115a/4 - Practice Midterm 1

8 October 2010

1. (a) Define a linear transformation.

   (b) Show that the set of F-linear transformations from V to W, where V and W are F-vector spaces, is an F-vector space, where addition is defined via the formula

   $$(T_0 + T_1)(v) = T_0(v) + T_1(v)$$

   and scalar multiplication is defined as

   $$(aT)(v) = aT(v).$$

   (c) Find a basis for the vector space of linear transformations $\mathbb{R}^1 \to \mathbb{R}^2$. You need to prove that the set you find is a basis.

2. Let $M_{2\times 2}(\mathbb{R})$ be the $\mathbb{R}$-vector space of 2×2 matrices with entries in $\mathbb{R}$.

   (a) Show that taking transpose is a linear transformation:

   $$^t : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}).$$

   Let $Id{-}^t$ denote the linear transformation sending a matrix A to $A - A^t$.

   (b) Find bases for the null space $N(Id{-}^t)$ and range $R(Id{-}^t)$. Again, this requires proof.

   (c) What are the nullity and rank of $Id{-}^t$.

3. Let V be a vector space, and let W and Z be subspaces of V such that for every vector v of V there are vectors $w \in W$ and $z \in Z$ such that $v = w + z$.

   (a) Show that if $span(S) = W$ and $span(T) = Z$ for some sets S and T of vectors in V, then $span(S \cup T) = V$.

   (b) Conclude that if W and Z are finite dimensional vector spaces with dimensions $dim(W) = m$ and $dim(Z) = n$, then V is finite dimensional, and $dim(V) \leq m+n$.

   (c) Prove or disprove (possibly by constructing a counterexample) that if S and T are bases for V and W, then $S \cup T$ is a basis for V.