061 - Final

8 June 2011

1. Prove that $3^n + 7^n - 2$ is divisible by 8 for all $n \geq 1$.

Solution Proof by induction. Write $f(n) = 3^n + 7^n - 2$. Base case: when $n = 1$, $f(1) = 8$. Now, suppose that $f(k)$ is divisible by 8 for $1 \leq k < n$. Write $f(n-1) = 8a$. Then,

$$\begin{aligned} f(n) = 3^n + 7^n - 2 = 3 \cdot 3^{n-1} + 7^n - 2 = 3(8a - 7^{n-1} + 2) + 7^n - 2 = 3 \cdot 8a + 7^n - 3 \cdot 7^{n-1} + 4 \\ = 3 \cdot 8a + (7-3) \cdot 7^{n-1} + 4 = 3 \cdot 8a + 4 \cdot 7^{n-1} + 4 = 3 \cdot 8a + 4(7^{n-1} + 1). \end{aligned}$$

Now, it is enough to show that 2 divides $7^{n-1} + 1$. But, 7^{n-1} is always odd so that $7^{n-1} + 1$ is always even.

2. Let X be a finite set with n elements. Determine, with proof, the number of reflexive binary relations there are on X.

Solution A binary relation on X is just a subset of $X \times X$. The subsets of $X \times X$ are the elements of the powerset $P(X \times X)$. Suppose that $X = \{x_1, \ldots, x_n\}$. A binary relation $R \in P(X \times X)$ is reflexive if and only if it contains the subset $D = \{(x_1, x_1), \ldots, (x_n, x_n)\}$. Let $Y = X \times X - D$. So, a reflexive binary relation R is uniquely determined by $R - D \subseteq Y$. As there are n elements of D and n^2 elements of $X \times X$, there are $n^2 - n$ elements of Y. The power set of Y corresponds bijectively to the set of reflexive binary relations on X. Thus, there are $2^{(n^2-n)}$ such relations.

3. How many rearrangements of MATHEMATICS are there where I is not next to C?

Solution There are

$$\frac{11!}{2!2!2!}$$

rearrangements of MATHEMATICS. Let $\Phi =$IC, a new symbol. Then, there are

$$\frac{10!}{2!2!2!}$$

rearrangements of the word MATHEMATΦS. Of course, these are exactly the rearrangements of MATHEMATICS where the I and C are next to each other and the I is before the C.

Similarly, there are

$$\frac{10!}{2!2!2!}$$

rearrangements where the I and C are next to each other and the C is before the I. Therefore, there are

$$\frac{11!}{2!2!2!} - 2\frac{10!}{2!2!2!}$$

rearrangements where the I and C are not next to each other.

4. Consider 5 card hands from a normal deck of 52 cards. Let the cards of each of the four suits be numbered 1 through 13. How many different hands are there with only cards with odd values and an odd number of suits?

Solution Suppose there is only 1 suit. There are 7 odd valued cards of each suit. So, there are $4\binom{7}{5}$ hands with 1 suit and only odd valued cards. If there are 3 suits, then there can be 3 cards from one suit and 1 from the other two suits, or 2 cards from one suit, 2 from a second suit, and 1 from the third suit. In the first case, there are $4\binom{3}{2}\binom{7}{3}\binom{7}{1}\binom{7}{1}$ possible hands. In the second case, there are $\binom{4}{2}\binom{2}{1}\binom{7}{2}\binom{7}{2}\binom{7}{1}$ hands. Summing up we get a total of

$$4\binom{7}{5} + 4\binom{3}{2}\binom{7}{3}\binom{7}{1}\binom{7}{1} + \binom{4}{2}\binom{2}{1}\binom{7}{2}\binom{7}{2}\binom{7}{1}$$

hands.

5. Suppose there are n people at a party and every person shake hands with at least one other person but not with themselves. Show that at the end of the party there are at least two people who have shaken hands with the same number of people.

Solution Let the people be labeled $x_1, \ldots, x_n$. Then, each person has shaken the hand of at most $n-1$ people. Let $f(x_n)$ be the number of handshakes. Then $f(x_1), \ldots, f(x_n)$ gives n numbers between 1 and $n-1$. Thus, two of those numbers are the same by the pigeonhole principle.

6. Prove that a simple graph is bipartite if every cycle has even length.

Solution We can assume that the graph G is connected. Let $T \subseteq G$ be a spanning tree, and pick a root vertex v of T. Then, let X be the set of vertices of T (equivalently of G) with even level, and let Y be the vertices of odd level. In T every edge is incident on a vertex of X and a vertex of Y. Let e be an edge incident on two vertices x and y that is in G but not in T, and let G' denote T plus e. Then G' is not a tree because it has too many edges, so there is a cycle in G' which includes the edge e. Deleting e from the cycle we get a path in T of odd length from x to y. Suppose that $x \in X$. Then it follows from the fact that the path is of odd length that y is of odd level and hence is in Y. Therefore, the partition X and Y of the vertices of G make G into a bipartite graph.

7. Let T be the full complete binary tree with 2^{100} terminal vertices. This means that the height of the tree is 100 and every vertex of level less than 100 has two children. A terminal vertex can be designated completely by the string $RLRRRLLRL\cdots$ of right and left branches taken to get to the terminal vertex from the root. How many of the children are obtained by paths from the root that never go right more than once in a row?

Solution This is just the 100th Fibonacci number

$$f_{100} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{100} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{100}.$$

8. Show that a simple connected planar graph has a vertex of degree at most 5.

Solution Let G be a simple connected planar graph. We may assume that G has at least 6 edges. Euler's formula applies: $f - e + v = 2$. On the other hand, since every face (including the outside one) bounds at least three edges, we have $2e \geq 3f$, since no edge can bound more than 2 faces. Therefore,

$$3f = 6 + 3e - 3v \leq 2e.$$

Rearranging, we see $e \geq 3v - 6$. Doubling this, we get $2e \geq 6v - 12$. But, $2e$ is also the sum of the degrees of all the vertices. Then, $\frac{2e}{v} \leq 6 - \frac{12}{v}$. Thus, the average degree of the vertices is less than 6. Hence, there is some vertex with degree less than 6.

9. Let T_1 and T_2 be the rooted trees with exactly one vertex. Construct full binary trees T_n inductively as follows. Let T_{n-1} be the left subtree of the root vertex, and let T_{n-2} be the right subtree of the root. How many terminal vertices does T_n have?

Solution Let a_n be the number of vertices of T_n. Then, there is an obvious recurrence relation $a_n = a_{n-1} + a_{n-2}$. Since $a_1 = a_2 = 1$, the a_n is just the nth Fibonacci number.

10. Let G be a connected graph, and suppose that the longest simple path in G has length n. Suppose that P_1 and P_2 are two simple paths in G with this maximal length. Prove that they have a common vertex.

Solution Since G is connected, every vertex of P_1 is connected by some path to every vertex of P_2. Assume that P_1 and P_2 don't have a common vertex, and let Q be a simple path from the beginning of P_1 to the beginning of P_2. Let S be the truncation of Q to a path from the last vertex v of P_1 on Q to the first vertex w on Q that is also on P_2. Then, no edge of S is on either P_1 or P_2. We construct a final path T as follows. Let x be the endpoint on P_1 farthest from v, and let T_0 be the part of P_1 that goes from x to v. Similarly, let T_2 be the part of P_2 that goes from w to the most distant endpoint of P_2. Let $T_1 = S$, and let T be the concatenation of T_0, T_1, and T_2. Then, since S has length at least 1 by hypothesis, T is a simple path longer than P_1 or P_2, which is a contradiction.