where $F(n)$ is the formula for the sum, first verify the equation for $n = 1$

$$a_1 = F(1)$$

(Basis Step). This is usually straightforward.

Now assume that the statement is true for n; that is, assume

$$a_1 + a_2 + \cdots + a_n = F(n).$$

Add a_{n+1} to both sides to get

$$a_1 + a_2 + \cdots + a_n + a_{n+1} = F(n) + a_{n+1}.$$

Finally, show that

$$F(n) + a_{n+1} = F(n+1).$$

To verify the preceding equation, use algebra to manipulate the left-hand side of the equation $[F(n) + a_{n+1}]$ until you get $F(n+1)$. *Look at* $F(n+1)$ *so you know where you're headed.* (It's somewhat like looking up the answer in the back of the book!) You've shown that

$$a_1 + a_2 + \cdots + a_{n+1} = F(n+1),$$

which is the Inductive Step. Now the proof is complete.

Proving an inequality is handled in a similar fashion. The difference is that instead of obtaining equality $[F(n) + a_{n+1} = F(n+1)$ in the preceding discussion], you obtain an *inequality*.

In general, the key to devising a proof by induction is to find case n "within" case $n+1$. Review the tiling problem (Example 7.6), which provides a striking example of case n "within" case $n+1$.

Section Review Exercises

1. State the Principle of Mathematical Induction.
2. Explain how a proof by mathematical induction proceeds.
3. Give a formula for the sum $1 + 2 + \cdots + n$.
4. What is the geometric sum? Give a formula for it.

Exercises

In Exercises 1–11, using induction, verify that each equation is true for every positive integer n.

1. $1 + 3 + 5 + \cdots + (2n-1) = n^2$
2. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$
3. $1(1!) + 2(2!) + \cdots + n(n!) = (n+1)! - 1$
4. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
5. $1^2 - 2^2 + 3^2 - \cdots + (-1)^{n+1}n^2 = \dfrac{(-1)^{n+1}n(n+1)}{2}$
6. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$
7. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$
8. $\dfrac{1}{2 \cdot 4} + \dfrac{1 \cdot 3}{2 \cdot 4 \cdot 6} + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots + \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} = \dfrac{1}{2} - \dfrac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)}$
9. $\dfrac{1}{2^2-1} + \dfrac{1}{3^2-1} + \cdots + \dfrac{1}{(n+1)^2-1} = \dfrac{3}{4} - \dfrac{1}{2(n+1)} - \dfrac{1}{2(n+2)}$
10. ★ $\cos x + \cos 2x + \cdots + \cos nx = \dfrac{\cos[(x/2)(n+1)]\sin(nx/2)}{\sin(x/2)}$ provided that $\sin(x/2) \neq 0$.
11. ★ $1 \sin x + 2 \sin 2x + \cdots + n \sin nx = \dfrac{\sin[(n+1)x]}{4\sin^2(x/2)} - \dfrac{(n+1)\cos[(2n+1)x/2]}{2\sin(x/2)}$ provided that $\sin(x/2) \neq 0$.

In Exercises 12–17, using induction, verify the inequality.

12. $\dfrac{1}{2n} \le \dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$, $n = 1, 2, \ldots$
13. ★ $\dfrac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \le \dfrac{1}{\sqrt{n+1}}$, $n = 1, 2, \ldots$
14. $2n + 1 \le 2^n$, $n = 3, 4, \ldots$
15. ★ $2^n \ge n^2$, $n = 4, 5, \ldots$

16. ★ $(a_1 a_2 \cdots a_{2^n})^{1/2^n} \le \dfrac{a_1 + a_2 + \cdots + a_{2^n}}{2^n}$, $n = 1, 2, \ldots$, and the a_i are positive numbers
17. $(1+x)^n \ge 1 + nx$, for $x \ge -1$ and $n \ge 1$
18. Use the geometric sum to prove that

$$r^0 + r^1 + \cdots + r^n < \frac{1}{1-r}$$

for all $n \ge 0$ and $0 < r < 1$.

19. ★ Prove that

$$1 \cdot r^1 + 2 \cdot r^2 + \cdots + nr^n < \frac{r}{(1-r)^2}$$

for all $n \ge 1$ and $0 < r < 1$. *Hint:* Using the result of the previous exercise, compare the sum of the terms in

$$\begin{array}{lllllll} r & r^2 & r^3 & r^4 & \cdots & & r^n \\ r^2 & r^3 & r^4 & \cdots & & r^n & \\ r^3 & r^4 & \cdots & & r^n & & \\ r^4 & \cdots & & & & & \\ \vdots & \vdots & & & & & \\ & r^n & & & & & \\ r^n & & & & & & \end{array}$$

in the diagonal direction (↙) with the sum of the terms by columns.

20. Prove that

$$\frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} < 2$$

for all $n \ge 1$.

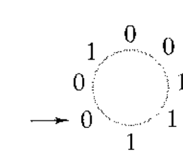
In Exercises 21–24, use induction to prove the statement.

21. $7^n - 1$ is divisible by 6, for all $n \ge 1$.
22. $11^n - 6$ is divisible by 5, for all $n \ge 1$.
23. $6 \cdot 7^n - 2 \cdot 3^n$ is divisible by 4, for all $n \ge 1$.
24. ★ $3^n + 7^n - 2$ is divisible by 8, for all $n \ge 1$.
25. By experimenting with small values of n, guess a formula for the given sum,

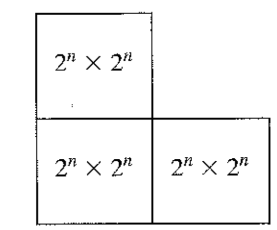
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)};$$

then use induction to verify your formula.

26. Use induction to show that n straight lines in the plane divide the plane into $(n^2 + n + 2)/2$ regions. Assume that no two lines are parallel and that no three lines have a common point.
27. Show that the regions of the preceding exercise can be colored red and green so that no two regions that share an edge are the same color.
28. Given n 0's and n 1's distributed in any manner whatsoever around a circle (see the following figure), show, using induction on n, that it is possible to start at some number and proceed clockwise around the circle to the original starting position so that, at any point during the cycle, we have seen at least as many 0's as 1's. In the following figure, a possible starting point is marked with an arrow.



29. Give a tiling of a 5×5 board with trominoes in which the upper-left square is missing.
30. Show a 5×5 deficient board that is impossible to tile with trominoes. Explain why your board cannot be tiled with trominoes.
31. Show that any $(2i) \times (3j)$ board, where i and j are positive integers, with no square missing, can be tiled with trominoes.
32. ★ Show that any 7×7 deficient board can be tiled with trominoes.
33. Show that any 11×11 deficient board can be tiled with trominoes. *Hint:* Subdivide the board into overlapping 7×7 and 5×5 boards and two 6×4 boards. Then, use Exercises 29, 31, and 32.
34. This exercise and the one that follows are due to Anthony Quas. A $2^n \times 2^n$ *L-shape*, $n \ge 0$, is a figure of the form



with no missing squares. Show that any $2^n \times 2^n$ L-shape can be tiled with trominoes.

35. Use the preceding exercise to give a different proof that any $2^n \times 2^n$ deficient board can be tiled with trominoes.

A straight tromino *is an object made up of three squares in a row:*

36. Which 4×4 deficient boards can be tiled with straight trominoes? *Hint:* Number the squares of the 4×4 board, left to right, top to bottom: 1, 2, 3, 1, 2, 3, and so on. Note that if there is a tiling, each straight tromino covers exactly one 2 and exactly one 3.
37. Which 5×5 deficient boards can be tiled with straight trominoes?
38. Which 8×8 deficient boards can be tiled with straight trominoes?
39. Use a loop invariant to prove that when the pseudocode

```
i = 1
pow = 1
while (i ≤ n) {
   pow = pow * a
   i = i + 1
}
```

terminates, pow is equal to a^n.

40. Prove that, after the following pseudocode terminates, $a[h] = val$; for all $p, i \le p < h$, $a[p] < val$; and for all $p, h < p \le j$, $a[p] \ge val$. In particular, val is in the position in the array $a[i], \ldots, a[j]$ where it would be if the array were sorted.

```
val = a[i]
h = i
for k = i + 1 to j
   if (a[k] < val) {
```

```
        h = h + 1
        swap(a[h], a[k])
    }
swap(a[i], a[h])
```

Hint: Use the loop invariant: For all $p, i < p \le h, a[p] < val$; and, for all $p, h < p < k, a[p] \ge val$. (A picture is helpful.)

This technique is called *partitioning*. This particular version is due to Nico Lomuto. Partitioning can be used to find the kth smallest element in an array and to construct a sorting algorithm called *quicksort*.

A 3D-septomino *is a three-dimensional* $2 \times 2 \times 2$ *cube with one* $1 \times 1 \times 1$ *corner cube removed. A* deficient cube *is a* $k \times k \times k$ *cube with one* $1 \times 1 \times 1$ *cube removed.*

41. Prove that a $2^n \times 2^n \times 2^n$ deficient cube can be tiled by 3D-septominoes.

42. Prove that if a $k \times k \times k$ deficient cube can be tiled by 3D-septominoes, then 7 divides one of $k-1, k-2, k-4$.

43. Suppose that $S_n = (n+2)(n-1)$ is (incorrectly) proposed as a formula for

$$2 + 4 + \cdots + 2n.$$

(a) Show that the Inductive Step is satisfied but that the Basis Step fails.

★(b) If S'_n is an arbitrary expression that satisfies the Inductive Step, what form must S'_n assume?

★ **44.** What is wrong with the following argument, which allegedly shows that any two positive integers are equal?

We use induction on n to "prove" that if a and b are positive integers and $n = \max\{a, b\}$, then $a = b$.

Basis Step ($n = 1$). If a and b are positive integers and $1 = \max\{a, b\}$, we must have $a = b = 1$.

Inductive Step Assume that if a' and b' are positive integers and $n = \max\{a', b'\}$, then $a' = b'$. Suppose that a and b are positive integers and that $n + 1 = \max\{a, b\}$. Now $n = \max\{a-1, b-1\}$. By the inductive hypothesis, $a - 1 = b - 1$. Therefore, $a = b$.

Since we have verified the Basis Step and the Inductive Step, by the Principle of Mathematical Induction, any two positive integers are equal!

45. What is wrong with the following "proof" that

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} \ne \frac{n^2}{n+1}$$

for all $n \ge 2$?

Suppose by way of contradiction that

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} = \frac{n^2}{n+1}. \qquad \textbf{(7.13)}$$

Then also

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} + \frac{n+1}{n+2} = \frac{(n+1)^2}{n+2}.$$

We could prove statement (7.13) by induction. In particular, the Inductive Step would give

$$\left(\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1}\right) + \frac{n+1}{n+2} = \frac{n^2}{n+1} + \frac{n+1}{n+2}.$$

Therefore,

$$\frac{n^2}{n+1} + \frac{n+1}{n+2} = \frac{(n+1)^2}{n+2}.$$

Multiplying each side of this last equation by $(n+1)(n+2)$ gives

$$n^2(n+2) + (n+1)^2 = (n+1)^3.$$

This last equation can be rewritten as

$$n^3 + 2n^2 + n^2 + 2n + 1 = n^3 + 3n^2 + 3n + 1$$

or

$$n^3 + 3n^2 + 2n + 1 = n^3 + 3n^2 + 3n + 1,$$

which is a contradiction. Therefore,

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} \ne \frac{n^2}{n+1},$$

as claimed.

46. Use mathematical induction to prove that

$$\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1} < \frac{n^2}{n+1}$$

for all $n \ge 2$. This inequality gives a correct proof of the statement of the preceding exercise.

In Exercises 47–51, suppose that $n > 1$ *people are positioned in a field (Euclidean plane) so that each has a unique nearest neighbor. Suppose further that each person has a pie that is hurled at the nearest neighbor. A* survivor *is a person that is not hit by a pie.*

47. Give an example to show that if n is even, there might be no survivor.

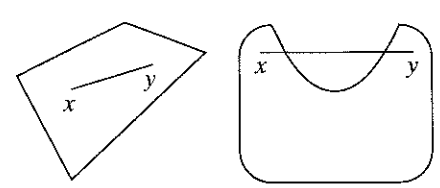
48. Give an example to show that there might be more than one survivor.

★ **49.** [Carmony] Use induction on n to show that if n is odd, there is always at least one survivor.

50. Prove or disprove: If n is odd, one of two persons farthest apart is a survivor.

51. Prove or disprove: If n is odd, a person who throws a pie the greatest distance is a survivor.

Exercises 52–55 deal with plane convex sets. A plane convex set, *subsequently abbreviated to "convex set," is a nonempty set X in the plane having the property that if x and y are any two points in X, the straight-line segment from x to y is also in X. The following figures illustrate.*



convex set   non convex set

52. Prove that if X and Y are convex sets and $X \cap Y$ (the points common to X and Y) is nonempty, $X \cap Y$ is a convex set.

★ **53.** Suppose that X_1, X_2, X_3, X_4 are convex sets, each three of which have a common point. Prove that all four sets have a common point.

★ **54.** Prove *Helly's Theorem:* Suppose that $X_1, X_2, \ldots, X_n$, $n \ge 4$, are convex sets, each three of which have a common point. Prove that all n sets have a common point.

55. Suppose that $n \ge 3$ points in the plane have the property that each three of them are contained in a circle of radius 1. Prove that there is a circle of radius 1 that contains all of the points.

56. If a and b are real numbers with $a < b$, an *open interval* (a, b) is the set of all real numbers x such that $a < x < b$. Prove that if $I_1, \ldots, I_n$ is a set of $n \ge 2$ open intervals such that each pair has a nonempty intersection, then

$$I_1 \cap I_2 \cap \cdots \cap I_n$$

(the points common to $I_1, \ldots, I_n$) is nonempty.

Flavius Josephus was a Jewish soldier and historian who lived in the first century (see [Graham, 1994; Schumer]). He was one of the leaders of a Jewish revolt against Rome in the year 66. The following year, he was among a group of trapped soldiers who decided to commit suicide rather than be captured. One version of the story is that, rather than being captured, they formed a circle and proceeded around the circle killing every third person. Josephus, being proficient in discrete math, figured out where he and a buddy should stand so they could avoid being killed.

Exercises 57–63 concern a variant of the Josephus Problem *in which every* second *person is eliminated. We assume that n people are arranged in a circle and numbered* $1, 2, \ldots, n$ *clockwise. Then, proceeding clockwise, 2 is eliminated, 4 is eliminated, and so on, until there is one survivor, denoted* $J(n)$.

57. Compute $J(4)$.

58. Compute $J(6)$.

59. Compute $J(10)$.

60. Use induction to show that $J(2^i) = 1$ for all $i \ge 1$.

61. Given a value of $n \ge 2$, let 2^i be the greatest power of 2 with $2^i \le n$. (Examples: If $n = 10$, $i = 3$. If $n = 16$, $i = 4$.) Let $j = n - 2^i$. (After subtracting 2^i, the greatest power of 2 less than or equal to n, from n, j is what is left over.) By using the result of Exercise 60 or otherwise, prove that

$$J(n) = 2j + 1.$$

62. Use the result of the Exercise 61 to compute $J(1000)$.

63. Use the result of the Exercise 61 to compute $J(100{,}000)$.

If $a_1, a_2, \ldots$ *is a sequence, we define the* difference operator Δ *to be*

$$\Delta a_n = a_{n+1} - a_n.$$

The formula of Exercise 64 can sometimes be used to find *a formula for a sum as opposed to using induction to* prove *a formula for a sum (see Exercises 65–67).*

64. Suppose that $\Delta a_n = b_n$. Show that

$$b_1 + b_2 + \cdots + b_n = a_{n+1} - a_1.$$

This formula is analogous to the calculus formula $\int_c^d f(x)\,dx = g(d) - g(c)$, where $Dg = f$ (D is the derivative operator). In the calculus formula, sum is replaced by integral, and Δ is replaced by derivative.

65. Let $a_n = n^2$, and compute Δa_n. Use Exercise 64 to find a formula for

$$1 + 2 + 3 + \cdots + n.$$

66. Use Exercise 64 to find a formula for

$$1(1!) + 2(2!) + \cdots + n(n!).$$

(Compare with Exercise 3.)

67. Use Exercise 64 to find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}.$$

(Compare with Exercise 25.)

68. Prove that if p and q are divisible by k, then $p + q$ is divisible by k.

Problem-Solving Corner Mathematical Induction

Problem

Define

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} \qquad \textbf{(1)}$$

for all $k \ge 1$. The numbers $H_1, H_2, \ldots$ are called the *harmonic numbers*. Prove that

$$H_{2^n} \ge 1 + \frac{n}{2} \qquad \textbf{(2)}$$

for all $n \ge 0$.

Attacking the Problem

It's often a good idea to begin attacking a problem by looking at some concrete examples of the expressions under consideration. Let's look at H_k for some small values of k. The smallest value of k for which H_k is defined is $k = 1$. In this case, the last term $1/k$ in the definition of H_k equals $1/1 = 1$. Since the first and last terms coincide,

$$H_1 = 1.$$

For $k = 2$, the last term $1/k$ in the definition of H_k equals $1/2$, so

$$H_2 = 1 + \frac{1}{2}.$$

Similarly, we find that

$$H_3 = 1 + \frac{1}{2} + \frac{1}{3},$$

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}.$$

We observe that H_1 appears as the first term of H_2, H_3, and H_4, that H_2 appears as the first two terms of H_3 and H_4, and that H_3 appears as the first three terms of H_4. In general, H_m appears as the first m terms of H_k if $m \le k$. This observation will help us later because the Inductive Step in a proof by induction must relate smaller instances of a problem to larger instances of the problem.

Problem-Solving Tips

To prove that two sets A and B are equal, written $A = B$, show that for every x, if $x \in A$, then $x \in B$, and if $x \in B$, then $x \in A$.

To prove that A is a subset of B, written $A \subseteq B$, show that for every x, if $x \in A$, then $x \in B$. Notice that if A is a subset of B, it is possible that $A = B$.

To prove that A is a proper subset of B, written $A \subset B$, show that A is a subset of B, as described in the previous paragraph, and that $A \neq B$, that is, that there exists an element $x \in B$, but $x \notin B$.

To visualize relationships among sets, use a Venn diagram. A Venn diagram can help determine whether a statement about sets is true or false.

A set of elements is determined by its members; order is irrelevant. On the other hand, ordered pairs and n-tuples take order into account.

Section Review Exercises

1. Define *set*.
2. What is set notation?
3. If X is a finite set, what is $|X|$?
4. How do we denote *x is an element of the set X*?
5. How do we denote *x is not an element of the set X*?
6. How do we denote the empty set?
7. Define $X = Y$, where X and Y are sets.
8. Define $X \subseteq Y$, where X and Y are sets.
9. Define *X is a proper subset of Y*.
10. What is the power set of X? How is it denoted?
11. If X has n elements, how many elements does the power set of X have?
12. Define X *union* Y. How is the union of X and Y denoted?
13. If $\mathcal{S}$ is a family of sets, how do we define the union of $\mathcal{S}$? How is the union denoted?
14. Define X *intersect* Y. How is the intersection of X and Y denoted?
15. If $\mathcal{S}$ is a family of sets, how do we define the intersection of $\mathcal{S}$? How is the intersection denoted?
16. Define *X and Y are disjoint sets*.
17. What is a pairwise disjoint family of sets?
18. Define the *difference* of sets X and Y. How is the difference denoted?
19. What is a universal set?
20. What is the complement of the set X? How is it denoted?
21. What is a Venn diagram?
22. Draw a Venn diagram of three sets and identify the set represented by each region.
23. State the associative laws for sets.
24. State the commutative laws for sets.
25. State the distributive laws for sets.
26. State the identity laws for sets.
27. State the complement laws for sets.
28. State the idempotent laws for sets.
29. State the bound laws for sets.
30. State the absorption laws for sets.
31. State the involution law for sets.
32. State the 0/1 laws for sets.
33. State De Morgan's laws for sets.
34. What is a partition of a set X?
35. Define the *Cartesian product* of sets X and Y. How is this Cartesian product denoted?
36. Define the *Cartesian product* of the sets $X_1, X_2, \ldots, X_n$. How is this Cartesian product denoted?

Exercises

In Exercises 1–16, let the universe be the set $U = \{1, 2, 3, \ldots, 10\}$. Let $A = \{1, 4, 7, 10\}$, $B = \{1, 2, 3, 4, 5\}$, and $C = \{2, 4, 6, 8\}$. List the elements of each set.

1. $A \cup B$
2. $B \cap C$
3. $A - B$
4. $B - A$
5. $\overline{A}$
6. $U - C$
7. $\overline{U}$
8. $A \cup \emptyset$
9. $B \cap \emptyset$
10. $A \cup U$
11. $B \cap U$
12. $A \cap (B \cup C)$
13. $\overline{B} \cap (C - A)$
14. $(A \cap B) - C$
15. $\overline{A \cap B} \cup C$
16. $(A \cup B) - (C - B)$

In Exercises 17–24, draw a Venn diagram and shade the given set.

17. $A \cap \overline{B}$
18. $\overline{A} - B$
19. $B \cup (B - A)$
20. $(A \cup B) - B$
21. $B \cap \overline{(C \cup A)}$
22. $(\overline{A} \cup B) \cap (\overline{C} - A)$
23. $((C \cap A) - \overline{(B - A)}) \cap C$
24. $(B - \overline{C}) \cup ((B - \overline{A}) \cap (C \cup B))$

Exercises 25–29 refer to a group of 191 students, of which 10 are taking French, business, and music; 36 are taking French and business; 20 are taking French and music; 18 are taking business and music; 65 are taking French; 76 are taking business; and 63 are taking music.

25. How many are taking French and music but not business?
26. How many are taking business and neither French nor music?
27. How many are taking French or business (or both)?
28. How many are taking music or French (or both) but not business?
29. How many are taking none of the three subjects?
30. A television poll of 151 persons found that 68 watched "Law and Disorder"; 61 watched "The East Wing"; 52 watched "The Tenors"; 16 watched both "Law and Disorder" and "The East Wing"; 25 watched both "Law and Disorder" and "The Tenors"; 19 watched both "The East Wing" and "The Tenors"; and 26 watched none of these shows. How many persons watched all three shows?
31. In a group of students, each student is taking a mathematics course or a computer science course or both. One-fifth of those taking a mathematics course are also taking a computer science course, and one-eighth of those taking a computer science course are also taking a mathematics course. Are more than one-third of the students taking a mathematics course?

In Exercises 32–35, let $X = \{1, 2\}$ and $Y = \{a, b, c\}$. List the elements in each set.

32. $X \times Y$
33. $Y \times X$
34. $X \times X$
35. $Y \times Y$

In Exercises 36–39, let $X = \{1, 2\}$, $Y = \{a\}$, and $Z = \{\alpha, \beta\}$. List the elements of each set.

36. $X \times Y \times Z$
37. $X \times Y \times Y$
38. $X \times X \times X$
39. $Y \times X \times Y \times Z$

In Exercises 40–43, list all partitions of the set.

40. $\{1\}$
41. $\{1, 2\}$
42. $\{a, b, c\}$
43. $\{a, b, c, d\}$

In Exercises 44–47, answer true or false.

44. $\{x\} \subseteq \{x\}$
45. $\{x\} \in \{x\}$
46. $\{x\} \in \{x, \{x\}\}$
47. $\{x\} \subseteq \{x, \{x\}\}$

In Exercises 48–52, determine whether each pair of sets is equal.

48. $\{1, 2, 3\}, \{1, 3, 2\}$
49. $\{1, 2, 2, 3\}, \{1, 2, 3\}$
50. $\{1, 1, 3\}, \{3, 3, 1\}$
51. $\{x \mid x^2 + x = 2\}, \{1, -2\}$
52. $\{x \mid x$ is a real number and $0 < x \leq 2\}, \{1, 2\}$
53. List the members of $\mathcal{P}(\{a, b\})$. Which are proper subsets of $\{a, b\}$?
54. List the members of $\mathcal{P}(\{a, b, c, d\})$. Which are proper subsets of $\{a, b, c, d\}$?
55. If X has 10 members, how many members does $\mathcal{P}(X)$ have? How many proper subsets does X have?
56. If X has n members, how many proper subsets does X have?
57. If X and Y are nonempty sets and $X \times Y = Y \times X$, what can we conclude about X and Y?

In each of Exercises 58–70, if the statement is true, prove it; otherwise, give a counterexample. The sets X, Y, and Z are subsets of a universal set U. Assume that the universe for Cartesian products is $U \times U$.

58. For all sets X and Y, either X is a subset of Y or Y is a subset of X.
59. $X \cap (Y - Z) = (X \cap Y) - (X \cap Z)$ for all sets X, Y, and Z.
60. $(X - Y) \cap (Y - X) = \emptyset$ for all sets X and Y.
61. $X - (Y \cup Z) = (X - Y) \cup Z$ for all sets X, Y, and Z.
62. $\overline{X - Y} = \overline{Y - X}$ for all sets X and Y.
63. $\overline{X \cap Y} \subseteq X$ for all sets X and Y.
64. $(X \cap Y) \cup (Y - X) = X$ for all sets X and Y.
65. $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$ for all sets X, Y, and Z.
66. $\overline{X \times Y} = \overline{X} \times \overline{Y}$ for all sets X and Y.
67. $X \times (Y - Z) = (X \times Y) - (X \times Z)$ for all sets X, Y, and Z.
68. $X - (Y \times Z) = (X - Y) \times (X - Z)$ for all sets X, Y, and Z.
69. $X \cap (Y \times Z) = (X \cap Y) \times (X \cap Z)$ for all sets X, Y, and Z.
70. $X \times \emptyset = \emptyset$ for every set X.

In Exercises 71–74, what relation must hold between sets A and B in order for the given condition to be true?

71. $A \cap B = A$
72. $A \cup B = A$
73. $\overline{A} \cap U = \emptyset$
74. $\overline{A \cap B} = \overline{B}$

The symmetric difference *of two sets A and B is the set*

$$A \triangle B = (A \cup B) - (A \cap B).$$

75. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, find $A \triangle B$.
76. Describe the symmetric difference of sets A and B in words.
77. Given a universe U, describe $A \triangle A$, $A \triangle \overline{A}$, $U \triangle A$, and $\emptyset \triangle A$.
78. Prove or disprove: If A, B, and C are sets satisfying $A \triangle C = B \triangle C$, then $A = B$.
79. Prove that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

80. Find a formula for $|A \cup B \cup C|$ similar to the formula of Exercise 79. Prove that your formula holds for all sets A, B, and C.
81. Let C be a circle and let $\mathcal{D}$ be the set of all diameters of C. What is $\cap \mathcal{D}$? (Here, by "diameter" we mean a line segment through the center of the circle with its endpoints on the circumference of the circle.)
82. Let P denote the set of integers greater than 1. For $i \geq 2$, define

$$X_i = \{ik \mid k \geq 2, k \in P\}.$$

Describe $P - \bigcup_{i=2}^{\infty} X_i$.

83. Prove the associative laws for sets [Theorem 1.12, part (a)].
84. Prove the commutative laws for sets [Theorem 1.12, part (b)].
85. Prove the second distributive law for sets [Theorem 1.12, part (c)].

is a function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$. Here, there is apparently no formula for membership; the definition just tells us which pairs make up the function.

On the other hand, a function may be defined by a formula. For example,

$$\{(n, n+2) \mid n \text{ is a positive integer}\}$$

defines a function from the set of positive integers to the set of positive integers. The "formula" for the mapping is "add 2."

The $f(x)$ notation may be used to indicate which element in the range is associated with an element x in the domain or to define a function. For example, for the function

$$f = \{(a, 1), (b, 3), (c, 2), (d, 1)\},$$

we could write $f(a) = 1$, $f(b) = 3$, and so on. Assuming that the domain of definition is the positive integers, the equation

$$g(n) = n + 2$$

defines the function

$$\{(n, n+2) \mid n \text{ is a positive integer}\}$$

from the set of positive integers to the set of positive integers.

To prove that a function f from X to Y is one-to-one, show that for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

To prove that a function f from X to Y is onto, show that for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$.

Section Review Exercises

1. What is a function from X to Y?
2. Explain how to use an arrow diagram to depict a function.
3. What is the graph of a function?
4. Given a set of points in the plane, how can we tell whether it is a function?
5. What is the value of x mod y?
6. What is a hash function?
7. What is a collision for a hash function?
8. What is a collision resolution policy?
9. What are pseudorandom numbers?
10. Explain how a linear congruential random number generator works, and give an example of a linear congruential random number generator.
11. What is the floor of x? How is the floor denoted?
12. What is the ceiling of x? How is the ceiling denoted?
13. Define *one-to-one function*. Give an example of a one-to-one function. Explain how to use an arrow diagram to determine whether a function is one-to-one.
14. Define *onto function*. Give an example of an onto function. Explain how to use an arrow diagram to determine whether a function is onto.
15. What is a bijection? Give an example of a bijection. Explain how to use an arrow diagram to determine whether a function is a bijection.
16. Define *inverse function*. Give an example of a function and its inverse. Given the arrow diagram of a function, how can we find the arrow diagram of the inverse function?
17. Define *composition of functions*. How is the composition of f and g denoted? Give an example of functions f and g and their composition. Given the arrow diagrams of two functions, how can we find the arrow diagram of the composition of the functions?
18. What is a binary operator? Give an example of a binary operator.
19. What is a unary operator? Give an example of a unary operator.

Exercises

Determine whether each set in Exercises 1–5 is a function from $X = \{1, 2, 3, 4\}$ to $Y = \{a, b, c, d\}$. If it is a function, find its domain and range, draw its arrow diagram, and determine if it is one-to-one, onto, or both. If it is both one-to-one and onto, give the description of the inverse function as a set of ordered pairs, draw its arrow diagram, and give the domain and range of the inverse function.

1. $\{(1, a), (2, a), (3, c), (4, b)\}$
2. $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$
3. $\{(1, c), (2, d), (3, a), (4, b)\}$
4. $\{(1, d), (2, d), (4, a)\}$
5. $\{(1, b), (2, b), (3, b), (4, b)\}$

Draw the graphs of the functions in Exercises 6–9. The domain of each function is the set of real numbers.

6. $f(x) = \lceil x \rceil - \lfloor x \rfloor$
7. $f(x) = x - \lfloor x \rfloor$
8. $f(x) = \lceil x^2 \rceil$
9. $f(x) = \lfloor x^2 - x \rfloor$

Determine whether each function in Exercises 10–15 is one-to-one. The domain of each function is the set of all real numbers. If the function is not one-to-one, prove it. Also, determine whether f is onto the set of all real numbers. If f is not onto, prove it.

10. $f(x) = 6x - 9$
11. $f(x) = 3x^2 - 3x + 1$
12. $f(x) = \sin x$
13. $f(x) = 2x^3 - 4$
14. $f(x) = 3^x - 2$
15. $f(x) = \dfrac{x}{1 + x^2}$
16. Give an example of a function different from those presented in the text that is one-to-one but not onto, and prove that your function has the required properties.
17. Give an example of a function different from those presented in the text that is onto but not one-to-one, and prove that your function has the required properties.
18. Give an example of a function different from those presented in the text that is neither one-to-one nor onto, and prove that your function has the required properties.

Each function in Exercises 19–24 is one-to-one on the specified domain X. By letting Y = range of f, we obtain a bijection from X to Y. Find each inverse function.

19. $f(x) = 4x + 2$, X = set of real numbers
20. $f(x) = 3^x$, X = set of real numbers
21. $f(x) = 3 \log_2 x$, X = set of positive real numbers
22. $f(x) = 3 + \dfrac{1}{x}$, X = set of nonzero real numbers
23. $f(x) = 4x^3 - 5$, X = set of real numbers
24. $f(x) = 6 + 2^{7x-1}$, X = set of real numbers
25. Given
$$g = \{(1, b), (2, c), (3, a)\},$$
a function from $X = \{1, 2, 3\}$ to $Y = \{a, b, c, d\}$, and
$$f = \{(a, x), (b, x), (c, z), (d, w)\},$$
a function from Y to $Z = \{w, x, y, z\}$, write $f \circ g$ as a set of ordered pairs and draw the arrow diagram of $f \circ g$.
26. Let f and g be functions from the positive integers to the positive integers defined by the equations
$$f(n) = 2n + 1, \qquad g(n) = 3n - 1.$$
Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.
27. Let f and g be functions from the positive integers to the positive integers defined by the equations
$$f(n) = n^2, \qquad g(n) = 2^n.$$
Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

28. Let f and g be functions from the positive real numbers to the positive real numbers defined by the equations
$$f(x) = \lfloor 2x \rfloor, \qquad g(x) = x^2.$$
Find the compositions $f \circ f$, $g \circ g$, $f \circ g$, and $g \circ f$.

In Exercises 29–34, decompose the function into simpler functions as in Example 2.43.

29. $f(x) = \log_2(x^2 + 2)$
30. $f(x) = \dfrac{1}{2x^2}$
31. $f(x) = \sin 2x$
32. $f(x) = 2 \sin x$
33. $f(x) = (3 + \sin x)^4$
34. $f(x) = \dfrac{1}{(\cos 6x)^3}$
35. Given
$$f = \{(x, x^2) \mid x \in X\};$$
a function from $X = \{-5, -4, \ldots, 4, 5\}$ to the set of integers, write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one or onto?
36. How many functions are there from $\{1, 2\}$ to $\{a, b\}$? Which are one-to-one? Which are onto?
37. Given
$$f = \{(a, b), (b, a), (c, b)\},$$
a function from $X = \{a, b, c\}$ to X:
(a) Write $f \circ f$ and $f \circ f \circ f$ as sets of ordered pairs.
(b) Define
$$f^n = f \circ f \circ \cdots \circ f$$
to be the n-fold composition of f with itself. Write f^9 and f^{623} as sets of ordered pairs.
38. Let f be the function from $X = \{0, 1, 2, 3, 4\}$ to X defined by
$$f(x) = 4x \bmod 5.$$
Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?
39. Let f be the function from $X = \{0, 1, 2, 3, 4, 5\}$ to X defined by
$$f(x) = 4x \bmod 6.$$
Write f as a set of ordered pairs and draw the arrow diagram of f. Is f one-to-one? Is f onto?
40. Verify the ISBN check character for this book.
41. Universal product codes (UPC) are the familiar bar codes that identify products so that they can be automatically priced at the checkout counter. A UPC is a 12-digit code in which the first digit characterizes the type of product (0 identifies an ordinary grocery item, 2 is an item sold by weight, 3 is a medical item, 4 is a special item, 5 is a coupon, and 6 and 7 are items not sold in retail stores). The next five digits identify the manufacturer, the next five digits identify the product, and the last digit is a check digit. (All UPC codes have a check digit. It is always present on the bar code, but it may not appear in the printed version.) For example, the UPC for a package of 10 Ortega taco shells is 0-54400-00800-5. The first zero identifies this as an ordinary grocery item, the next five digits 54400 identify the manufacturer Nabisco Foods, and the next five digits 00800 identify the product as a package of 10 Ortega taco shells.