to obtain

$$\begin{aligned} C(k,k) &+ C(k+1,k) + C(k+2,k) + \cdots + C(n,k) \\ &= 1 + C(k+2,k+1) - C(k+1,k+1) + C(k+3,k+1) \\ &\quad - C(k+2,k+1) + \cdots + C(n+1,k+1) - C(n,k+1) \\ &= C(n+1,k+1). \end{aligned}$$

Exercise 48, Section 6.3, shows another way to prove (6.7.4).

Example 6.7.9 Use equation (6.7.4) to find the sum

$$1 + 2 + \cdots + n.$$

We may write

$$\begin{aligned} 1 + 2 + \cdots + n &= C(1,1) + C(2,1) + \cdots + C(n,1) \\ &= C(n+1,2) \qquad \text{by equation (6.7.4)} \\ &= \frac{(n+1)n}{2}. \end{aligned}$$

Section Review Exercises

1. State the Binomial Theorem.
2. Explain how the Binomial Theorem is derived.
3. What is Pascal's triangle?
4. State the formulas that can be used to generate Pascal's triangle.

Exercises

1. Expand $(x + y)^4$ using the Binomial Theorem.
2. Expand $(2c - 3d)^5$ using the Binomial Theorem.

In Exercises 3–9, find the coefficient of the term when the expression is expanded.

3. x^4y^7; $(x + y)^{11}$
4. s^6t^6; $(2s - t)^{12}$
5. $x^2y^3z^5$; $(x + y + z)^{10}$
6. $w^2x^3y^2z^5$; $(2w + x + 3y + z)^{12}$
7. a^2x^3; $(a + x + c)^2(a + x + d)^3$
8. a^2x^3; $(a + ax + x)(a + x)^4$
9. a^3x^4; $(a + \sqrt{ax} + x)^2(a + x)^5$

In Exercises 10–12, find the number of terms in the expansion of each expression.

10. $(x + y + z)^{10}$
11. $(w + x + y + z)^{12}$
★12. $(x + y + z)^{10}(w + x + y + z)^2$
13. Find the next row of Pascal's triangle given the row

$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1.$$

14. (a) Show that $C(n, k) < C(n, k + 1)$ if and only if $k < (n - 1)/2$.
(b) Use part (a) to deduce that the maximum of $C(n, k)$ for $k = 0, 1, \ldots, n$ is $C(n, \lfloor n/2 \rfloor)$.
15. Use the Binomial Theorem to show that

$$0 = \sum_{k=0}^{n} (-1)^k C(n, k).$$

16. Use induction on n to prove the Binomial Theorem.
17. Prove Theorem 6.7.6 by using Theorem 6.2.17.
18. Give a combinatorial argument to show that

$$C(n, k) = C(n, n - k).$$

★19. Prove equation (6.7.4) by giving a combinatorial argument.
20. Find the sum

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n - 1)n.$$

★21. Use equation (6.7.4) to derive a formula for

$$1^2 + 2^2 + \cdots + n^2.$$

22. Use the Binomial Theorem to show that

$$\sum_{k=0}^{n} 2^k C(n, k) = 3^n.$$

23. Suppose that n is even. Prove that

$$\sum_{k=0}^{n/2} C(n, 2k) = 2^{n-1} = \sum_{k=1}^{n/2} C(n, 2k - 1).$$

24. Prove

$$(a + b + c)^n = \sum_{0 \le i+j \le n} \frac{n!}{i!\,j!\,(n - i - j)!} a^i b^j c^{n-i-j}.$$

25. Use Exercise 24 to write the expansion of $(x + y + z)^3$.
26. Prove

$$3^n = \sum_{0 \le i+j \le n} \frac{n!}{i!\,j!\,(n - i - j)!}.$$

★27. Give a combinatorial argument to prove that

$$\sum_{k=0}^{n} C(n, k)^2 = C(2n, n).$$

28. Prove

$$n(1 + x)^{n-1} = \sum_{k=1}^{n} C(n, k) k x^{k-1}.$$

29. Use the result of Exercise 28 to show that

$$n2^{n-1} = \sum_{k=1}^{n} kC(n, k). \qquad \textbf{(6.7.5)}$$

★30. Prove equation (6.7.5) by induction.
31. A *smoothing sequence* $b_0, \ldots, b_{k-1}$ is a (finite) sequence satisfying $b_i \ge 0$ for $i = 0, \ldots, k - 1$, and $\sum_{i=0}^{k-1} b_i = 1$. A *smoothing of the (infinite) sequence* $a_1, a_2, \ldots$ *by the smoothing sequence* $b_0, \ldots, b_{k-1}$ is the sequence $\{a'_j\}$ defined by

$$a'_j = \sum_{i=0}^{k-1} a_{i+j} b_i.$$

The idea is that averaging smooths noisy data.

The *binomial smoother of size* k is the sequence

$$\frac{B_0}{2^n}, \ldots, \frac{B_{k-1}}{2^n},$$

where $B_0, \ldots, B_{k-1}$ is row n of Pascal's triangle (row 0 being the top row).

Let c_0, c_1 be the smoothing sequence defined by $c_0 = c_1 = 1/2$. Show that if a sequence a is smoothed by c, the resulting sequence is smoothed by c, and so on k times; then, the sequence that results can be obtained by one smoothing of a by the binomial smoother of size $k + 1$.

32. In Example 6.1.6 we showed that there are 3^n ordered pairs (A, B) satisfying $A \subseteq B \subseteq X$, where X is an n-element set. Derive this result by considering the cases $|A| = 0$, $|A| = 1, \ldots, |A| = n$, and then using the Binomial Theorem.
33. Show that

$$\sum_{k=m}^{n} C(k, m) H_k = C(n + 1, m + 1) \left(H_{n+1} - \frac{1}{m + 1} \right)$$

for all $n \ge m$, where H_k, the kth harmonic number, is defined as

$$H_k = \sum_{i=1}^{k} \frac{1}{i}.$$

34. Prove that

$$\sum_{i=1}^{n} \frac{1}{C(n, i)} = \frac{n + 1}{2^n} \sum_{i=0}^{n-1} \frac{2^i}{i + 1},$$

for all $n \in \mathbf{Z}^+$.

35. Prove that

$$\left(\frac{m}{m + n} \right)^m \left(\frac{n}{m + n} \right)^n C(m + n, m) < 1$$

for all $m, n \in \mathbf{Z}^+$. *Hint*: Consider the term for $k = m$ in the binomial theorem expansion of $(x + y)^{m+n}$ for appropriate x and y.

★36. Prove that for all $k \in \mathbf{Z}^{nonneg}$,

$$\sum_{i=1}^{n} i^k = \frac{n^{k+1}}{k + 1} + C_k n^k + C_{k-1} n^{k-1} + \cdots + C_1 n + C_0,$$

for all $n \in \mathbf{Z}^+$. *Hint*: Use Strong Induction on k; Exercise 67, Section 2.4, with $a_k = i^{k+1}$; and the Binomial Theorem.

6.8 → The Pigeonhole Principle

The **Pigeonhole Principle** (also known as the *Dirichlet Drawer Principle* or the *Shoe Box Principle*) is sometimes useful in answering the question: Is there an item having a given property? When the Pigeonhole Principle is successfully applied, the principle tells us only that the object exists; the principle will not tell us how to find the object or how many there are.

The first version of the Pigeonhole Principle that we will discuss asserts that if n pigeons fly into k pigeonholes and $k < n$, some pigeonhole contains at least two pigeons (see Figure 6.8.1). The reason this statement is true can be seen by arguing by contradiction. If the conclusion is false, each pigeonhole contains at most one pigeon and, in this case, we can account for at most k pigeons. Since there are n pigeons and $n > k$, we have a contradiction.

which is a contradiction. Therefore, there are at least k values, $a_1, \ldots, a_k \in X$, such that

$$f(a_1) = f(a_2) = \cdots = f(a_k).$$

Our last example illustrates the use of the third form of the Pigeonhole Principle.

Example 6.8.5 A useful feature of black-and-white pictures is the average brightness of the picture. Let us say that two pictures are similar if their average brightness differs by no more than some fixed value. Show that among six pictures, there are either three that are mutually similar or three that are mutually dissimilar.

Denote the pictures $P_1, P_2, \ldots, P_6$. Each of the five pairs

$$(P_1, P_2), \quad (P_1, P_3), \quad (P_1, P_4), \quad (P_1, P_5), \quad (P_1, P_6),$$

has the value "similar" or "dissimilar." By the third form of the Pigeonhole Principle, there are at least $\lceil 5/2 \rceil = 3$ pairs with the same value; that is, there are three pairs

$$(P_1, P_i), \quad (P_1, P_j), \quad (P_1, P_k)$$

all similar or all dissimilar. Suppose that each pair is similar. (The case that each pair is dissimilar is Exercise 14.) If any pair

$$(P_i, P_j), \quad (P_i, P_k), \quad (P_j, P_k) \qquad \textbf{(6.8.5)}$$

is similar, then these two pictures together with P_1 are mutually similar and we have found three mutually similar pictures. Otherwise, each of the pairs (6.8.5) is dissimilar and we have found three mutually dissimilar pictures. ◀

Section Review Exercises

1. State three forms of the Pigeonhole Principle.
2. Give an example of the use of each form of the Pigeonhole Principle.

Exercises

1. Prove that if five cards are chosen from an ordinary 52-card deck, at least two cards are of the same suit.
2. Prove that among a group of six students, at least two received the same grade on the final exam. (The grades assigned were chosen from A, B, C, D, F.)
3. Suppose that each person in a group of 32 people receives a check in January. Prove that at least two people receive checks on the same day.
4. Prove that among 35 students in a class, at least two have first names that start with the same letter.
5. Prove that if f is a function from the finite set X to the finite set Y and $|X| > |Y|$, then f is not one-to-one.
6. Suppose that six distinct integers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Prove that at least two of the six have a sum equal to 11. *Hint:* Consider the partition $\{1, 10\}$, $\{2, 9\}$, $\{3, 8\}$, $\{4, 7\}$, $\{5, 6\}$.
7. Thirteen persons have first names Dennis, Evita, and Ferdinand and last names Oh, Pietro, Quine, and Rostenkowski. Show that at least two persons have the same first and last names.
8. Eighteen persons have first names Alfie, Ben, and Cissi and last names Dumont and Elm. Show that at least three persons have the same first and last names.
9. Professor Euclid is paid every other week on Friday. Show that in some month she is paid three times.
10. Is it possible to interconnect five processors so that exactly two processors are directly connected to an identical number of processors? Explain.
11. An inventory consists of a list of 115 items, each marked "available" or "unavailable." There are 60 available items. Show that there are at least two available items in the list exactly four items apart.
12. An inventory consists of a list of 100 items, each marked "available" or "unavailable." There are 55 available items. Show that there are at least two available items in the list exactly nine items apart.

★13. An inventory consists of a list of 80 items, each marked "available" or "unavailable." There are 50 available items. Show that there are at least two unavailable items in the list either three or six items apart.

14. Complete Example 6.8.5 by showing that if the pairs (P_1, P_i), (P_1, P_j), (P_1, P_k) are dissimilar, there are three pictures that are mutually similar or mutually dissimilar.

15. Does the conclusion to Example 6.8.5 necessarily follow if there are fewer than six pictures? Explain.

16. Does the conclusion to Example 6.8.5 necessarily follow if there are more than six pictures? Explain.

Answer Exercises 17–20 to give an argument that shows that if X is any $(n+2)$-element subset of $\{1, 2, \ldots, 2n+1\}$ and m is the greatest element in X, there exist distinct i and j in X with $m = i + j$.

For each element $k \in X - \{m\}$, let

$$a_k = \begin{cases} k & \text{if } k \le \dfrac{m}{2} \\ m - k & \text{if } k > \dfrac{m}{2}. \end{cases}$$

17. How many elements are in the domain of a?
18. Show that the range of a is contained in $\{1, 2, \ldots, n\}$.
19. Explain why Exercises 17 and 18 imply that $a_i = a_j$ for some $i \ne j$.
20. Explain why Exercise 19 implies that there exist distinct i and j in X with $m = i + j$.
21. Give an example of an $(n+1)$-element subset X of $\{1, 2, \ldots, 2n+1\}$ having the property: For no distinct $i, j \in X$ do we have $i + j \in X$.

Answer Exercises 22–25 to give an argument that proves the following result.

A sequence $a_1, a_2, \ldots, a_{n^2+1}$ of $n^2 + 1$ distinct numbers contains either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

Suppose by way of contradiction that every increasing or decreasing subsequence has length n or less. Let b_i be the length of a longest increasing subsequence starting at a_i, and let c_i be the length of a longest decreasing subsequence starting at a_i.

22. Show that the ordered pairs (b_i, c_i), $i = 1, \ldots, n^2 + 1$, are distinct.
23. How many ordered pairs (b_i, c_i) are there?
24. Explain why $1 \le b_i \le n$ and $1 \le c_i \le n$.
25. What is the contradiction?

Answer Exercises 26–29 to give an argument that shows that in a group of 10 persons there are at least two such that either the difference or sum of their ages is divisible by 16. Assume that the ages are given as whole numbers.

Let $a_1, \ldots, a_{10}$ denote the ages. Let $r_i = a_i \bmod 16$ and let

$$s_i = \begin{cases} r_i & \text{if } r_i \le 8 \\ 16 - r_i & \text{if } r_i > 8. \end{cases}$$

26. Show that $s_1, \ldots, s_{10}$ range in value from 0 to 8.
27. Explain why $s_j = s_k$ for some $j \ne k$.
28. Suppose that $s_j = s_k$ for some $j \ne k$. Explain why if $s_j = r_j$ and $s_k = r_k$ or $s_j = 16 - r_j$ and $s_k = 16 - r_k$, then 16 divides $a_j - a_k$.
29. Show that if the conditions in Exercise 28 fail, then 16 divides $a_j + a_k$.
30. Show that in the decimal expansion of the quotient of two integers, eventually some block of digits repeats. *Examples:*

$$\frac{1}{6} = 0.1\underline{6}66\ldots, \qquad \frac{217}{660} = 0.32\underline{878}787\ldots.$$

★31. Twelve basketball players, whose uniforms are numbered 1 through 12, stand around the center ring on the court in an arbitrary arrangement. Show that some three consecutive players have the sum of their numbers at least 20.

★32. For the situation of Exercise 31, find and prove an estimate for how large the sum of some four consecutive players' numbers must be.

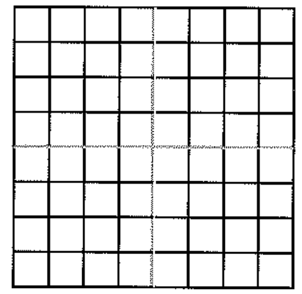
★33. Let f be a one-to-one function from $X = \{1, 2, \ldots, n\}$ onto X. Let $f^k = f \circ f \circ \cdots \circ f$ denote the k-fold composition of f with itself. Show that there are distinct positive integers i and j such that $f^i(x) = f^j(x)$ for all $x \in X$. Show that for some positive integer k, $f^k(x) = x$ for all $x \in X$.

★34. A 3×7 rectangle is divided into 21 squares each of which is colored red or black. Prove that the board contains a nontrivial rectangle (not $1 \times k$ or $k \times 1$) whose four corner squares are all black or all red.

★35. Prove that if p ones and q zeros are placed around a circle in an arbitrary manner, where p, q, and k are positive integers satisfying $p \ge kq$, the arrangement must contain at least k consecutive ones.

★36. Write an algorithm that, given a sequence a, finds the length of a longest increasing subsequence of a.

37. A $2k \times 2k$ grid is divided into $4k^2$ squares and four $k \times k$ subgrids. The following figure shows the grid for $k = 4$:



Show that it is impossible to mark k squares in the upper-left, $k \times k$ subgrid and k squares in the lower-right, $k \times k$ subgrid so that no two marked squares are in the same row, column, or diagonal of the $2k \times 2k$ grid.

This is a variant of the *n-queens problem,* which we discuss in detail in Section 9.3.

Exercises

In Exercises 1–3, find a recurrence relation and initial conditions that generate a sequence that begins with the given terms.

1. 3, 7, 11, 15, ...
2. 3, 6, 9, 15, 24, 39, ...
3. 1, 1, 2, 4, 16, 128, 4096, ...

In Exercises 4–8, assume that a person invests \$2000 at 14 percent interest compounded annually. Let A_n represent the amount at the end of n years.

4. Find a recurrence relation for the sequence $A_0, A_1, \ldots$.
5. Find an initial condition for the sequence $A_0, A_1, \ldots$.
6. Find A_1, A_2, and A_3.
7. Find an explicit formula for A_n.
8. How long will it take for a person to double the initial investment?

If a person invests in a tax-sheltered annuity, the money invested, as well as the interest earned, is not subject to taxation until withdrawn from the account. In Exercises 9–12, assume that a person invests \$2000 each year in a tax-sheltered annuity at 10 percent interest compounded annually. Let A_n represent the amount at the end of n years.

9. Find a recurrence relation for the sequence $A_0, A_1, \ldots$.
10. Find an initial condition for the sequence $A_0, A_1, \ldots$.
11. Find A_1, A_2, and A_3.
12. Find an explicit formula for A_n.

In Exercises 13–17, assume that a person invests \$3000 at 12 percent annual interest compounded quarterly. Let A_n represent the amount at the end of n years.

13. Find a recurrence relation for the sequence $A_0, A_1, \ldots$.
14. Find an initial condition for the sequence $A_0, A_1, \ldots$.
15. Find A_1, A_2, and A_3.
16. Find an explicit formula for A_n.
17. How long will it take for a person to double the initial investment?
18. Let S_n denote the number of n-bit strings that do not contain the pattern 000. Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.

Exercises 19–21 refer to the sequence S where S_n denotes the number of n-bit strings that do not contain the pattern 00.

19. Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.
20. Show that $S_n = f_{n+2}, n = 1, 2, \ldots$, where f denotes the Fibonacci sequence.
21. By considering the number of n-bit strings with exactly i 0's and Exercise 20, show that

$$f_{n+2} = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} C(n+1-i, i), \qquad n = 1, 2, \ldots,$$

where f denotes the Fibonacci sequence.

Exercises 22–24 refer to the sequence $S_1, S_2, \ldots$, where S_n denotes the number of n-bit strings that do not contain the pattern 010.

22. Compute S_1, S_2, S_3, and S_4.
23. By considering the number of n-bit strings that do not contain the pattern 010 that have no leading 0's (i.e., that begin with 1); that have one leading 0 (i.e., that begin 01); that have two leading 0's; and so on, derive the recurrence relation

$$S_n = S_{n-1} + S_{n-3} + S_{n-4} + S_{n-5} + \cdots + S_1 + 3. \qquad \textbf{(7.1.14)}$$

24. By replacing n by $n-1$ in (7.1.14), write a formula for S_{n-1}. Subtract the formula for S_{n-1} from the formula for S_n and use the result to derive the recurrence relation

$$S_n = 2S_{n-1} - S_{n-2} + S_{n-3}.$$

In Exercises 25–33, $C_0, C_1, C_2, \ldots$ denotes the sequence of Catalan numbers.

25. Given that $C_0 = C_1 = 1$ and $C_2 = 2$, compute C_3, C_4, and C_5 by using the recurrence relation of Example 7.1.7.
26. Show that the Catalan numbers are given by the recurrence relation

$$(n+2)C_{n+1} = (4n+2)C_n, \qquad n \geq 0,$$

and initial condition $C_0 = 1$.

27. Prove that $n + 2 < C_n$ for all $n \geq 4$.

★28. Prove that C_n is prime if and only if $n = 2$ or $n = 3$. *Hint:* First, use proof by contradiction to prove that if $n \geq 5$, C_n is not prime. Use Exercises 26 and 27 and Exercise 25, Section 5.3, to derive a contradiction. Now examine the cases $n = 0, 1, 2, 3, 4$.

29. Prove that

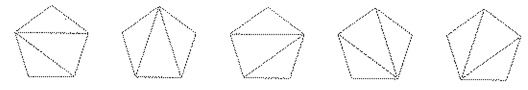
$$C_n \geq \frac{4^{n-1}}{n^2} \qquad \text{for all } n \geq 1.$$

30. Derive a recurrence relation and an initial condition for the number of ways to parenthesize the product

$$a_1 * a_2 * \cdots * a_n, \qquad n \geq 2.$$

Examples: There is one way to parenthesize $a_1 * a_2$, namely, $(a_1 * a_2)$. There are two ways to parenthesize $a_1 * a_2 * a_3$, namely, $((a_1 * a_2) * a_3)$ and $(a_1 * (a_2 * a_3))$. Deduce that the number of ways to parenthesize the product of n elements is C_{n-1}, $n \geq 2$.

★31. Derive a recurrence relation and an initial condition for the number of ways to divide a convex $(n+2)$-sided polygon, $n \geq 1$, into triangles by drawing $n-1$ lines through the corners that do not intersect in the interior of the polygon. (A polygon is *convex* if any line joining two points in the polygon lies wholly in the polygon.) For example, there are five ways to divide a convex pentagon into triangles by drawing two nonintersecting lines through the corners:



Deduce that the number of ways to divide a convex $(n+2)$-sided polygon into triangles by drawing $n-1$ nonintersecting lines through the corners is C_n, $n \geq 1$.

32. How many parenthesized expressions are there containing n distinct binary operators, $n+1$ distinct variables, and $n-1$ pairs of parentheses? For example, if $n = 2$, and we choose $*$ and $+$ as the operators and x, y, and z as the variables, some of the expressions are

$$(x * y)+z, \quad x * (y+z), \quad x * (z+y), \quad x+(z * y), \quad z * (y+x).$$

33. Consider routes from the lower-left corner to the upper-right corner in an $(n+1) \times (n+1)$ grid in which we are restricted to traveling only to the right or upward. By dividing the routes into classes based on when, after leaving the lower-left corner, the route first meets the diagonal line from the lower-left corner to the upper-right corner, derive the recurrence relation

$$C_n = \frac{1}{2}C(2(n+1), n+1) - \sum_{k=0}^{n-1} C_k C(2(n-k), n-k).$$

In Exercises 34 and 35, let S_n denote the number of routes from the lower-left corner of an $n \times n$ grid to the upper-right corner in which we are restricted to traveling to the right, upward, or diagonally northeast [i.e., from (i, j) to $(i+1, j+1)$] and in which we are allowed to touch but not go above a diagonal line from the lower-left corner to the upper-right corner. The numbers $S_0, S_1, \ldots$ are called the Schröder *numbers.*

34. Show that $S_0 = 1$, $S_1 = 2$, $S_2 = 6$, and $S_3 = 22$.
35. Derive a recurrence relation for the sequence of Schröder numbers.
36. Write explicit solutions for the Tower of Hanoi puzzle for $n = 3, 4$.
37. To what values do the price and quantity tend in Example 7.1.9 when $b < k$?
38. Show that when $b < k$ in Example 7.1.9, the price tends to that given by the intersection of the supply and demand curves.
39. Show that when $b > k$ in Example 7.1.9, the differences between successive prices increase.

Exercises 40–46 refer to Ackermann's function $A(m, n)$.

40. Compute $A(2, 2)$ and $A(2, 3)$.
41. Use induction to show that

$$A(1, n) = n + 2, \qquad n = 0, 1, \ldots.$$

42. Use induction to show that

$$A(2, n) = 3 + 2n, \qquad n = 0, 1 \ldots.$$

43. Guess a formula for $A(3, n)$ and prove it by using induction.

★44. Prove that $A(m, n) > n$ for all $m \geq 0$, $n \geq 0$ by induction on m. The inductive step will use induction on n.

45. By using Exercise 44 or otherwise, prove that $A(m, n) > 1$ for all $m \geq 1$, $n \geq 0$.
46. By using Exercise 44 or otherwise, prove that $A(m, n) < A(m, n+1)$ for all $m \geq 0$, $n \geq 0$.

What we and others have called Ackermann's function is actually derived from Ackermann's original function defined by

$$\begin{aligned}
AO(0, y, z) &= z + 1, && y, z \geq 0,\\
AO(1, y, z) &= y + z, && y, z \geq 0,\\
AO(2, y, z) &= yz, && y, z \geq 0,\\
AO(x+3, y, 0) &= 1, && x, y \geq 0,\\
AO(x+3, y, z+1) &= \\
&\quad AO(x+2, y, AO(x+3, y, z)), && x, y, z \geq 0.
\end{aligned}$$

Exercises 47–50 refer to the function AO and to Ackermann's function A.

47. Show that $A(x, y) = AO(x, 2, y+3) - 3$ for $y \geq 0$ and $x = 0, 1, 2$.
48. Show that $AO(x, 2, 1) = 2$ for $x \geq 2$.
49. Show that $AO(x, 2, 2) = 4$ for $x \geq 2$.

★50. Show that $A(x, y) = AO(x, 2, y+3) - 3$ for $x, y \geq 0$.

51. A network consists of n nodes. Each node has communications facilities and local storage. Periodically, all files must be shared. A *link* consists of two nodes sharing files. Specifically, when nodes A and B are linked, A transmits all its files to B and B transmits all its files to A. Only one link exists at a time, and after a link is established and the files are shared, the link is deleted. Let a_n be the minimum number of links required by n nodes so that all files are known to all nodes.
    (a) Show that $a_2 = 1$, $a_3 \leq 3$, $a_4 \leq 4$.
    (b) Show that $a_n \leq a_{n-1} + 2$, $n \geq 3$.
52. If P_n denotes the number of permutations of n distinct objects, find a recurrence relation and an initial condition for the sequence $P_1, P_2, \ldots$.
53. Suppose that we have n dollars and that each day we buy either orange juice (\$1), milk (\$2), or beer (\$2). If R_n is the number of ways of spending all the money, show that

$$R_n = R_{n-1} + 2R_{n-2}.$$

Order is taken into account. For example, there are 11 ways to spend \$4: *MB, BM, OOM, OOB, OMO, OBO, MOO, BOO, OOOO, MM, BB.*

54. Suppose that we have n dollars and that each day we buy either tape (\$1), paper (\$1), pens (\$2), pencils (\$2), or binders (\$3). If R_n is the number of ways of spending all the money, derive a recurrence relation for the sequence $R_1, R_2, \ldots$.

55. Let R_n denote the number of regions into which the plane is divided by n lines. Assume that each pair of lines meets in a point, but that no three lines meet in a point. Derive a recurrence relation for the sequence $R_1, R_2, \ldots$.

Exercises 56 and 57 refer to the sequence S_n defined by

$$S_1 = 0, \quad S_2 = 1, \quad S_n = \frac{S_{n-1} + S_{n-2}}{2}, \quad n = 3, 4, \ldots.$$

56. Compute S_3 and S_4.

★57. Guess a formula for S_n and use induction to show that it is correct.

★58. Let F_n denote the number of functions f from $X = \{1, \ldots, n\}$ into X having the property that if i is in the range of f, then $1, 2, \ldots, i-1$ are also in the range of f. (Set $F_0 = 1$.) Show that the sequence $F_0, F_1, \ldots$ satisfies the recurrence relation

$$F_n = \sum_{j=0}^{n-1} C(n, j) F_j.$$

59. If α is a bit string, let $C(\alpha)$ be the maximum number of consecutive 0's in α. [*Examples:* $C(10010) = 2$, $C(00110001) = 3$.] Let S_n be the number of n-bit strings α with $C(\alpha) \le 2$. Develop a recurrence relation for $S_1, S_2, \ldots$.

60. The sequence $g_1, g_2, \ldots$ is defined by the recurrence relation

$$g_n = g_{n-1} + g_{n-2} + 1, \qquad n \ge 3,$$

and initial conditions

$$g_1 = 1, \qquad g_2 = 3.$$

By using mathematical induction or otherwise, show that

$$g_n = 2f_{n+1} - 1, \qquad n \ge 1,$$

where $f_1, f_2, \ldots$ is the Fibonacci sequence.

61. Consider the formula

$$u_n = \begin{cases} u_{3n+1} & \text{if } n \text{ is odd and greater than 1} \\ u_{n/2} & \text{if } n \text{ is even and greater than 1} \end{cases}$$

and initial condition $u_1 = 1$. Explain why the formula is *not* a recurrence relation. A longstanding, open conjecture is that for every positive integer n, u_n is defined and equal to 1. Compute u_n for $n = 2, \ldots, 7$.

62. Define the sequence $t_1, t_2, \ldots$ by the recurrence relation

$$t_n = t_{n-1} t_{n-2}, \qquad n \ge 3,$$

and initial conditions

$$t_1 = 1, \qquad t_2 = 2.$$

What is wrong with the following "proof" that $t_n = 1$ for all $n \ge 1$?

Basis Step For $n = 1$, we have $t_1 = 1$; thus, the Basis Step is verified.

Inductive Step Assume that $t_k = 1$ for $k < n$. We must prove that $t_n = 1$. Now

$$\begin{aligned} t_n &= t_{n-1} t_{n-2} \\ &= 1 \cdot 1 \qquad \text{by the inductive assumption} \\ &= 1. \end{aligned}$$

The Inductive Step is complete.

63. Derive a recurrence relation for $C(n, k)$, the number of k-element subsets of an n-element set. Specifically, write $C(n+1, k)$ in terms of $C(n, i)$ for appropriate i.

64. Derive a recurrence relation for $S(k, n)$, the number of ways of choosing k items, allowing repetitions, from n available types. Specifically, write $S(k, n)$ in terms of $S(k-1, i)$ for appropriate i.

65. Let $S(n, k)$ denote the number of functions from $\{1, \ldots, n\}$ onto $\{1, \ldots, k\}$. Show that $S(n, k)$ satisfies the recurrence relation

$$S(n, k) = k^n - \sum_{i=1}^{k-1} C(k, i) S(n, i).$$

66. The *Lucas sequence* $L_1, L_2, \ldots$ (named after Édouard Lucas, the inventor of the Tower of Hanoi puzzle) is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, \qquad n \ge 3,$$

and the initial conditions

$$L_1 = 1, \qquad L_2 = 3.$$

(a) Find the values of L_3, L_4, and L_5.

(b) Show that

$$L_{n+2} = f_{n+1} + f_{n+3}, \qquad n \ge 1,$$

where $f_1, f_2, \ldots$ denotes the Fibonacci sequence.

67. Establish the recurrence relation

$$s_{n+1,k} = s_{n,k-1} + n s_{n,k}$$

for Stirling numbers of the first kind (see Exercise 87, Section 6.2).

68. Establish the recurrence relation

$$S_{n+1,k} = S_{n,k-1} + k S_{n,k}$$

for Stirling numbers of the second kind (see Exercise 88, Section 6.2).

★69. Show that

$$S_{n,k} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i (k-i)^n C(k, i),$$

where $S_{n,k}$ denotes a Stirling number of the second kind (see Exercise 88, Section 6.2).

70. Assume that a person invests a sum of money at r percent compounded annually. Explain the rule of thumb: To estimate the time to double the investment, divide 70 by r.

71. Derive a recurrence relation for the number of multiplications needed to evaluate an $n \times n$ determinant by the cofactor method.

A rise/fall permutation *is a permutation p of $1, 2, \ldots, n$ satisfying*

$$p(i) < p(i+1) \qquad \text{for } i = 1, 3, 5, \ldots$$

and

$$p(i) > p(i+1) \qquad \text{for } i = 2, 4, 6, \ldots.$$

For example, there are five rise/fall permutations of 1, 2, 3, 4:

1, 3, 2, 4; 1, 4, 2, 3; 2, 3, 1, 4;
2, 4, 1, 3; 3, 4, 1, 2.

Let E_n denote the number of rise/fall permutations of $1, 2, \ldots, n$. (Define $E_0 = 1$.) The numbers $E_0, E_1, E_2, \ldots$ are called the Euler numbers.

72. List all rise/fall permutations of 1, 2, 3. What is the value of E_3?

73. List all rise/fall permutations of 1, 2, 3, 4, 5. What is the value of E_5?

74. Show that in a rise/fall permutation of $1, 2, \ldots, n$, n must occur in position $2i$, for some i.

★75. Use Exercise 74 to derive the recurrence relation

$$E_n = \sum_{j=1}^{\lfloor n/2 \rfloor} C(n-1, 2j-1) E_{2j-1} E_{n-2j}.$$

★76. By considering where 1 must occur in a rise/fall permutation, derive the recurrence relation

$$E_n = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} C(n-1, 2j) E_{2j} E_{n-2j-1}.$$

★77. Prove that

$$E_n = \frac{1}{2} \sum_{j=1}^{n-1} C(n-1, j) E_j E_{n-j-1}.$$

7.2 ➜ Solving Recurrence Relations

To solve a recurrence relation involving the sequence $a_0, a_1, \ldots$ is to find an explicit formula for the general term a_n. In this section we discuss two methods of solving recurrence relations: **iteration** and a special method that applies to **linear homogeneous recurrence relations with constant coefficients.** For more powerful methods, such as methods that make use of generating functions, consult [Brualdi].

To solve a recurrence relation involving the sequence $a_0, a_1, \ldots$ by iteration, we use the recurrence relation to write the nth term a_n in terms of certain of its predecessors $a_{n-1}, \ldots, a_0$. We then successively use the recurrence relation to replace each of $a_{n-1}, \ldots$ by certain of their predecessors. We continue until an explicit formula is obtained. The iterative method was used to solve the recurrence relation of Example 7.1.3.

Example 7.2.1 ▶ We can solve the recurrence relation

$$a_n = a_{n-1} + 3, \tag{7.2.1}$$

subject to the initial condition

$$a_1 = 2,$$

by iteration. Replacing n by $n-1$ in (7.2.1), we obtain

$$a_{n-1} = a_{n-2} + 3.$$

If we substitute this expression for a_{n-1} into (7.2.1), we obtain

$$\begin{aligned} a_n &= \boxed{a_{n-1}} + 3 \\ &\quad\downarrow \\ &= \boxed{a_{n-2} + 3} + 3 \\ &= a_{n-2} + 2 \cdot 3. \end{aligned} \tag{7.2.2}$$

Replacing n by $n-2$ in (7.2.1), we obtain

$$a_{n-2} = a_{n-3} + 3.$$

Section Review Exercises

1. Explain how to solve a recurrence relation by iteration.
2. What is an nth-order, linear homogeneous recurrence relation with constant coefficients?
3. Give an example of a second-order, linear homogeneous recurrence relation with constant coefficients.
4. Explain how to solve a second-order, linear homogeneous recurrence relation with constant coefficients.

Exercises

Tell whether or not each recurrence relation in Exercises 1–10 is a linear homogeneous recurrence relation with constant coefficients. Give the order of each linear homogeneous recurrence relation with constant coefficients.

1. $a_n = -3a_{n-1}$
2. $a_n = 2na_{n-1}$
3. $a_n = 2na_{n-2} - a_{n-1}$
4. $a_n = a_{n-1} + n$
5. $a_n = 7a_{n-2} - 6a_{n-3}$
6. $a_n = a_{n-1} + 1 + 2^{n-1}$
7. $a_n = (\lg 2n)a_{n-1} - [\lg(n-1)]a_{n-2}$
8. $a_n = 6a_{n-1} - 9a_{n-2}$
9. $a_n = -a_{n-1} - a_{n-2}$
10. $a_n = -a_{n-1} + 5a_{n-2} - 3a_{n-3}$

In Exercises 11–26, solve the given recurrence relation for the initial conditions given.

11. Exercise 1; $a_0 = 2$
12. Exercise 2; $a_0 = 1$
13. Exercise 4; $a_0 = 0$
14. $a_n = 2^n a_{n-1}$; $a_0 = 1$
15. $a_n = 6a_{n-1} - 8a_{n-2}$; $a_0 = 1$, $a_1 = 0$
16. $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 5$, $a_1 = 16$
17. $a_n = 2a_{n-1} + 8a_{n-2}$; $a_0 = 4$, $a_1 = 10$
18. $2a_n = 7a_{n-1} - 3a_{n-2}$; $a_0 = a_1 = 1$
19. Exercise 6; $a_0 = 0$
20. Exercise 8; $a_0 = a_1 = 1$
21. $a_n = -8a_{n-1} - 16a_{n-2}$; $a_0 = 2$, $a_1 = -20$
22. $9a_n = 6a_{n-1} - a_{n-2}$; $a_0 = 6$, $a_1 = 5$
23. The Lucas sequence

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 3; \qquad L_1 = 1, \quad L_2 = 3$$

24. Exercise 53, Section 7.1
25. Exercise 55, Section 7.1
26. The recurrence relation preceding Exercise 56, Section 7.1
27. The population of Utopia increases 5 percent per year. In 2000 the population was 10,000. What was the population in 1970?
28. Assume that the deer population of Rustic County is 0 at time $n = 0$. Suppose that at time n, $100n$ deer are introduced into Rustic County and that the population increases 20 percent each year. Write a recurrence relation and an initial condition that define the deer population at time n and then solve the recurrence relation. The following formula may be of use:

$$\sum_{i=1}^{n-1} ix^{i-1} = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}.$$

Exercises 29–33 concern Toots and Sly, who flip fair pennies. If the pennies are both heads or both tails, Toots wins. If one penny is a head and the other a tail, Sly wins. Toots starts with T pennies, and Sly starts with S pennies.

29. Let p_n denote the probability that Toots wins all of Sly's pennies if Toots starts with n pennies. Write a recurrence relation for p_n.
30. What is the value of p_0?
31. What is the value of p_{S+T}?
32. Solve your recurrence relation of Exercise 29.
33. Find the probability that Toots wins all of Sly's pennies.

Sometimes a recurrence relation that is not a linear homogeneous equation with constant coefficients can be transformed into a linear homogeneous equation with constant coefficients. In Exercises 34 and 35, make the given substitution and solve the resulting recurrence relation, then find the solution to the original recurrence relation.

34. Solve the recurrence relation

$$\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}}$$

with initial conditions $a_0 = a_1 = 1$ by making the substitution $b_n = \sqrt{a_n}$.

35. Solve the recurrence relation

$$a_n = \sqrt{\frac{a_{n-2}}{a_{n-1}}}$$

with initial conditions $a_0 = 8$, $a_1 = 1/(2\sqrt{2})$ by taking the logarithm of both sides and making the substitution $b_n = \lg a_n$.

In Exercises 36–38, solve the recurrence relation for the initial conditions given.

36. $a_n = -2na_{n-1} + 3n(n-1)a_{n-2}$; $a_0 = 1$, $a_1 = 2$

★37. $c_n = 2 + \sum_{i=1}^{n-1} c_i$, $n \geq 2$; $c_1 = 1$

★38. $A(n, m) = 1 + A(n-1, m-1) + A(n-1, m)$, $n - 1 \geq m \geq 1$, $n \geq 2$; $A(n, 0) = A(n, n) = 1$, $n \geq 0$

39. Show that

$$f_{n+1} \geq \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}, \qquad n \geq 1,$$

where f denotes the Fibonacci sequence.

40. The equation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n) \qquad \textbf{(7.2.20)}$$

is called a **second-order, linear inhomogeneous recurrence relation with constant coefficients.**

Let $g(n)$ be a solution of (7.2.20). Show that any solution U of (7.2.20) is of the form

$$U_n = V_n + g(n), \qquad \textbf{(7.2.21)}$$

where V is a solution of the homogeneous equation (7.2.13).

If $f(n) = C$ in (7.2.20), it can be shown that $g(n) = C'$ in (7.2.21) if 1 is not a root of

$$t^2 - c_1 t - c_2 = 0, \qquad \textbf{(7.2.22)}$$

$g(n) = C'n$ if 1 is a root of (7.2.22) of multiplicity one, and $g(n) = C'n^2$ if 1 is a root of (7.2.22) of multiplicity two. Similarly, if $f(n) = Cn$, it can be shown that $g(n) = C_1'n + C_0'$ if 1 is not a root of (7.2.22), $g(n) = C_1'n^2 + C_0'n$ if 1 is a root of multiplicity one, and $g(n) = C_1'n^3 + C_0'n^2$ if 1 is a root of multiplicity two. If $f(n) = Cn^2$, it can be shown that $g(n) = C_2'n^2 + C_1'n + C_0'$ if 1 is not a root of (7.2.22), $g(n) = C_2'n^3 + C_1'n^2 + C_0'$ if 1 is a root of multiplicity one, and $g(n) = C_2'n^4 + C_1'n^3 + C_0'n^2$ if 1 is a root of multiplicity two. If $f(n) = C^n$, it can be shown that $g(n) = C'C^n$ if C is not a root of (7.2.22), $g(n) = C'nC^n$ if C is a root of multiplicity one, and $g(n) = C'n^2C^n$ if C is a root of multiplicity two. The constants can be determined by substituting $g(n)$ into the recurrence relation and equating coefficients on the two sides of the resulting equation. As examples, the constant terms on the two sides of the equation must be equal, and the coefficient of n on the left side of the equation must equal the coefficient of n on the right side of the equation. Use these facts together with Exercise 40 to find the general solutions of the recurrence relations of Exercises 41–46.

41. $a_n = 6a_{n-1} - 8a_{n-2} + 3$
42. $a_n = 7a_{n-1} - 10a_{n-2} + 16n$
43. $a_n = 2a_{n-1} + 8a_{n-2} + 81n^2$
44. $2a_n = 7a_{n-1} - 3a_{n-2} + 2^n$
45. $a_n = -8a_{n-1} - 16a_{n-2} + 3n$
46. $9a_n = 6a_{n-1} - a_{n-2} + 5n^2$
47. The equation

$$a_n = f(n)a_{n-1} + g(n)a_{n-2} \qquad \textbf{(7.2.23)}$$

is called a **second-order, linear homogeneous recurrence relation.** The coefficients $f(n)$ and $g(n)$ are not necessarily constant. Show that if S and T are solutions of (7.2.23), then $bS + dT$ is also a solution of (7.2.23).

48. Suppose that both roots of

$$t^2 - c_1 t - c^2 = 0$$

are equal to r, and suppose that a_n satisfies

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}, \qquad a_0 = C_0, \qquad a_1 = C_1.$$

Show that there exist constants b and d such that

$$a_n = br^n + dnr^n, \qquad n = 0, 1, \ldots,$$

thus completing the proof of Theorem 7.2.14.

49. Let a_n be the minimum number of links required to solve the n-node communication problem (see Exercise 51, Section 7.1). Use iteration to show that $a_n \leq 2n - 4$, $n \geq 4$.

The n-disk, four-peg Tower of Hanoi puzzle has the same rules as the three-peg puzzle; the only difference is that there is an extra peg. Exercises 50–53 refer to the following algorithm to solve the n-disk, four-peg Tower of Hanoi puzzle.

Assume that the pegs are numbered 1, 2, 3, 4 and that the problem is to move the disks, which are initially stacked on peg 1, to peg 4. If $n = 1$, move the disk to peg 4 and stop. If $n > 1$, let k_n be the largest integer satisfying

$$\sum_{i=1}^{k_n} i \leq n.$$

Fix k_n disks at the bottom of peg 1. Recursively invoke this algorithm to move the $n - k_n$ disks at the top of peg 1 to peg 2. During this part of the algorithm, the k_n bottom disks on peg 1 remain fixed. Next, move the k_n disks on peg 1 to peg 4 by invoking the optimal three-peg algorithm (see Example 7.1.8) and using only pegs 1, 3, and 4. Finally, again recursively invoke this algorithm to move the $n - k_n$ disks on peg 2 to peg 4. During this part of the algorithm, the k_n disks on peg 4 remain fixed. Let $T(n)$ denote the number of moves required by this algorithm.

This algorithm, although not known to be optimal, uses as few moves as any other algorithm that has been proposed for the four-peg problem.

50. Derive the recurrence relation

$$T(n) = 2T(n - k_n) + 2^{k_n} - 1.$$

51. Compute $T(n)$ for $n = 1, \ldots, 10$. Compare these values with the optimal number of moves to solve the three-peg problem.

★52. Let

$$r_n = n - \frac{k_n(k_n+1)}{2}.$$

Using induction or otherwise, prove that

$$T(n) = (k_n + r_n - 1)2^{k_n} + 1.$$

★53. Show that $T(n) = O(4^{\sqrt{n}})$.

★54. Give an optimal algorithm to solve a variant of the Tower of Hanoi puzzle in which a stack of disks, except for the beginning and ending stacks, is allowed as long as the largest disk in the stack is on the bottom (the disks above the largest disk in the stack can be ordered in any way whatsoever). The problem is to transfer the disks to another peg by moving one disk at a time beginning with all the disks stacked on one peg in order from largest (bottom) to smallest as in the original puzzle. The end position will be the same as in the original puzzle—the disks will be stacked in order from largest (bottom) to smallest. Prove that your algorithm is optimal. This problem is due to John McCarthy.