Example 14

Let $T: R^2 \to R^2$ be the linear transformation defined by

$$T(a_1, a_2) = (2a_2 - a_1, 3a_1),$$

and suppose that $U: R^2 \to R^2$ is linear. If we know that $U(1,2) = (3,3)$ and $U(1,1) = (1,3)$, then $U = T$. This follows from the corollary and from the fact that $\{(1,2),(1,1)\}$ is a basis for R^2. ◆

HW4

due 22 October

EXERCISES

1. Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over F), and T is a function from V to W.
   (a) If T is linear, then T preserves sums and scalar products.
   (b) If $T(x+y) = T(x) + T(y)$, then T is linear.
   (c) T is one-to-one if and only if the only vector x such that $T(x) = 0$ is $x = 0$.
   (d) If T is linear, then $T(0_V) = 0_W$.
   (e) If T is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$.
   (f) If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W.
   (g) If $T, U: V \to W$ are both linear and agree on a basis for V, then $T = U$.
   (h) Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T: V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

For Exercises 2 through 6, prove that T is a linear transformation, and find bases for both N(T) and R(T). Then compute the nullity and rank of T, and verify the dimension theorem. Finally, use the appropriate theorems in this section to determine whether T is one-to-one or onto.

2. $T: R^3 \to R^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.
3. $T: R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$.
4. $T: M_{2\times 3}(F) \to M_{2\times 2}(F)$ defined by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}.$$

5. $T: P_2(R) \to P_3(R)$ defined by $T(f(x)) = xf(x) + f'(x)$.

6. $T: M_{n\times n}(F) \to F$ defined by $T(A) = \text{tr}(A)$. Recall (Example 4, Section 1.3) that

$$\text{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

7. Prove properties 1, 2, 3, and 4 on page 65.
8. Prove that the transformations in Examples 2 and 3 are linear.
9. In this exercise, $T: R^2 \to R^2$ is a function. For each of the following parts, state why T is *not* linear.
   (a) $T(a_1, a_2) = (1, a_2)$
   (b) $T(a_1, a_2) = (a_1, a_1^2)$
   (c) $T(a_1, a_2) = (\sin a_1, 0)$
   (d) $T(a_1, a_2) = (|a_1|, a_2)$
   (e) $T(a_1, a_2) = (a_1 + 1, a_2)$
10. Suppose that $T: R^2 \to R^2$ is linear, $T(1,0) = (1,4)$, and $T(1,1) = (2,5)$. What is $T(2,3)$? Is T one-to-one?
11. Prove that there exists a linear transformation $T: R^2 \to R^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?
12. Is there a linear transformation $T: R^3 \to R^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?
13. Let V and W be vector spaces, let $T: V \to W$ be linear, and let $\{w_1, w_2, \ldots, w_k\}$ be a linearly independent subset of R(T). Prove that if $S = \{v_1, v_2, \ldots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \ldots, k$, then S is linearly independent.
14. Let V and W be vector spaces and $T: V \to W$ be linear.
   (a) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.
   (b) Suppose that T is one-to-one and that S is a subset of V. Prove that S is linearly independent if and only if $T(S)$ is linearly independent.
   (c) Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and T is one-to-one and onto. Prove that $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is a basis for W.
15. Recall the definition of $P(R)$ on page 10. Define

$$T: P(R) \to P(R) \quad \text{by} \quad T(f(x)) = \int_0^x f(t)\,dt.$$

Prove that T linear and one-to-one, but not onto.

16. Let $\mathsf{T}: \mathsf{P}(R) \to \mathsf{P}(R)$ be defined by $\mathsf{T}(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.

17. Let V and W be finite-dimensional vector spaces and $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be linear.

(a) Prove that if $\dim(\mathsf{V}) < \dim(\mathsf{W})$, then T cannot be onto.
(b) Prove that if $\dim(\mathsf{V}) > \dim(\mathsf{W})$, then T cannot be one-to-one.

18. Give an example of a linear transformation $\mathsf{T}: \mathsf{R}^2 \to \mathsf{R}^2$ such that $\mathsf{N}(\mathsf{T}) = \mathsf{R}(\mathsf{T})$.

19. Give an example of distinct linear transformations T and U such that $\mathsf{N}(\mathsf{T}) = \mathsf{N}(\mathsf{U})$ and $\mathsf{R}(\mathsf{T}) = \mathsf{R}(\mathsf{U})$.

20. Let V and W be vector spaces with subspaces V_1 and W_1, respectively. If $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ is linear, prove that $\mathsf{T}(\mathsf{V}_1)$ is a subspace of W and that $\{x \in \mathsf{V}: \mathsf{T}(x) \in \mathsf{W}_1\}$ is a subspace of V.

21. Let V be the vector space of sequences described in Example 5 of Section 1.2. Define the functions $\mathsf{T}, \mathsf{U}: \mathsf{V} \to \mathsf{V}$ by

$$\mathsf{T}(a_1, a_2, \ldots) = (a_2, a_3, \ldots) \quad \text{and} \quad \mathsf{U}(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots).$$

T and U are called the **left shift** and **right shift** operators on V, respectively.

(a) Prove that T and U are linear.
(b) Prove that T is onto, but not one-to-one.
(c) Prove that U is one-to-one, but not onto.

22. Let $\mathsf{T}: \mathsf{R}^3 \to R$ be linear. Show that there exist scalars a, b, and c such that $\mathsf{T}(x, y, z) = ax + by + cz$ for all $(x, y, z) \in \mathsf{R}^3$. Can you generalize this result for $\mathsf{T}: \mathsf{F}^n \to F$? State and prove an analogous result for $\mathsf{T}: \mathsf{F}^n \to \mathsf{F}^m$.

23. Let $\mathsf{T}: \mathsf{R}^3 \to R$ be linear. Describe geometrically the possibilities for the null space of T. *Hint:* Use Exercise 22.

The following definition is used in Exercises 24–27 and in Exercise 30.

Definition. *Let V be a vector space and W_1 and W_2 be subspaces of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$. (Recall the definition of direct sum given in the exercises of Section 1.3.) A function $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ is called the* ***projection on*** *W_1* ***along*** *W_2 if, for $x = x_1 + x_2$ with $x_1 \in \mathsf{W}_1$ and $x_2 \in \mathsf{W}_2$, we have $\mathsf{T}(x) = x_1$.*

24. Let $\mathsf{T}: \mathsf{R}^2 \to \mathsf{R}^2$. Include figures for each of the following parts.

(a) Find a formula for $\mathsf{T}(a, b)$, where T represents the projection on the y-axis along the x-axis.
(b) Find a formula for $\mathsf{T}(a, b)$, where T represents the projection on the y-axis along the line $L = \{(s, s): s \in R\}$.

25. Let $\mathsf{T}: \mathsf{R}^3 \to \mathsf{R}^3$.

(a) If $\mathsf{T}(a, b, c) = (a, b, 0)$, show that T is the projection on the xy-plane along the z-axis.
(b) Find a formula for $\mathsf{T}(a, b, c)$, where T represents the projection on the z-axis along the xy-plane.
(c) If $\mathsf{T}(a, b, c) = (a - c, b, 0)$, show that T is the projection on the xy-plane along the line $L = \{(a, 0, a): a \in R\}$.

26. Using the notation in the definition above, assume that $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ is the projection on W_1 along W_2.

(a) Prove that T is linear and $\mathsf{W}_1 = \{x \in \mathsf{V}: \mathsf{T}(x) = x\}$.
(b) Prove that $\mathsf{W}_1 = \mathsf{R}(\mathsf{T})$ and $\mathsf{W}_2 = \mathsf{N}(\mathsf{T})$.
(c) Describe T if $\mathsf{W}_1 = \mathsf{V}$.
(d) Describe T if W_1 is the zero subspace.

27. Suppose that W is a subspace of a finite-dimensional vector space V.

(a) Prove that there exists a subspace W′ and a function $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ such that T is a projection on W along W′.
(b) Give an example of a subspace W of a vector space V such that there are *two* projections on W along two (distinct) subspaces.

The following definitions are used in Exercises 28–32.

Definitions. *Let V be a vector space, and let $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ be linear. A subspace W of V is said to be* **T-*invariant*** *if $\mathsf{T}(x) \in \mathsf{W}$ for every $x \in \mathsf{W}$, that is, $\mathsf{T}(\mathsf{W}) \subseteq \mathsf{W}$. If W is T-invariant, we define the* ***restriction of*** *T* ***on*** *W to be the function $\mathsf{T}_\mathsf{W}: \mathsf{W} \to \mathsf{W}$ defined by $\mathsf{T}_\mathsf{W}(x) = \mathsf{T}(x)$ for all $x \in \mathsf{W}$.*

Exercises 28-32 assume that W is a subspace of a vector space V and that $\mathsf{T}: \mathsf{V} \to \mathsf{V}$ is linear. *Warning:* Do not assume that W is T-invariant or that T is a projection unless explicitly stated.

28. Prove that the subspaces $\{0\}$, V, R(T), and N(T) are all T-invariant.

29. If W is T-invariant, prove that T_W is linear.

30. Suppose that T is the projection on W along some subspace W′. Prove that W is T-invariant and that $\mathsf{T}_\mathsf{W} = \mathsf{I}_\mathsf{W}$.

31. Suppose that $\mathsf{V} = \mathsf{R}(\mathsf{T}) \oplus \mathsf{W}$ and W is T-invariant. (Recall the definition of *direct sum* given in the exercises of Section 1.3.)

(a) Prove that $W \subseteq N(T)$.
(b) Show that if V is finite-dimensional, then $W = N(T)$.
(c) Show by example that the conclusion of (b) is not necessarily true if V is not finite-dimensional.

32. Suppose that W is T-invariant. Prove that $N(T_W) = N(T) \cap W$ and $R(T_W) = T(W)$.

33. Prove Theorem 2.2 for the case that β is infinite, that is, $R(T) = \text{span}(\{T(v) : v \in \beta\})$.

34. Prove the following generalization of Theorem 2.6: Let V and W be vector spaces over a common field, and let β be a basis for V. Then for any function $f: \beta \to W$ there exists exactly one linear transformation $T: V \to W$ such that $T(x) = f(x)$ for all $x \in \beta$.

Exercises 35 and 36 assume the definition of *direct sum* given in the exercises of Section 1.3.

35. Let V be a finite-dimensional vector space and $T: V \to V$ be linear.
(a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$.
(b) Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.
Be careful to say in each part where finite-dimensionality is used.

36. Let V and T be as defined in Exercise 21.
(a) Prove that $V = R(T) + N(T)$, but V is not a direct sum of these two spaces. Thus the result of Exercise 35(a) above cannot be proved without assuming that V is finite-dimensional.
(b) Find a linear operator T_1 on V such that $R(T_1) \cap N(T_1) = \{0\}$ but V is not a direct sum of $R(T_1)$ and $N(T_1)$. Conclude that V being finite-dimensional is also essential in Exercise 35(b).

37. A function $T: V \to W$ between vector spaces V and W is called **additive** if $T(x + y) = T(x) + T(y)$ for all $x, y \in V$. Prove that if V and W are vector spaces over the field of rational numbers, then any additive function from V into W is a linear transformation.

38. Let $T: C \to C$ be the function defined by $T(z) = \bar{z}$. Prove that T is additive (as defined in Exercise 37) but not linear.

39. Prove that there is an additive function $T: R \to R$ (as defined in Exercise 37) that is not linear. *Hint:* Let V be the set of real numbers regarded as a vector space over the field of rational numbers. By the corollary to Theorem 1.13 (p. 60), V has a basis β. Let x and y be two distinct vectors in β, and define $f: \beta \to V$ by $f(x) = y$, $f(y) = x$, and $f(z) = z$ otherwise. By Exercise 34, there exists a linear transformation $T: V \to V$ such that $T(u) = f(u)$ for all $u \in \beta$. Then T is additive, but for $c = y/x$, $T(cx) \neq cT(x)$.

The following exercise requires familiarity with the definition of *quotient space* given in Exercise 31 of Section 1.3.

40. Let V be a vector space and W be a subspace of V. Define the mapping $\eta: V \to V/W$ by $\eta(v) = v + W$ for $v \in V$.
(a) Prove that η is a linear transformation from V onto V/W and that $N(\eta) = W$.
(b) Suppose that V is finite-dimensional. Use (a) and the dimension theorem to derive a formula relating dim(V), dim(W), and dim(V/W).
(c) Read the proof of the dimension theorem. Compare the method of solving (b) with the method of deriving the same result as outlined in Exercise 35 of Section 1.6.

2.2 THE MATRIX REPRESENTATION OF A LINEAR TRANSFORMATION

Until now, we have studied linear transformations by examining their ranges and null spaces. In this section, we embark on one of the most useful approaches to the analysis of a linear transformation on a finite-dimensional vector space: the representation of a linear transformation by a matrix. In fact, we develop a one-to-one correspondence between matrices and linear transformations that allows us to utilize properties of one to study properties of the other.

We first need the concept of an *ordered basis* for a vector space.

Definition. *Let V be a finite-dimensional vector space. An* **ordered basis** *for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.*

Example 1

In F^3, $\beta = \{e_1, e_2, e_3\}$ can be considered an ordered basis. Also $\gamma = \{e_2, e_1, e_3\}$ is an ordered basis, but $\beta \neq \gamma$ as ordered bases. ◆

For the vector space F^n, we call $\{e_1, e_2, \ldots, e_n\}$ the **standard ordered basis** for F^n. Similarly, for the vector space $P_n(F)$, we call $\{1, x, \ldots, x^n\}$ the **standard ordered basis** for $P_n(F)$.

Now that we have the concept of ordered basis, we can identify abstract vectors in an n-dimensional vector space with n-tuples. This identification is provided through the use of *coordinate vectors*, as introduced next.

EXERCISES

1. Label the following statements as true or false. Assume that V and W are finite-dimensional vector spaces with ordered bases β and γ, respectively, and $\mathsf{T}, \mathsf{U}\colon \mathsf{V} \to \mathsf{W}$ are linear transformations.
   (a) For any scalar a, $a\mathsf{T} + \mathsf{U}$ is a linear transformation from V to W.
   (b) $[\mathsf{T}]_\beta^\gamma = [\mathsf{U}]_\beta^\gamma$ implies that $\mathsf{T} = \mathsf{U}$.
   (c) If $m = \dim(\mathsf{V})$ and $n = \dim(\mathsf{W})$, then $[\mathsf{T}]_\beta^\gamma$ is an $m \times n$ matrix.
   (d) $[\mathsf{T} + \mathsf{U}]_\beta^\gamma = [\mathsf{T}]_\beta^\gamma + [\mathsf{U}]_\beta^\gamma$.
   (e) $\mathcal{L}(\mathsf{V}, \mathsf{W})$ is a vector space.
   (f) $\mathcal{L}(\mathsf{V}, \mathsf{W}) = \mathcal{L}(\mathsf{W}, \mathsf{V})$.

2. Let β and γ be the standard ordered bases for R^n and R^m, respectively. For each linear transformation $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^m$, compute $[\mathsf{T}]_\beta^\gamma$.
   (a) $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$.
   (b) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^2$ defined by $\mathsf{T}(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$.
   (c) $\mathsf{T}\colon \mathsf{R}^3 \to R$ defined by $\mathsf{T}(a_1, a_2, a_3) = 2a_1 + a_2 - 3a_3$.
   (d) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$ defined by

$$\mathsf{T}(a_1, a_2, a_3) = (2a_2 + a_3, -a_1 + 4a_2 + 5a_3, a_1 + a_3).$$

   (e) $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^n$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = (a_1, a_1, \ldots, a_1)$.
   (f) $\mathsf{T}\colon \mathsf{R}^n \to \mathsf{R}^n$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = (a_n, a_{n-1}, \ldots, a_1)$.
   (g) $\mathsf{T}\colon \mathsf{R}^n \to R$ defined by $\mathsf{T}(a_1, a_2, \ldots, a_n) = a_1 + a_n$.

3. Let $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ be defined by $\mathsf{T}(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for R^2 and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$. Compute $[\mathsf{T}]_\beta^\gamma$. If $\alpha = \{(1,2), (2,3)\}$, compute $[\mathsf{T}]_\alpha^\gamma$.

4. Define

$$\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{P}_2(R) \quad \text{by} \quad \mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad \gamma = \{1, x, x^2\}.$$

Compute $[\mathsf{T}]_\beta^\gamma$.

5. Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\beta = \{1, x, x^2\},$$

and

$$\gamma = \{1\}.$$

   (a) Define $\mathsf{T}\colon \mathsf{M}_{2\times 2}(F) \to \mathsf{M}_{2\times 2}(F)$ by $\mathsf{T}(A) = A^t$. Compute $[\mathsf{T}]_\alpha$.
   (b) Define

$$\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{M}_{2\times 2}(R) \quad \text{by} \quad \mathsf{T}(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

   where $'$ denotes differentiation. Compute $[\mathsf{T}]_\beta^\alpha$.
   (c) Define $\mathsf{T}\colon \mathsf{M}_{2\times 2}(F) \to F$ by $\mathsf{T}(A) = \operatorname{tr}(A)$. Compute $[\mathsf{T}]_\alpha^\gamma$.
   (d) Define $\mathsf{T}\colon \mathsf{P}_2(R) \to R$ by $\mathsf{T}(f(x)) = f(2)$. Compute $[\mathsf{T}]_\beta^\gamma$.
   (e) If

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix},$$

   compute $[A]_\alpha$.
   (f) If $f(x) = 3 - 6x + x^2$, compute $[f(x)]_\beta$.
   (g) For $a \in F$, compute $[a]_\gamma$.

6. Complete the proof of part (b) of Theorem 2.7.

7. Prove part (b) of Theorem 2.8.

8.† Let V be an n-dimensional vector space with an ordered basis β. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{F}^n$ by $\mathsf{T}(x) = [x]_\beta$. Prove that T is linear.

9. Let V be the vector space of complex numbers over the field R. Define $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ by $\mathsf{T}(z) = \overline{z}$, where $\overline{z}$ is the complex conjugate of z. Prove that T is linear, and compute $[\mathsf{T}]_\beta$, where $\beta = \{1, i\}$. (Recall by Exercise 38 of Section 2.1 that T is not linear if V is regarded as a vector space over the field C.)

10. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$. Define $v_0 = \mathit{0}$. By Theorem 2.6 (p. 72), there exists a linear transformation $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ such that $\mathsf{T}(v_j) = v_j + v_{j-1}$ for $j = 1, 2, \ldots, n$. Compute $[\mathsf{T}]_\beta$.

11. Let V be an n-dimensional vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be a linear transformation. Suppose that W is a T-invariant subspace of V (see the exercises of Section 2.1) having dimension k. Show that there is a basis β for V such that $[\mathsf{T}]_\beta$ has the form

$$\begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A is a $k \times k$ matrix and O is the $(n-k) \times k$ zero matrix.

12. Let V be a finite-dimensional vector space and T be the projection on W along W', where W and W' are subspaces of V. (See the definition in the exercises of Section 2.1 on page 76.) Find an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

13. Let V and W be vector spaces, and let T and U be nonzero linear transformations from V into W. If $\mathsf{R(T)} \cap \mathsf{R(U)} = \{\mathit{0}\}$, prove that $\{\mathsf{T}, \mathsf{U}\}$ is a linearly independent subset of $\mathcal{L}(\mathsf{V}, \mathsf{W})$.

14. Let $\mathsf{V} = \mathsf{P}(R)$, and for $j \geq 1$ define $\mathsf{T}_j(f(x)) = f^{(j)}(x)$, where $f^{(j)}(x)$ is the jth derivative of $f(x)$. Prove that the set $\{\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_n\}$ is a linearly independent subset of $\mathcal{L}(\mathsf{V})$ for any positive integer n.

15. Let V and W be vector spaces, and let S be a subset of V. Define $S^0 = \{\mathsf{T} \in \mathcal{L}(\mathsf{V}, \mathsf{W}) \colon \mathsf{T}(x) = \mathit{0} \text{ for all } x \in S\}$. Prove the following statements.

(a) S^0 is a subspace of $\mathcal{L}(\mathsf{V}, \mathsf{W})$.
(b) If $S_1$and S_2 are subsets of V and $S_1 \subseteq S_2$, then $S_2^0 \subseteq S_1^0$.
(c) If V_1 and V_2 are subspaces of V, then $(\mathsf{V}_1 + \mathsf{V}_2)^0 = \mathsf{V}_1^0 \cap \mathsf{V}_2^0$.

16. Let V and W be vector spaces such that $\dim(\mathsf{V}) = \dim(\mathsf{W})$, and let $\mathsf{T} \colon \mathsf{V} \to \mathsf{W}$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[\mathsf{T}]_\beta^\gamma$ is a diagonal matrix.

2.3 COMPOSITION OF LINEAR TRANSFORMATIONS AND MATRIX MULTIPLICATION

In Section 2.2, we learned how to associate a matrix with a linear transformation in such a way that both sums and scalar multiples of matrices are associated with the corresponding sums and scalar multiples of the transformations. The question now arises as to how the matrix representation of a composite of linear transformations is related to the matrix representation of each of the associated linear transformations. The attempt to answer this question leads to a definition of matrix multiplication. We use the more convenient notation of UT rather than $\mathsf{U} \circ \mathsf{T}$ for the composite of linear transformations U and T. (See Appendix B.)

Our first result shows that the composite of linear transformations is linear.

Theorem 2.9. *Let V, W, and Z be vector spaces over the same field F, and let $\mathsf{T} \colon \mathsf{V} \to \mathsf{W}$ and $\mathsf{U} \colon \mathsf{W} \to \mathsf{Z}$ be linear. Then $\mathsf{UT} \colon \mathsf{V} \to \mathsf{Z}$ is linear.*

Proof. Let $x, y \in \mathsf{V}$ and $a \in F$. Then

$$\begin{aligned}\mathsf{UT}(ax + y) &= \mathsf{U}(\mathsf{T}(ax + y)) = \mathsf{U}(a\mathsf{T}(x) + \mathsf{T}(y)) \\ &= a\mathsf{U}(\mathsf{T}(x)) + \mathsf{U}(\mathsf{T}(y)) = a(\mathsf{UT})(x) + \mathsf{UT}(y). \quad \blacksquare\end{aligned}$$

The following theorem lists some of the properties of the composition of linear transformations.

Theorem 2.10. *Let V be a vector space. Let $\mathsf{T}, \mathsf{U}_1, \mathsf{U}_2 \in \mathcal{L}(\mathsf{V})$. Then*
(a) $\mathsf{T}(\mathsf{U}_1 + \mathsf{U}_2) = \mathsf{TU}_1 + \mathsf{TU}_2$ *and* $(\mathsf{U}_1 + \mathsf{U}_2)\mathsf{T} = \mathsf{U}_1\mathsf{T} + \mathsf{U}_2\mathsf{T}$
(b) $\mathsf{T}(\mathsf{U}_1\mathsf{U}_2) = (\mathsf{TU}_1)\mathsf{U}_2$
(c) $\mathsf{TI} = \mathsf{IT} = \mathsf{T}$
(d) $a(\mathsf{U}_1\mathsf{U}_2) = (a\mathsf{U}_1)\mathsf{U}_2 = \mathsf{U}_1(a\mathsf{U}_2)$ *for all scalars a.*

Proof. Exercise. $\blacksquare$

A more general result holds for linear transformations that have domains unequal to their codomains. (See Exercise 8.)

Let $\mathsf{T} \colon \mathsf{V} \to \mathsf{W}$ and $\mathsf{U} \colon \mathsf{W} \to \mathsf{Z}$ be linear transformations, and let $A = [\mathsf{U}]_\beta^\gamma$ and $B = [\mathsf{T}]_\alpha^\beta$, where $\alpha = \{v_1, v_2, \ldots, v_n\}$, $\beta = \{w_1, w_2, \ldots, w_m\}$, and $\gamma = \{z_1, z_2, \ldots, z_p\}$ are ordered bases for V, W, and Z, respectively. We would like to define the product AB of two matrices so that $AB = [\mathsf{UT}]_\alpha^\gamma$. Consider the matrix $[\mathsf{UT}]_\alpha^\gamma$. For $1 \leq j \leq n$, we have

$$\begin{aligned}(\mathsf{UT})(v_j) = \mathsf{U}(\mathsf{T}(v_j)) &= \mathsf{U}\left(\sum_{k=1}^m B_{kj} w_k\right) = \sum_{k=1}^m B_{kj} \mathsf{U}(w_k) \\ &= \sum_{k=1}^m B_{kj} \left(\sum_{i=1}^p A_{ik} z_i\right) = \sum_{i=1}^p \left(\sum_{k=1}^m A_{ik} B_{kj}\right) z_i \\ &= \sum_{i=1}^p C_{ij} z_i,\end{aligned}$$

where

$$C_{ij} = \sum_{k=1}^m A_{ik} B_{kj}.$$

This computation motivates the following definition of matrix multiplication.

Definition. *Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. We define the **product** of A and B, denoted AB, to be the $m \times p$ matrix such that*

$$(AB)_{ij} = \sum_{k=1}^n A_{ik} B_{kj} \quad \text{for } 1 \leq i \leq m, \quad 1 \leq j \leq p.$$

Note that $(AB)_{ij}$ is the sum of products of corresponding entries from the ith row of A and the jth column of B. Some interesting applications of this definition are presented at the end of this section.

Thus persons 1, 3, 4, and 5 dominate (can send messages to) all the others in at most two stages, while persons 1, 2, 3, and 4 are dominated by (can receive messages from) all the others in at most two stages.

EXERCISES

1. Label the following statements as true or false. In each part, V, W, and Z denote vector spaces with ordered (finite) bases α, β, and γ, respectively; $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ and $\mathsf{U}\colon \mathsf{W} \to \mathsf{Z}$ denote linear transformations; and A and B denote matrices.

(a) $[\mathsf{UT}]_\alpha^\gamma = [\mathsf{T}]_\alpha^\beta[\mathsf{U}]_\beta^\gamma$.
(b) $[\mathsf{T}(v)]_\beta = [\mathsf{T}]_\alpha^\beta[v]_\alpha$ for all $v \in \mathsf{V}$.
(c) $[\mathsf{U}(w)]_\beta = [\mathsf{U}]_\alpha^\beta[w]_\beta$ for all $w \in \mathsf{W}$.
(d) $[\mathsf{I_V}]_\alpha = I$.
(e) $[\mathsf{T}^2]_\alpha^\beta = ([\mathsf{T}]_\alpha^\beta)^2$.
(f) $A^2 = I$ implies that $A = I$ or $A = -I$.
(g) $\mathsf{T} = \mathsf{L}_A$ for some matrix A.
(h) $A^2 = O$ implies that $A = O$, where O denotes the zero matrix.
(i) $\mathsf{L}_{A+B} = \mathsf{L}_A + \mathsf{L}_B$.
(j) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$.

2. (a) Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute $A(2B+3C), (AB)D$, and $A(BD)$.

(b) Let

$$A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 4 & 0 & 3 \end{pmatrix}.$$

Compute A^t, A^tB, BC^t, CB, and CA.

3. Let $g(x) = 3 + x$. Let $\mathsf{T}\colon \mathsf{P}_2(R) \to \mathsf{P}_2(R)$ and $\mathsf{U}\colon \mathsf{P}_2(R) \to \mathsf{R}^3$ be the linear transformations respectively defined by

$$\mathsf{T}(f(x)) = f'(x)g(x) + 2f(x) \quad \text{and} \quad U(a + bx + cx^2) = (a+b, c, a-b).$$

Let β and γ be the standard ordered bases of $\mathsf{P}_2(R)$ and R^3, respectively.

(a) Compute $[\mathsf{U}]_\beta^\gamma, [\mathsf{T}]_\beta$, and $[\mathsf{UT}]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.
(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[\mathsf{U}(h(x))]_\gamma$. Then use $[\mathsf{U}]_\beta^\gamma$ from (a) and Theorem 2.14 to verify your result.

4. For each of the following parts, let T be the linear transformation defined in the corresponding part of Exercise 5 of Section 2.2. Use Theorem 2.14 to compute the following vectors:

(a) $[\mathsf{T}(A)]_\alpha$, where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$.
(b) $[\mathsf{T}(f(x))]_\alpha$, where $f(x) = 4 - 6x + 3x^2$.
(c) $[\mathsf{T}(A)]_\gamma$, where $A = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$.
(d) $[\mathsf{T}(f(x))]_\gamma$, where $f(x) = 6 - x + 2x^2$.

5. Complete the proof of Theorem 2.12 and its corollary.

6. Prove (b) of Theorem 2.13.

7. Prove (c) and (f) of Theorem 2.15.

8. Prove Theorem 2.10. Now state and prove a more general result involving linear transformations with domains unequal to their codomains.

9. Find linear transformations $\mathsf{U}, \mathsf{T}\colon \mathsf{F}^2 \to \mathsf{F}^2$ such that $\mathsf{UT} = \mathsf{T}_0$ (the zero transformation) but $\mathsf{TU} \neq \mathsf{T}_0$. Use your answer to find matrices A and B such that $AB = O$ but $BA \neq O$.

10. Let A be an $n \times n$ matrix. Prove that A is a diagonal matrix if and only if $A_{ij} = \delta_{ij}A_{ij}$ for all i and j.

11. Let V be a vector space, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{V}$ be linear. Prove that $\mathsf{T}^2 = \mathsf{T}_0$ if and only if $\mathsf{R}(\mathsf{T}) \subseteq \mathsf{N}(\mathsf{T})$.

12. Let V, W, and Z be vector spaces, and let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ and $\mathsf{U}\colon \mathsf{W} \to \mathsf{Z}$ be linear.

(a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?
(b) Prove that if UT is onto, then U is onto. Must T also be onto?
(c) Prove that if U and T are one-to-one and onto, then UT is also.

13. Let A and B be $n \times n$ matrices. Recall that the trace of A is defined by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ and $\operatorname{tr}(A) = \operatorname{tr}(A^t)$.