leaves $w_i^+(x_0, B^{n+1})$ and Λ, where $x_0 \in B^{n+1}$, form a continuous family $\{\mathcal{T}_{a(i)}^+\}_a$ of disjoint curve segments that approach T_i in the C^0 topology, with each $\mathcal{T}_{a(i)}^+$ corresponding to some $w_i^+(x_0, B^1 \times \{0\})$. The intersections between the unstable-like leaves $w_i^+(B^{n+1}, y_0)$ and Λ, where $y_0 \in B^{n+1}$, form a continuous family of disjoint curve segments $\{\mathcal{S}_{b(i)}^+\}_b$ that are topologically crossing T_i, with each $\mathcal{S}_{b(i)}^+$ corresponding to some $w_{i-1,i}(B^1 \times \{0\}, y_0)$. We require that the curve $w_i^+(B^1 \times \{0\}, 0) \subseteq F^{m_i^+}(S(T_{i-1}))$. Thus, one of the curves $\mathcal{S}_{b_0(i)}^+$ from the family $\{\mathcal{S}_{b(i)}^+\}_b$ is contained in the image under $F^{m_i^+}$ of $S(T_{i-1})$.

Since to construct W_i^+ we used the linearized coordinates near Λ, each stable-like leaf $w_i^+(x_0, B^{n+1})$ of W_i^+ is a union of fibers of the form

$$w_i^+(x_0, B^{n+1}) = \bigcup_{p\in\mathcal{T}_{a(i)}^+} \left(W^s_{\rm loc}(p) \cap W_i^+\right),$$

for some curve segment $\mathcal{T}_{a(i)}$ in R_i^+. Also, each unstable-like leaf $w_i^+(B^{n+1}, y_0)$ of W_i^+ is a union of fibers of the form

$$w_i^+(B^{n+1}, y_0) = \bigcup_{q \in \mathcal{S}^+_{b(i)}} \left(W^u_{\rm loc}(q) \cap W_i^+\right),$$

for some curve segment $\mathcal{S}^+_{b(i)}$ in R_i^+.

The last requirement that we impose on W^+ is that each curve segment $\mathcal{S}^+_{b(i)}$ in R_i^+ has its points lying on the tori $T_{j_l^-(i)}$ and $T_{j_l^+(i)}$. That is, the rectangle R_i^+ has a pair of sides made of the endpoints of the curves $\mathcal{S}^+_{b(i)}$ on opposite sides of T_i in Λ, and lying on some invariant tori neighboring T_i. This completes the construction of a homeomorphism $w_i^+ : B^{n+1} \times B^{n+1} \to M$ defining a second window W_i^+ contained in an (ε_i^+)-neighborhood of $F^{m^+}(x^+) \in T_i$. The stable-like leaf $w_i^+(0, B^{n+1})$ of W_i^+ is contained in $W^s(T_i)$. The window $W_{i-1,i}$ is correctly aligned with the window W_i^+ under $F^{m_i^+}$.

In the case when when T_i is at the boundary of a Birkhoff zone of instability specified in (A4), we define the rectangle R_i^+ in the same way as above, except that instead of having a pair of sides lying on some invariant tori $T_{j^+(i)}$ with $T_{j^-(i)}$, we just have them lie on opposite sides of T_i in Λ. This ends the construction of W_i^+.

The above construction concerns the propagation of the window $W_{i-1,i}$ forward in time to $F^{m_i^+}(W_{i-1,i})$ that is correctly aligned with W_i^+.

In a similar fashion we propagate the window $W_{i-1,i}$ backwards in time to $F^{-m^-_{i-1}}(W_{i-1,i})$ about the point $F^{-m^-_{i-1}}(x_{i-1,i}) \in W^u(T_{i-1})$, and construct a window W^-_{i-1} about the point $F^{-m^-_{i-1}}(x_{i-1,i}) \in T_{i-1}$, such that W^-_{i-1} is correctly aligned with the window $F^{-m^-_{i-1}}(W_{i-1,i})$. Moreover, the window W^-_{i-1} is chosen to be inside an (ε_{i-1})-neighborhood of T_{i-1}.

We now list the key features of W^-_{i-1}, that are analogous to the corresponding features of W_i^+. The window W^-_{i-1} is the image of a homeomorphism $w^-_{i-1} : B^{n+1} \times B^{n+1} \to M$, which is a restriction of h to $B^{n+1} \times B^{n+1}$, rescaled appropriately. The unstable-like leaf $w^-_{i-1}(0, B^{n+1})$ of W^-_{i-1} is contained in $W^u(T_{i-1})$. The intersection between the image of w^-_{i-1} and Λ is a 2-dimensional topological rectangle R^-_{i-1} that contains a segment of T_{i-1}. The intersections between the unstable-like leaves $w^-_{i-1}(B^{n+1}, y_0)$ and Λ, where $y_0 \in B^{n+1}$, form a continuous family of disjoint curve segments $\{\mathcal{T}^-_{a(i-1)}\}_a$ that approach that approach T_{i-1} in the C^0-topology, with each $\mathcal{T}^-_{a(i-1)}$ corresponding to some curve $w_{i-1,i}(B^1 \times \{0\}, y_0)$. The intersections between the stable-like leaves $w^-_{i-1}(x_0, B^{n+1})$ and Λ, where $x_0 \in B^{n+1}$, form a continuous family $\{\mathcal{S}^+_{b(i-1)}\}_a$ of disjoint curve segments that are topologically crossing T_{i-1}, with each $\mathcal{S}^-_{b(i-1)}$ corresponding to some $w^-_{i-1}(x_0, B^1 \times \{0\})$. It is also required that the curve $w^-_{i-1}(0, B^1 \times \{0\})$ is contained in $F^{m^-_{i-1}}(S^{-1}(T_i))$, and each curve segment $\mathcal{S}^+_{b(i-1)}$ in R^-_{i-1} has its points lying on a pair of tori $T_{j_l^-(i-1)}$ and $T_{j_l^+(i-1)}$, each within $(\varepsilon_{i-1}/2)$ from T_{i-1} (so the rectangle R^-_{i-1} has a pair of sides lying on $T_{j_l^-(i-1)}$ and $T_{j_l^+(i-1)}$).

We have that each stable-like leaf $w^-_{i-1}(x_0, B^{n+1})$ of W^-_{i-1} is a union of fibers of the form

$$w^-_{i-1}(x_0, B^{n+1}) = \bigcup_{p\in\mathcal{S}^-_{b(i-1)}} \left(W^s_{\rm loc}(p)\cap W^-_{i-1}\right),$$

for some curve segment $\mathcal{S}^-_{b(i-1)}$ in R^-_{i-1}. Similarly, each unstable-like leaf $w^-_{i-1}(B^{n+1}, y_0)$ of W^-_{i-1} is a union of fibers of the form

$$w^-_{i-1}(B^{n+1}, y_0) = \bigcup_{q\in\mathcal{T}^+_{a(i-1)}} \left(W^u_{\rm loc}(q)\cap W^-_{i-1}\right),$$

for some curve segment $\mathcal{T}^+_{a(i-1)}$ in R^-_{i-1}. In the special case when T_{i-1} is at the boundary of a Birkhoff zone of instability specified in (A4), the rectangle R^-_{i-1} is constructed to have the pair of sides made of the endpoints of the curves $\mathcal{S}^+_{b(i-1)}$ just lying on opposite sides of T_{i-1} in Λ, rather than lying on some invariant tori neighboring T_{i-1}.

4.2. **Construction of windows along a transition chain.** We consider a transition chain of tori $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$, where k is some positive integer, as prescribed by assumption (A3). We have that $W^u(T_{i-1})$ has a topologically crossing intersection with $W^u(T_i)$ at a point $x_{i-1,i}$, where T_{i-1} and T_i are any two consecutive tori in the above sequence. The point $x_{i-1,i}$ lies on the unstable manifold of $x^-_{i-1}\in T_{i-1}$ and also on the stable manifold of $x^+_i \in T_i$. Each torus in the transition chain, except for the tori at both ends, can be approximated from both sides, relative to the C^0-topology, by other smooth tori.

We would like to construct a finite sequence of windows along these tori such that any two consecutive windows in the sequence are correctly aligned under some power of F. We will perform this construction inductively starting at $T_{i_{k-1}+1}$, at the beginning of the transition chain.

The initial step of the construction is as described in Subsection 4.1. This consists in constructing a window $W_{i_{k-1}+1,i_{k-1}+2}$ about the heteroclinic point $x_{i_{k-1}+1,i_{k-1}+2}$, and two windows, $W^-_{i_k+1}$ about the point $F^{-m^-_{i_{k-1}+1}}(x^-_{i_{k-1}+1}) \in T_{i_{k-1}+1}$, and $W^+_{i_k+2}$ about the point $F^{-m^+_{i_{k-1}+2}}(x^+_{i_{k-1}+2}) \in T_{i_{k-1}+2}$, such that $W^-_{i_{k-1}+1}$ is correctly aligned with $W_{i_{k-1}+1,i_{k-1}+2}$ under some iterate $F^{m^-_{i_{k-1}+1}}$, and $W_{i_{k-1}+1,i_{k-1}+2}$ is correctly aligned with $W^+_{i_{k-1}+2}$ under some iterate $F^{m^+_{i_{k-1}+2}}$. Also, all points of the window $W^-_{i_{k-1}+1}$ are within $(\varepsilon_{i_{k-1}+1})$ from $T_{i_{k-1}+1}$, and all points of the window $W^+_{i_{k-1}+2}$ are within $(\varepsilon_{i_{k-1}+2})$ from $T_{i_{k-1}+2}$.

We now assume that we arrived with the inductive construction at some heteroclinic connection between T_{i-1} and T_i, where the tori T_{i-1} and T_i are two consecutive tori in the chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$. The inductive construction yields a window W^+_i within (ε_i) from T_i, which is of the form

$$W^+_i = \bigcup_{(I,\phi)\in R^+_i} Q^+_i(I,\phi),$$

where the rectangle R^+_i has a pair of sides lying on some invariant tori $T_{j^-(i)}$ and $T_{j^+(i)}$ on opposite sides of T_i, and $Q^+_i(I,\phi)$ is a topological $(2n)$-dimensional rectangle corresponding to the hyperbolic directions, for each $(I,\phi)\in R^+_i$. The unstable-like leaves of this window are leaves of the form $w_i^+(B^n \times B^1, y_0)$, that intersect R_i^+ along curves $\mathcal{S}^+_{b(i)}$ that are topologically crossing T_i and have their endpoints on the tori $T_{j^-(i)}$ and $T_{j^+(i)}$.

We consider the subsequent heteroclinic connection in the chain, between T_i and T_{i+1}. About the corresponding heteroclinic point $x_{i,i+1}$ we construct a test window $W_{i,i+1}$. As in Subsection 4.1, we construct the windows W_i^- about T_i, and the window W^+_{i+1} about the T_{i+1}, such that W_i^- is correctly aligned with $F^{-m_i^-}(W_{i,i+1})$, and $F^{m^+_{i+1}}(W_{i,i+1})$ is correctly aligned with W^+_{i+1}. The window W_i^- is contained in an (ε_i)-neighborhood of T_i, and is of the form

$$W_i^- = \bigcup_{(I,\phi)\in R_i^-} Q_i^-(I,\phi),$$

where the rectangle R_i^- has a pair of sides on $T_{j'^-(i)}$ and $T_{j'^+(i)}$, and $Q_i^-(I,\phi)$ is a topological $(2n)$-dimensional rectangle corresponding to the hyperbolic directions, for each $(I,\phi)\in R_i^-$. The stable-like leaves of this window are leaves of the form $w_i^-(x_0, B^n\times B^1)$, that intersect R_i^- along curves $\mathcal{S}^-_{b(i)}$ that are topologically crossing T_i and have their endpoints on the same neighboring tori $T_{j^-(i)}$ and $T_{j^-(i)}$ of T_i as in the construction of W_i^+.

We will use the twist map property of F restricted to Λ to make the windows W_i^+ and W_i^- correctly aligned under some iterate of F. For this purpose, we will first make R_i^+ correctly aligned under R_i^- under some iterate of F.

By the twist condition, and by the fact that F is topologically transitive on T_i (assumption (A2)), there exists m_i such that each curve $\mathcal{S}^+_{b(i)}$ in R_i^+ connecting $T_{j^-(i)}$ and $T_{j^+(i)}$ is mapped by F^{m_i} onto a curve that intersects in a topologically crossing manner each curve $\mathcal{S}^-_{b(i)}$ in R_i^- connecting $T_{j^-(i)}$ and $T_{j^+(i)}$. Since the curves $\mathcal{S}^+_{b(i)}$ in R_i^+ connecting $T_{j^-(i)}$ and $T_{j^+(i)}$ represent the unstable-like directions of R_i^+, and the curves $\mathcal{S}^-_{b(i)}$ in R_i^- connecting $T_{j^-(i)}$ and $T_{j^+(i)}$ represent the stable-like directions of R_i^-, we obtain that the window R_i^+ is correctly aligned with the window R_i^- under F^{m_i}. See Figure 3. We need to impose additional restrictions W_i^+ (implicitly on the $(2n)$-dimensional rectangles $Q_i^+(I,\phi)$), and on W_i^- (implicitly on $Q_i^-(I,\phi)$), such that the whole window W_i^+ is correctly aligned with the window W_i^- under F^{m_i}. In this order, the image of each leaf of the type $w_i^+(B^n\times B^1, x_0)$ under F^{m_i} should topologically cross each leaf of the type $w_i^-(x_0, B^n\times B^1)$. The alignment in the hyperbolic direction is due to the fact that the unstable directions will contract exponentially and the stable directions will expand exponentially. Since we have control of the iterates of W_i^+ under F only if these iterates remain close to Λ, the unstable-like leaves of W_i^+ may need to be enlarged, and the stable-like leaves of W_i^+ may need to be shrunk, in their hyperbolic directions.

Changing (adjusting) the size of the leaves in W_i^+ imposes similar changes in the test window $W_{i,i+1}$ and the corresponding window W^+_{i+1} (without changing the order of the iterates m_i^- and m^+_{i+1} of F). This completes the induction step.

Thus, these adjustment get propagated forward along the transition chain $T_{i_{k-1}+1}, T_{i_{k-1}+2},\ldots,T_{i_k-1},T_{i_k}$. When the upper boundary of the Birkhoff zone of instability between T_{i_k} and T_{i_k+1} is reached, we need to cross it.

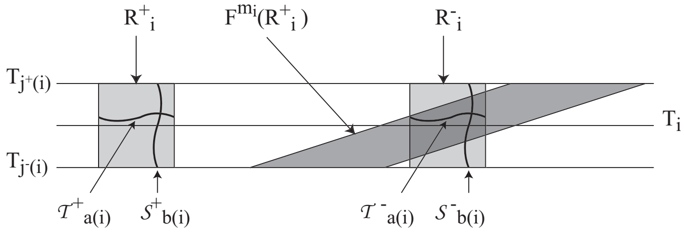

FIGURE 3. Windows in the annulus correctly aligned under the twist map.

We emphasize that this construction uses as an essential feature the fact that the tori in the transition chain can be approximated from both sides, relative to the C^0-topology, by other tori. We note that this type of tori also play an important role in the variational method in [15, 16] for proving the existence of drift orbits.

4.3. **Construction of windows across a Birkhoff zone of instability.** We consider a Birkhoff zones of instability bounded by the tori T_{i_k} and T_{i_k+1} for some k, as prescribed in assumption (A4).

The torus T_{i_k} is the last torus in a transition chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$. We assume that correctly windows have already been constructed inductively along this transition chain. These windows form a sequence of the type:

$$W^-_{i_{k-1}+1}, W_{i_{k-1}+1, i_{k-1}+2}, W^+_{i_{k-1}+2}, \ldots, W^-_{i_k-1}, W_{i_k-1,i_k}, W^+_{i_k},$$

where each window is correctly aligned with the subsequent window in the sequence under some iterate of F.

We consider the torus T_{i_k+1} at the other boundary of the Birkhoff zone of instability. Corresponding to the heteroclinic connection between T_{i_k+1} and T_{i_k+2}, the next invariant torus in the sequence $(T_i)_{i\in\mathbb{Z}}$, we can construct the correctly aligned windows $W^-_{i_k+1}$ near T_{i_k+1}, W_{i_k+1,i_k+2} near x_{i_k+1,i_k+2}, and $W^+_{i_k+2}$ near T_{i_k+2}, as in Subsection 4.1.

We want to make the window $W^+_{i_k}$ on the one side of the Birkhoff zone of instability correctly aligned with the window $W^-_{i_k+1}$ on the other side of the Birkhoff zone of instability, under some iterate of F. We will use the existence of Birkhoff connecting orbits that go from near one boundary of the Birkhoff zone of instability to near the other boundary of the zone.

For this reason, we first consider the topological rectangles in $R^+_{i_k}$ corresponding to $W^+_{i_k}$, and $R^-_{i_k+1}$ corresponding to $W^-_{i_k+1}$, both rectangles being in Λ. First we want to have make these rectangles correctly aligned. By construction, the rectangle $R^+_{i_k}$ contains a curve segment $\mathcal{S}_{b_0(i_k)}$ (corresponding to some curve $w^+_{i_k}(B^1\times\{0\}, y_0)$ contained in a unstable-like leaf of $W^+_{i_k}$) that is contained in $F^{m^+_{i_k}}(S(T_{i_k-1}))$. Also, the rectangle $R^-_{i_k+1}$ contains a curve segment $\mathcal{S}_{b_0(i_k+1)}$ (corresponding to some curve $w^+_{i_k+1}(x_0, B^1\times\{0\})$ contained in a stable-like leaf of $W^+_{i_k}$) that is contained in $F^{m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$.

Condition (A3) implies that the intersection between $S(T_{i_k-1})$ and T_{i_k} encloses an open neighborhood in Λ of some curve segment of T_{i_k}. It follows that intersection of the iterate $F^{m^+_{i_k}}(S(T_{i_k-1}))$ of $S(T_{i_k-1})$ and T_{i_k} also encloses an open

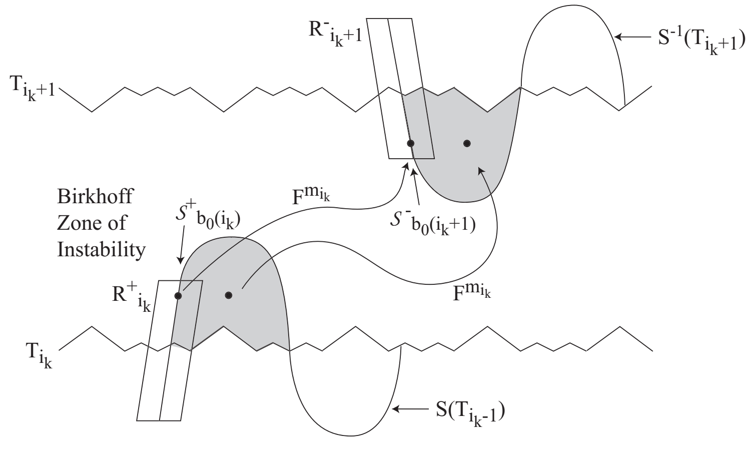

FIGURE 4. Birkhoff connecting orbits.

neighborhood in Λ of some curve segment of T_{i_k}; let us call this latter neighborhood U. The boundary part of U inside the annulus bounded by T_{i_k} and T_{i_k+1} is entirely contained in $F^{m^+_{i_k}}(S(T_{i_k-1}))$. Similarly, the intersection between $S^{-1}(T_{i_k+2})$ and T_{i_k+1} encloses an open neighborhood in Λ of some curve segment of T_{i_k+1}. Therefore the intersection of the iterate $F^{m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$ of $S^{-1}(T_{i_k+2})$ and T_{i_k+1} encloses an open neighborhood V in Λ of some curve segment of T_{i_k+1}. The boundary part of V inside the annulus bounded by T_{i_k+1} and T_{i_k+2} is entirely contained in $F^{m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$. By condition (A2) the map F is topologically transitive on both boundary tori T_{i_k+1} and T_{i_k+2}. This implies that there exits a Birkhoff connecting orbit from U to V in Λ, i.e., there exist $x \in U$ and $m_i > 0$ such that $F^m(x) \in V$. Thus, there exist a point $x_{i_k} \in F^{m^+_{i_k}}(S(T_{i_k-1}))$ whose image $F^{m_{i_k}}$ under $F_{m_{i_k}}$ is a point $x_{i_{k+1}} \in F^{m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$. Moreover, the point x_{i_k} can be chosen so that the intersection at y_{i_k+1} between $F^{m_{i_k}}(F^{m^+_{i_k}}(S(T_{i_k-1})))$ and $F^{m^-_{i_k+1}}(S^{-1}(T_{i_k+2}))$ is topologically crossing (otherwise there will be no point in the open set U that goes inside the open set V). See Figure 4.

We now need to perform a series of adjustment to the rectangles $R^+_{i_k}$ and $R^-_{i_k+1}$ in Λ, so that $R^+_{i_k}$ is correctly aligned with $R^-_{i_k+1}$ under $F^{m_{i_k}}$. First, if necessary, we need to extend the original rectangle $R^+_{i_k}$ along the curve $\mathcal{S}^+_{b_0(i_k)}$ such that $R^+_{i_k}$ contains the point $x_{i_k} \in \mathcal{S}^+_{b_0(i_k)}$. We shall similarly extend the rectangle $R^-_{i_k+1}$ along the curve $\mathcal{S}^+_{b_0(i_k+1)}$ such that $R^-_{i_k+1}$ contains the point $y_{i_k+1} \in \mathcal{S}^+_{b_0(i_k+1)}$. Thus the point $x_{i_k} \in R^+_{i_k}$ is taken by $F^{m_{i_k}}$ to the point $y_{i_k+1} \in R^-_{i_k+1}$. We already know, from above, that the image of $\mathcal{S}^+_{b_0(i_k)}$ under $F^{m_{i_k}}$ intersects $\mathcal{S}^+_{b_0(i_k+1)}$ at y_{i_k+1} in a topologically crossing manner. Then, if necessary, we shrink $R^+_{i_k}$ along its stable-like leaves, and shrink $R^-_{i_k+1}$ along its unstable-like leaves, such that the image of each of the curves $\mathcal{S}^+_{b(i_k)}$ under $F^{m_{i_k}}$ intersects each of the curves $\mathcal{S}^-_{b(i_k+1)}$ in a topologically crossing manner. This is possible due to the stability property of

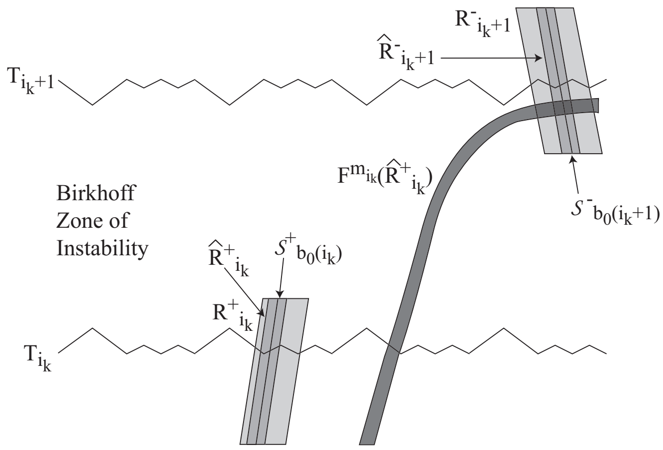

FIGURE 5. Correctly aligned windows across a Birkhoff zone of instability.

topological crossing. Moreover, we construct the rectangle $\hat{R}^+_{i_k}$ such that its stable-like leaves $\mathcal{T}^+_{a(i_k)}$ have their endpoints lying on the images under $F^{m^+_{i_k}} \circ S$ of some invariant tori $T'_{j^-(i_k-1)}$ and $T'_{j^+(i_k-1)}$ neighboring T_{i_k-1}. Such neighboring tori that are sufficiently close to the torus T_{i_k-1} exist due to condition (A5). In summary, the rectangle $R^+_{i_k}$ has a pair of edges on opposite sides of T_{i_k} in Λ, and the other pais of edges lying on some iterate of the images under the scattering map of a pair of tori near the previous torus in the sequence. Similarly, the rectangle $\hat{R}^-_{i_k+1}$ is constructed such that its unstable-like leaves $\mathcal{T}^-_{a(i_k+1)}$ have their endpoints lying on the images under $F^{-m^-_{i_k+1}} \circ S^{-1}$ of some tori $T'_{j^-(i_k+2)}$ and $T'_{j^+(i_k+2)}$ neighboring T_{i_k+2}.

Since the unstable-like leaves of $R^+_{i_k}$ are the curves $\mathcal{S}^+_{b(i_k)}$, and the stable-like leaves of $R^-_{i_k+1}$ are the curves $\mathcal{S}^-_{b(i_k+1)}$, then it follows that $R^+_{i_k}$ is correctly aligned with $R^-_{i_k+1}$ under $F^{m_{i_k}}$. We will denote the rectangles $R^+_{i_k}$ and $R^-_{i_k+1}$ adjusted as above by $\hat{R}^+_{i_k}$ and $\hat{R}^-_{i_k+1}$. See Figure 5.

The adjustment of the rectangle $R^-_{i_k+1}$ requires an appropriate adjustment of the corresponding test window $W^-_{i_k+1}$. Namely, the window $W^-_{i_k+1}$ will be replaced by a window $\hat{W}^-_{i_k+1}$ of the form

$$\hat{W}^-_{i_k+1} = \bigcup_{(I,\phi)\in\hat{R}^-_{i_k+1}} Q^-_{i_k+1}(I,\phi),$$

where the rectangle $\hat{R}^-_{i_k+1}$ is the adjusted rectangle from above, and, for each $(I,\phi) \in \hat{R}^-_{i_k+1}$ the $(2n)$-dimensional topological rectangle $Q^-_{i_k+1}(I,\phi)$ is the same rectangle as the one corresponding to the original window $W^-_{i_k+1}$. Consequently, the windows W_{i_k+1,i_k+2} near x_{i_k+1,i_k+2}, and $W^+_{i_k+2}$ near T_{i_k+2} have to be replaced by corresponding windows $\hat{W}_{i_k+1,i_k+2}$ and $\hat{W}^+_{i_k+2}$ respectively, so that $\hat{W}^-_{i_k+1}$ is correctly aligned with $\hat{W}_{i_k+1,i_k+2}$ under $F^{m^-_{i_k+1}}$, and $\hat{W}_{i_k+1,i_k+2}$ is correctly aligned with $\hat{W}^+_{i_k+2}$ under $F^{m^+_{i_k+2}}$. We stress that the orders of the iterates $F^{m^-_{i_k+1}}$ and

$F^{m^+_{i_k+2}}$ for these correct alignments do not change from the ones for the test windows $W^-_{i_k+1}$, W_{i_k+1,i_k+2} and $W^+_{i_k+2}$, since the adjustments do not involve the hyperbolic directions of these windows. At this stage, the construction of windows about the heteroclinic connection between T_{i_k+1} and T_{i_k+2} is completed. To simplify the notation, we drop the $\widehat{}$ symbol from the notation of the adjusted windows, so from now on we will denote them still by $W^-_{i_k+1}$, W_{i_k+1,i_k+2} and $W^+_{i_k+2}$. Then, beginning with T_{i_k+2}, the construction of correctly aligned windows is continued inductively forward along the transition chain $T_{i_k+1}, T_{i_k+2}, \dots, T_{i_{k+1}-1}, T_{i_{k+1}}$.

We have also performed an adjustment of the rectangle $R^+_{i_k}$ about the torus T_{i_k} at the other boundary of the Birkhoff zone of instability. This requires an appropriate adjustment of the corresponding test window $W^+_{i_k}$. The window $W^+_{i_k}$ will be replaced by a window $\hat{W}^-_{i_k}$ of the form

$$\hat{W}^+_{i_k} = \bigcup_{(I,\phi)\in\hat{R}^+_{i_k}} Q^+_{i_k}(I,\phi),$$

where, for each $(I,\phi) \in \hat{R}^+_{i_k}$ the $(2n)$-dimensional topological rectangle $Q^+_{i_k}(I,\phi)$ is the same rectangle as the one corresponding to the original window $W^+_{i_k}$. The previously constructed windows W_{i_k-1,i_k} near x_{i_k-1,i_k}, and $W^-_{i_k+2}$ near T_{i_k+2} will be replaced by corresponding windows $\hat{W}_{i_k-1,i_k}$ and $\hat{W}^-_{i_k-1}$ respectively, so that $\hat{W}^-_{i_k-1}$ is correctly aligned with $\hat{W}_{i_k-1,i_k}$ under $F^{m^-_{i_k-1}}$, and $\hat{W}_{i_k-1,i_k}$ is correctly aligned with $\hat{W}^+_{i_k}$ under $F^{m^+_{i_k}}$. The orders of the iterates $F^{m^-_{i_k-1}}$ and $F^{m^+_{i_k}}$ remain the same as before. The intersection between $\hat{W}^-_{i_k-1}$ and Λ is a topological rectangle $\hat{R}^-_{i_k-1}$. Since on $\hat{R}^+_{i_k}$ we imposed that its stable-like leaves lie on some iterate of the images under the scattering map of a pair of tori near T_{i_k-1}, in order to ensure the correct alignment of windows, we choose the rectangle $\hat{R}^-_{i_k-1}$ such that its stable-like leaves $\mathcal{S}^-_{b(i_k-1)}$ have their endpoints lying on a pair of invariant tori $T''_{j^-(i_k-1)}$ and $T''_{j^+(i_k-1)}$ neighboring T_{i_k-1}, that are sufficiently close to T_{i_k-1}. This is possible due to condition (A5).

Now we consider the rectangle $R^+_{i_k-1}$ corresponding to the window $W^+_{i_k-1}$. By construction, its unstable-like leaves $\mathcal{S}^+_{b(i_k-1)}$ have their endpoints lying on opposite sides of T_{i_k-1}, on a pair of invariant tori $T_{j^-(i_k-1)}$ and $T_{j^+(i_k-1)}$. If the annulus bounded by $T_{j^-(i_k-1)}$ and $T_{j^+(i_k-1)}$ is contained in the annulus bounded by $T''_{j^-(i_k-1)}$ and $T''_{j^+(i_k-1)}$, then the rectangle $R^-_{i_k-1}$ can be made correctly aligned with the rectangle $\hat{R}^+_{i_k-1}$ under some sufficiently large iterate $F^{m'_{i_k-1}}$, as in Subsection 4.2. Consequently, the window $W^-_{i_k-1}$ is correctly aligned with the window $\hat{W}^+_{i_k-1}$ under $F^{m'_{i_k-1}}$. Note that here we use the fact that F restricted to T_{i_k-1} is topologically transitive, as specified in (A2). If the annulus bounded by $T_{j^-(i_k-1)}$ and $T_{j^+(i_k-1)}$ is not contained in the annulus bounded by $T''_{j^-(i_k-1)}$ and $T''_{j^+(i_k-1)}$, then the rectangle $R^+_{i_k-1}$ needs to be shaved-off so that its unstable-like leaves have their endpoints lying on $T''_{j^-(i_k-1)}$ and $T''_{j^+(i_k-1)}$. Hence $R^+_{i_k-1}$ will be replaced with a rectangle $\hat{R}^+_{i_k-1}$ whose unstable-like leaves $\hat{\mathcal{S}}^+_{b(i_k-1)}$ have their endpoints lying on $T''_{j^-(i_k-1)}$ and $T''_{j^+(i_k-1)}$. Thus $\hat{R}^-_{i_k-1}$ is correctly aligned with the rectangle $\hat{R}^+_{i_k-1}$ under some sufficiently large iterate $F^{m'_{i_k-1}}$. An important remark in this case is

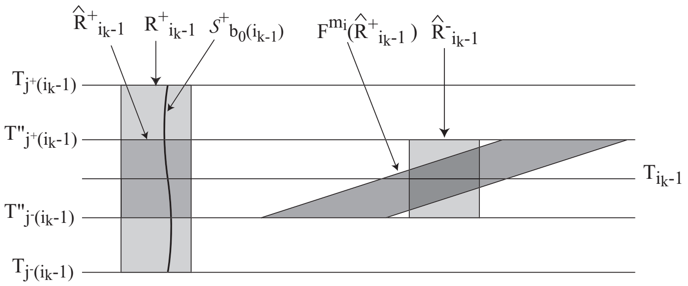

FIGURE 6. Correctly aligned windows across a Birkhoff zone of instability.

that the rectangle $\hat{R}^-_{i_k-1}$ can be chosen as a stable-like sub-rectangle of R_{i_k}, since we only need to shrink the unstable-like leaves of R_{i_k}. See Figure 6. Consequently, the window $W^-_{i_k-1}$ will be adjusted to a window $\hat{W}^-_{i_k-1}$ that is a vertical sub-window of $W^-_{i_k-1}$. The key observation now is that the series of adjustments stops here as we do not need to modify any of the previously constructed windows. Indeed, by Lemma 3.6, since W_{i_k-2,i_k-1} is correctly aligned with $W^+_{i_k-1}$, replacing $W^+_{i_k-1}$ by a vertical sub-window $\hat{W}^+_{i_k-1}$ does not destroy the previous correct alignment. So we have $W^-_{i_k-2}$ correctly aligned with W_{i_k-2,i_k-1}, and W_{i_k-2,i_k-1} correctly aligned with $\hat{W}^+_{i_k-1}$; also $\hat{W}^-_{i_k-1}$ is correctly aligned with $\hat{W}_{i_k-1,i_k}$, and $\hat{W}_{i_k-1,i_k}$ is correctly aligned with $\hat{W}^+_{i_k}$. To simplify the notation, we drop the $\widehat{\ }$ symbol from the notation of the adjusted windows.

In conclusion, at the end of this step, we have obtained the following:

(i) A sequence of correctly aligned windows

$$W^-_{i_{k-1}+1}, W_{i_{k-1}+1,i_{k-1}+2}, W^+_{i_{k-1}+2}, \ldots, W^-_{i_k-1}, W_{i_k-1,i_k}, W^+_{i_k},$$

along the transition chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$.

(ii) A sequence of correctly aligned windows

$$W^-_{i_k+1}, W_{i_k+1,i_k+2}, W^+_{i_k+2}, \ldots, W^-_{i_{k+1}-1}, W_{i_{k+1}-1,i_{k+1}}, W^+_{i_{k+1}},$$

along the transition chain $T_{i_k+1}, T_{i_k+2}, \ldots, T_{i_{k+1}-1}, T_{i_{k+1}}$.

(iii) The window $W^+_{i_k}$ by one boundary of the Birkhoff zone of instability between T_{i_k} and T_{i_k+1} is correctly aligned with the window $W^-_{i_k+1}$ by the other boundary of the Birkhoff zone of instability.

4.4. **Construction of windows along the transition chains and across the Birkhoff zones of instability.** To summarize, in Subsection 4.1, we described the construction of correctly aligned windows about a topologically crossing heteroclinic connection. In Subsection 4.2 we described the inductive construction along a topological transition chain $T_{i_{k-1}+1}, T_{i_{k-1}+2}, \ldots, T_{i_k-1}, T_{i_k}$, starting at $T_{i_{k-1}+1}$ and moving forward along the transition chain. In Subsection 4.3 we continued this inductive construction across a Birkhoff zone of instability and along the subsequent transition chain $T_{i_k+1}, T_{i_k+2}, \ldots, T_{i_{k+1}-1}, T_{i_{k+1}}$. This process required the revision of the last windows about T_{i_k-1} and T_{i_k}, while the rest of the windows remained unchanged. Thus starting from some initial torus $T_{i_{k-1}+1}$ and moving forward, we can construct correctly aligned windows along infinitely many topological transition chains interspersed with Birkhoff zones of instability. Such construction does not need to revise the windows constructed at the initial step by $T_{i_{k-1}+1}$. Thus, a similar construction can be performed backwards in time, along infinitely many topological transition chains interspersed with Birkhoff zones of instability. In conclusion, one obtains a bi-infinite sequence of correctly aligned windows of the type $W_i^-, W_i^+, W_{i,i+1}$, with the windows W_i^-, W_i^+ contained in an (ε_i)-neighborhood of T_i. The Shadowing Lemma-type of result Theorem 3.4 implies the existence of an orbit (z_i) that (ε_i)-shadows the tori (T_i).

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