Example 8.1.13 ▶

The graph in Figure 8.1.14 is *not* bipartite. It is often easiest to prove that a graph is not bipartite by arguing by contradiction.

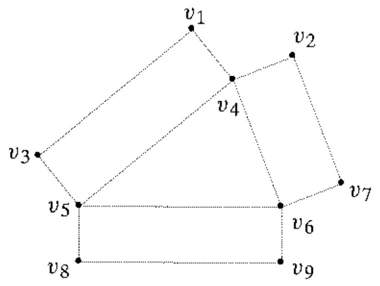


Figure 8.1.14 A graph that is not bipartite.

Suppose that the graph in Figure 8.1.14 is bipartite. Then the vertex set can be partitioned into two subsets V_1 and V_2 such that each edge is incident on one vertex in V_1 and one vertex in V_2. Now consider the vertices v_4, v_5, and v_6. Since v_4 and v_5 are adjacent, one is in V_1 and the other in V_2. We may assume that v_4 is in V_1 and that v_5 is in V_2. Since v_5 and v_6 are adjacent and v_5 is in V_2, v_6 is in V_1. Since v_4 and v_6 are adjacent and v_4 is in V_1, v_6 is in V_2. But now v_6 is in both V_1 and V_2, which is a contradiction since V_1 and V_2 are disjoint. Therefore, the graph in Figure 8.1.14 is not bipartite. ◀

Example 8.1.14 ▶

The complete graph K_1 on one vertex is bipartite. We may let V_1 be the set containing the one vertex and V_2 be the empty set. Then each edge (namely none!) is incident on one vertex in V_1 and one vertex in V_2. ◀

Definition 8.1.15 ▶

The *complete bipartite graph on m and n vertices,* denoted $K_{m,n}$, is the simple graph whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which the edge set consists of all edges of the form (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$. ◀

Example 8.1.16 ▶

The complete bipartite graph on two and four vertices, $K_{2,4}$, is shown in Figure 8.1.15.

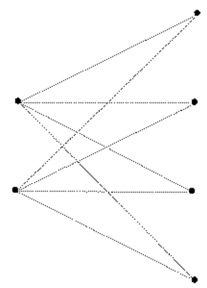


Figure 8.1.15 The complete bipartite graph $K_{2,4}$. ◀

Problem-Solving Tips

To model a given situation as a graph, you must first decide what the vertices represent. Then an edge between two vertices represents some kind of relation. For example, if several teams play soccer games, we could let the vertices represent the teams. We could then put an edge between two vertices (teams) if the two teams represented by the two vertices played at least one game. The graph would then show which teams have played each other.

To determine whether a graph is bipartite, try to separate the vertices into two disjoint sets V_1 and V_2 so that each edge is incident on one vertex in one set and one vertex in the other set. If you succeed, the graph is bipartite and you have discovered the sets V_1 and V_2. If you fail, the graph is not bipartite. To try to separate the vertices into two disjoint sets, pick a start vertex v. Put $v \in V_1$. Put all vertices adjacent to v in V_2. Pick a vertex $w \in V_2$. Put all vertices adjacent to w in V_1. Pick a vertex $v' \in V_2$, $v' \neq v$. Put all vertices adjacent to v' in V_2. Continue in this way. If you can put each vertex into either V_1 or V_2, but not both, the graph is bipartite. If at some point, you are forced to put a vertex into both V_1 *and* V_2, the graph is not bipartite.

Section Review Exercises

1. Define *undirected graph.*
2. Give an example of something in the real world that can be modeled by an undirected graph.
3. Define *directed graph.*
4. Give an example of something in the real world that can be modeled by a directed graph.
5. What does it mean for an edge to be *incident on a vertex*?
6. What does it mean for a vertex to be *incident on an edge*?
7. What does it mean for v and w to be *adjacent vertices*?
8. What are parallel edges?
9. What is a loop?
10. What is an isolated vertex?
11. What is a simple graph?
12. What is a weighted graph?
13. Give an example of something in the real world that can be modeled by a weighted graph.
14. Define *length of path* in a weighted graph.
15. What is a similarity graph?
16. Define *n-cube.*
17. What is a serial computer?
18. What is a serial algorithm?
19. What is a parallel computer?
20. What is a parallel algorithm?
21. What is the complete graph on n vertices? How is it denoted?
22. Define *bipartite graph.*
23. What is the complete bipartite graph on m and n vertices? How is it denoted?

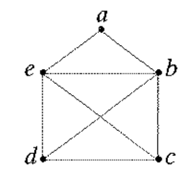
Exercises

In a tournament, the Snow beat the Pheasants once, the Skyscrapers beat the Tuna once, the Snow beat the Skyscrapers twice, the Pheasants beat the Tuna once, and the Pheasants beat the Snow once. In Exercises 1–4, use a graph to model the tournament. The teams are the vertices. Describe the kind of graph used (e.g., undirected graph, directed graph, simple graph).

1. There is an edge between teams if the teams played.
2. There is an edge between teams for each game played.
3. There is an edge from team t_i to team t_j if t_i beat t_j at least one time.
4. There is an edge from team t_i to team t_j for each victory of t_i over t_j.

Explain why none of the graphs in Exercises 5–7 has a path from a to a that passes through each edge exactly one time.

5.



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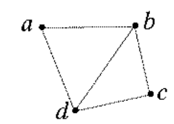

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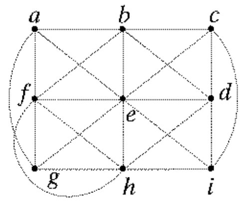
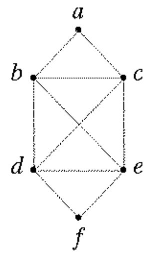


Show that each graph in Exercises 8–10 has a path from a to a that passes through each edge exactly one time by finding such a path by inspection.

8.



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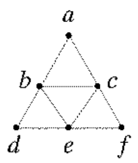

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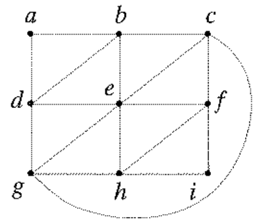
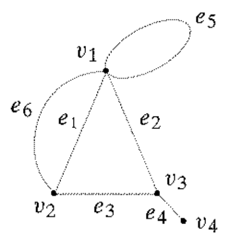


For each graph $G = (V, E)$ in Exercises 11–13, find V, E, all parallel edges, all loops, all isolated vertices, and tell whether G is a simple graph. Also, tell on which vertices edge e_1 is incident.

11.



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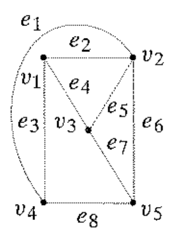
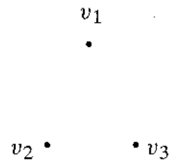


13.



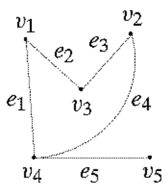
14. Draw K_3 and K_5.

15. Find a formula for the number of edges in K_n.

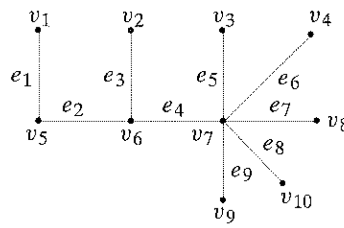
16. Give an example of a bipartite graph different from those in the examples of this section. Specify the disjoint vertex sets.

State which graphs in Exercises 17–23 are bipartite graphs. If the graph is bipartite, specify the disjoint vertex sets.

17.



18.



19. Figure 8.1.2

20. Figure 8.1.5

21. Exercise 11

22. Exercise 12

23. Exercise 13

24. Draw $K_{2,3}$ and $K_{3,3}$.

25. Find a formula for the number of edges in $K_{m,n}$.

26. Most authors require that V_1 and V_2 be nonempty in Definition 8.1.11. According to these authors, which of the graphs in Examples 8.1.12–8.1.14 are bipartite?

In Exercises 27–29, find a path of minimum length from v to w in the graph of Figure 8.1.7 that passes through each vertex exactly one time.

27. $v = b, w = e$

28. $v = c, w = d$

29. $v = a, w = b$

30. Paul Erdős (1913–1996) was one of the most prolific mathematicians of all time. He was the author or co-author of nearly 1500 papers. Mathematicians who co-authored a paper with Erdős are said to have *Erdős number one*. Mathematicians who did not co-author a paper with Erdős but who co-authored a paper with a mathematician whose Erdős number is one are said to have *Erdős number two*. Higher Erdős numbers are defined similarly. For example, the author of this book has Erdős number five. Johnsonbaugh co-authored a paper with Tadao Murata, Murata co-authored a paper with A. T. Amin, Amin co-authored a paper with Peter J. Slater, Slater co-authored a paper with Frank Harary, and Harary co-authored a paper with Erdős. Develop a graph model for Erdős numbers. In your model, what is an Erdős number?

31. Is the graph model for Bacon numbers (see Example 8.1.6) a simple graph?

32. Draw the similarity graph that results from setting $S = 40$ in Example 8.1.7. How many classes are there?

33. Draw the similarity graph that results from setting $S = 50$ in Example 8.1.7. How many classes are there?

34. In general, is "is similar to" an equivalence relation?

35. Suggest additional properties for Example 8.1.7 that might be useful in comparing programs.

36. How might one automate the selection of S to group data into classes using a similarity graph?

37. Draw a 2-cube.

38. Draw a picture like that in Figure 8.1.11 to show how a 3-cube may be constructed from two 2-cubes.

39. Prove that the recursive construction in Example 8.1.8 actually yields an n-cube.

40. How many edges are incident on a vertex in an n-cube?

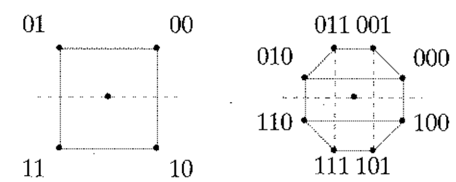
41. How many edges are in an n-cube?

★42. In how many ways can the vertices of an n-cube be labeled $0, \ldots, 2^n - 1$ so that there is an edge between two vertices if and only if the binary representation of their labels differs in exactly one bit?

[Bain] invented an algorithm to draw the n-cube in the plane. In the algorithm, all vertices are on the unit circle in the xy-plane. The angle of a point is the angle from the positive x-axis counterclockwise to the ray from the origin to the point. The input is n.

1. *If $n = 0$, put one unlabeled vertex at $(-1, 0)$ and stop.*
2. *Recursively invoke this algorithm with input $n - 1$.*
3. *Move each vertex so that its new angle is half the current angle, maintaining edge connections.*
4. *Reflect each vertex and edge in the x-axis.*
5. *Connect each vertex above the x-axis to its mirror image below the x-axis.*
6. *Prefix 0 to the label of each vertex above the x-axis, and prefix 1 to the label of each vertex below the x-axis.*

The following figures show how the algorithm draws the 2-cube and 3-cube.

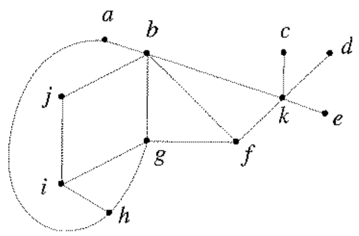


43. Show how the algorithm constructs the 2-cube from the 1-cube.

44. Show how the algorithm constructs the 3-cube from the 2-cube.

45. Show how the algorithm constructs the 4-cube from the 3-cube.

Exercises 46–48 refer to the following graph. The vertices represent offices. An edge connects two offices if there is a communication link between the two. Notice that any office can communicate with any other either directly through a communication link or by having others relay the message.

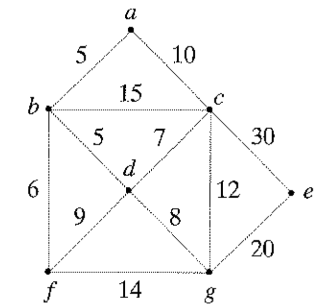


46. Show, by giving an example, that communication among all offices is still possible even if some communication links are broken.

47. What is the maximum number of communication links that can be broken with communication among all offices still possible?

48. Show a configuration in which the maximum number of communication links are broken with communication among all offices still possible.

49. In the following graph the vertices represent cities and the numbers on the edges represent the costs of building the indicated roads. Find a least-expensive road system that connects all the cities.



In a precedence graph, the vertices model certain actions. For example, a vertex might model a statement in a computer program. There is an edge from vertex v to vertex w if the action modeled by v must occur before the action modeled by w. Draw a precedence graph for each computer program in Exercises 50–52.

50. $x = 1$
$y = 2$
$z = x + y$
$z = z + 1$

51. $x = 1$
$y = 2$
$z = y + 2$
$w = x + 5$
$x = z + w$

52. $x = 1$
$y = 2$
$z = 3$
$a = x + y$
$b = y + z$
$c = x + z$
$c = c + 1$
$x = a + b + c$

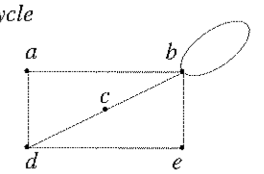
53. Let $\mathcal{G}$ denote the set of simple graphs $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbf{Z}^+$. Define a function f from $\mathcal{G}$ to $\mathbf{Z}^{nonneg}$ by the rule $f(G) = |E|$. Is f one-to-one? Is f onto? Explain.

11. Give an example of a subgraph.
12. What is a component of a graph?
13. Give an example of a component of a graph.
14. If a graph is connected, how many components does it have?
15. Define *degree of vertex v*.
16. What is an Euler cycle?
17. State a necessary and sufficient condition that a graph have an Euler cycle.
18. Give an example of a graph that has an Euler cycle. Specify the Euler cycle.
19. Give an example of a graph that does *not* have an Euler cycle. Prove that it does not have an Euler cycle.
20. What is the relationship between the sum of the degrees of the vertices in a graph and the number of edges in the graph?
21. In any graph, must the number of vertices of odd degree be even?
22. State a necessary and sufficient condition that a graph have a path with no repeated edges from v to w ($v \neq w$) containing all the edges and vertices.
23. If a graph G contains a cycle from v to v, must G contain a simple cycle from v to v?

Exercises

In Exercises 1–9, tell whether the given path in the graph is

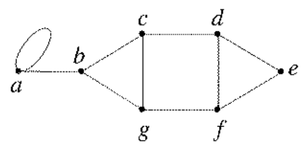
(a) A simple path
(b) A cycle
(c) A simple cycle



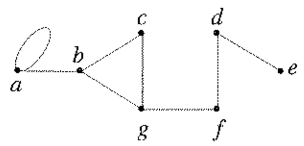
1. (b, b)	2. (e, d, c, b)
3. (a, d, c, d, e)	4. (d, c, b, e, d)
5. $(b, c, d, a, b, e, d, c, b)$	6. (b, c, d, e, b, b)
7. (a, d, c, b, e)	8. (d)
9. (d, c, b)	

In Exercises 10–18, draw a graph having the given properties or explain why no such graph exists.

10. Six vertices each of degree 3
11. Five vertices each of degree 3
12. Four vertices each of degree 1
13. Six vertices; four edges
14. Four edges; four vertices having degrees 1, 2, 3, 4
15. Four vertices having degrees 1, 2, 3, 4
16. Simple graph; six vertices having degrees 1, 2, 3, 4, 5, 5
17. Simple graph; five vertices having degrees 2, 3, 3, 4, 4
18. Simple graph; five vertices having degrees 2, 2, 4, 4, 4
19. Find all the simple cycles in the following graph.

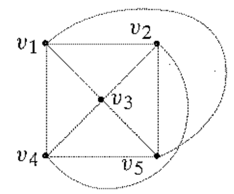


20. Find all simple paths from a to e in the graph of Exercise 19.
21. Find all connected subgraphs of the following graph containing all of the vertices of the original graph and having as few edges as possible. Which are simple paths? Which are cycles? Which are simple cycles?

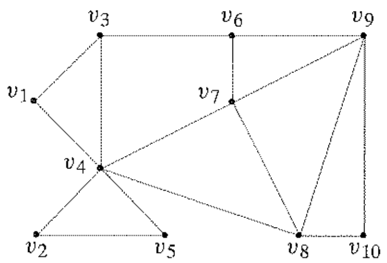


Find the degree of each vertex for the following graphs.

22.

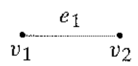


23.



In Exercises 24–27, find all subgraphs having at least one vertex of the graph given.

24.



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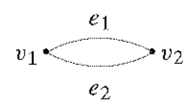
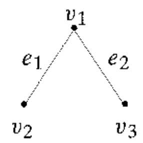
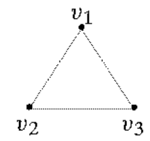


26.



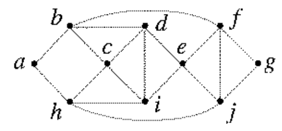
★27.



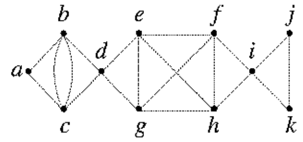
In Exercises 28–33, decide whether the graph has an Euler cycle. If the graph has an Euler cycle, exhibit one.

28. Exercise 21	29. Exercise 22
30. Exercise 23	31. Figure 8.2.4

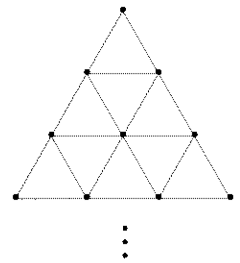
32.



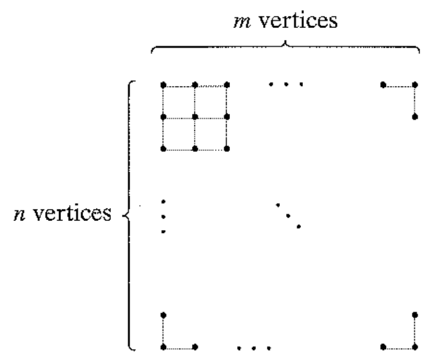
33.



34. The following graph is continued to an arbitrary, finite depth. Does the graph contain an Euler cycle? If the answer is yes, describe one.



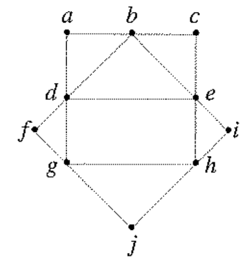
35. When does the complete graph K_n contain an Euler cycle?
36. When does the complete bipartite graph $K_{m,n}$ contain an Euler cycle?
37. For which values of m and n does the graph contain an Euler cycle?



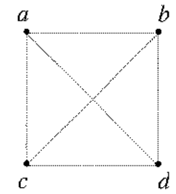
38. For which values of n does the n-cube contain an Euler cycle?

In Exercises 39 and 40, verify that the number of vertices of odd degree in the graph is even.

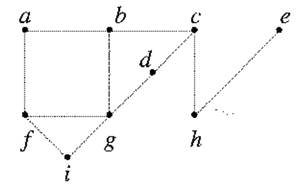
39.



40.



41. For the graph of Exercise 39, find a path with no repeated edges from d to e containing all the edges.
42. Let G be a connected graph with four vertices v_1, v_2, v_3, and v_4 of odd degree. Show that there are paths with no repeated edges from v_1 to v_2 and from v_3 to v_4 such that every edge in G is in exactly one of the paths.
43. Illustrate Exercise 42 using the following graph.



44. State and prove a generalization of Exercise 42 where the number of vertices of odd degree is arbitrary.

In Exercises 45 and 46, tell whether each assertion is true or false. If false, give a counterexample and if true, prove it.

45. Let G be a graph and let v and w be distinct vertices. If there is a path from v to w, there is a simple path from v to w.
46. If a graph contains a cycle that includes all the edges, the cycle is an Euler cycle.
47. Let G be a connected graph. Suppose that an edge e is in a cycle. Show that G with e removed is still connected.
48. Give an example of a connected graph such that the removal of any edge results in a graph that is not connected. (Assume that removing an edge does not remove any vertices.)

★49. Can a knight move around a chessboard and return to its original position making every move exactly once? (A move is considered to be made when the move is made in either direction.)

50. Show that if G' is a connected subgraph of a graph G, then G' is contained in a component.
51. Show that if a graph G is partitioned into connected subgraphs so that each edge and each vertex in G belong to one of the subgraphs, the subgraphs are components.

52. Let G be a directed graph and let G' be the undirected graph obtained from G by ignoring the direction of edges in G. Assume that G is connected. If v is a vertex in G, we say the *parity of v is even* if the number of edges of the form (v, w) is even; *odd parity* is defined similarly. Prove that if v and w are vertices in G having odd parity, it is possible to change the orientation of certain edges in G so that v and w have even parity and the parity of all other vertices in G is unchanged.

★**53.** Show that the maximum number of edges in a simple, disconnected graph with n vertices is $(n-1)(n-2)/2$.

★**54.** Show that the maximum number of edges in a simple, bipartite graph with n vertices is $\lfloor n^2/4 \rfloor$.

A vertex v in a connected graph G is an articulation point if the removal of v and all edges incident on v disconnects G.

55. Give an example of a graph with six vertices that has exactly two articulation points.

56. Give an example of a graph with six vertices that has no articulation points.

57. Show that a vertex v in a connected graph G is an articulation point if and only if there are vertices w and x in G having the property that every path from w to x passes through v.

Let G be a directed graph and let v be a vertex in G. The indegree of v, in(v), is the number of edges of the form (w, v). The outdegree of v, out(v), is the number of edges of the form (v, w). A directed Euler cycle in G is a sequence of edges of the form

$$(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n),$$

where $v_0 = v_n$, every edge in G occurs exactly one time, and all vertices appear.

58. Show that a directed graph G contains a directed Euler cycle if and only if the undirected graph obtained by ignoring the directions of the edges of G is connected and $\text{in}(v) = \text{out}(v)$ for every vertex v in G.

A de Bruijn sequence for n (in 0's and 1's) is a sequence

$$a_1, \quad \ldots, \quad a_{2^n}$$

of 2^n bits having the property that if s is a bit string of length n, for some m,

$$s = a_m a_{m+1} \cdots a_{m+n-1}. \qquad \textbf{(8.2.2)}$$

In (8.2.2), we define $a_{2^n+i} = a_i$ for $i = 1, \ldots, 2^n - 1$.

59. Verify that 00011101 is a de Bruijn sequence for $n = 3$.

60. Let G be a directed graph with vertices corresponding to all bit strings of length $n-1$. A directed edge exists from vertex $x_1 \cdots x_{n-1}$ to $x_2 \cdots x_n$. Show that a directed Euler cycle in G corresponds to a de Bruijn sequence.

★**61.** Show that there is a de Bruijn sequence for every $n = 1, 2, \ldots$.

★**62.** A *closed path* is a path from v to v. Show that a connected graph G is bipartite if and only if every closed path in G has even length.

63. How many paths of length $k \geq 1$ are there in K_n?

64. Show that there are

$$\frac{n(n-1)[(n-1)^k - 1]}{n-2}$$

paths whose lengths are between 1 and k, inclusive, in K_n, $n > 2$.

65. Let v and w be distinct vertices in K_n. Let p_m denote the number of paths of length m from v to w in K_n, $1 \leq m \leq n$.

(a) Derive a recurrence relation for p_m.

(b) Find an explicit formula for p_m.

66. Let v and w be distinct vertices in K_n, $n \geq 2$. Show that the number of simple paths from v to w is

$$(n-2)! \sum_{k=0}^{n-2} \frac{1}{k!}.$$

★**67.** [Requires calculus] Show that there are $\lfloor n!e - 1 \rfloor$ simple paths in K_n. ($e = 2.71828\ldots$ is the base of the natural logarithm.)

68. Let G be a graph. Define a relation R on the set V of vertices of G as vRw if there is a path from v to w. Prove that R is an equivalence relation on V.

69. Prove that a connected graph with one or two vertices, each of which has even degree, has an Euler cycle.

Let G be a connected graph. The distance between vertices v and w in G, dist(v, w), is the length of a shortest path from v to w. The diameter of G is

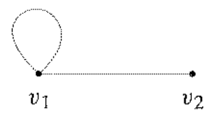
$$d(G) = \max\{dist(v, w) \mid v \text{ and } w \text{ are vertices in } G\}.$$

70. Find the diameter of the graph of Figure 8.2.10.

71. Find the diameter of the n-cube. In the context of parallel computation, what is the meaning of this value?

72. Find the diameter of K_n, the complete graph on n vertices.

73. Show that the number of paths in the following graph from v_1 to v_1 of length n is equal to the $(n+1)$st Fibonacci number f_{n+1}.



74. Let G be a simple graph with n vertices in which every vertex has degree k and

$$k \geq \frac{n-3}{2} \quad \text{if } n \bmod 4 = 1$$
$$k \geq \frac{n-1}{2} \quad \text{if } n \bmod 4 \neq 1.$$

Show that G is connected.

A cycle in a simple directed graph [i.e., a directed graph in which there is at most one edge of the form (v, w) and no edges of the form (v, v)] is a sequence of three or more vertices

$$(v_0, v_1, \ldots, v_n)$$

in which (v_{i-1}, v_i) is an edge for $i = 1, \ldots, n$ and $v_0 = v_n$. A directed acyclic graph (dag) is a simple directed graph with no cycles.

75. Show that a dag has at least one vertex with no out edges [i.e., there is at least one vertex v such that there are no edges of the form (v, w)].

76. Show that the maximum number of edges in an n-vertex dag is $n(n-1)/2$.

77. An *independent set* in a graph G is a subset S of the vertices of G having the property that no two vertices in S are adjacent. (Note that $\varnothing$ is an independent set for any graph.) Prove the following result due to [Prodinger].

Let P_n be the graph that is a simple path with n vertices. Prove that the number of independent sets in P_n is equal to f_{n+2}, $n = 1, 2, \ldots$, where $\{f_n\}$ is the Fibonacci sequence.

78. Let G be a graph. Suppose that for every pair of distinct vertices v_1 and v_2 in G, there is a unique vertex w in G such that v_1 and w are adjacent and v_2 and w are adjacent.

(a) Prove that if v and w are nonadjacent vertices in G, then $\delta(v) = \delta(w)$.

(b) Prove that if there is a vertex of degree $k > 1$ and no vertex is adjacent to all other vertices, then the degree of every vertex is k.

Problem-Solving Corner: Graphs

Problem

Is it possible in a department of 25 persons, racked by dissension, for each person to get along with exactly five others?

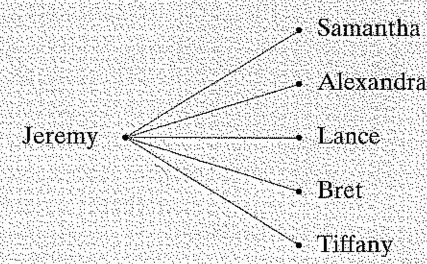
Attacking the Problem

Where do we start? Since this problem is in Chapter 8, which deals with graphs, it would probably be a good idea to try to model the problem as a graph. If this problem were not associated with a particular section or chapter in the book, we might try several approaches—one of which might be to model the problem as a graph. Many discrete problems can be solved by modeling them using graphs. This is not to say that this is the only approach possible. Most of the time by taking different approaches, we can solve a single problem in many ways. (A nice example is [Wagon].)

Finding a Solution

A fundamental issue in building a graph model is to figure out what the graph is—what are the vertices, and what are the edges? In this problem, there's not much choice; we have persons and dissension. Let's try letting the vertices be the people. It's very common in a graph model for the edges to indicate a *relationship* between the vertices. Here the relationship is "gets along with," so we'll put an edge between two vertices (people) if they get along.

Now suppose that each person gets along with exactly five others. For example, in the figure that follows, which shows part of our graph, Jeremy gets along with Samantha, Alexandra, Lance, Bret, and Tiffany, and no others.



It follows that the degree of every vertex is 5. Now let's take stock of the situation: We have 25 vertices and each vertex has degree 5. Before reading on, try to determine whether this is possible.

Corollary 8.2.22 says that the number of vertices of odd degree is even. We have a contradiction because the number of vertices of odd degree is *odd*. Therefore, it is not possible in a department of 25 persons racked by dissension for each person to get along with exactly five others.

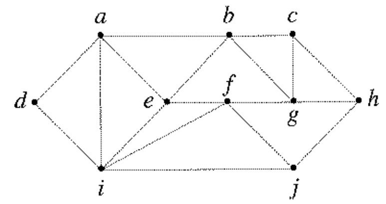
Formal Solution

No. It is not possible in a department of 25 persons racked by dissension for each person to get along with exactly five others. Suppose by way of contradiction that it is possible. Consider a graph where the vertices are the persons and an edge connects two vertices (people) if the people get along. Since every vertex has odd degree, the number of vertices of odd degree is odd, which is a contradiction.

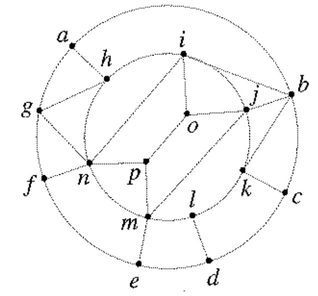
Exercises

Find a Hamiltonian cycle in each graph.

1.

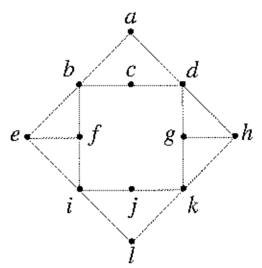


2.

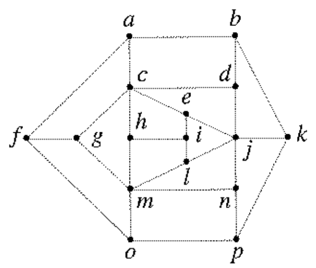


Show that none of the graphs contains a Hamiltonian cycle.

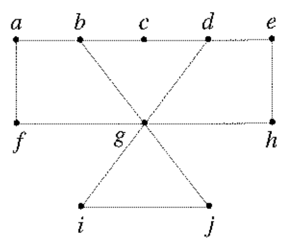
3.



4.

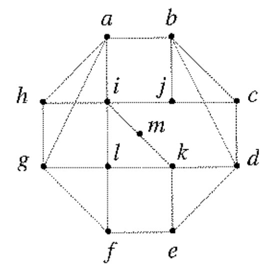


5.

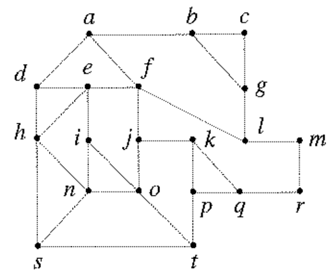


Determine whether or not each graph contains a Hamiltonian cycle. If there is a Hamiltonian cycle, exhibit it; otherwise, give an argument that shows there is no Hamiltonian cycle.

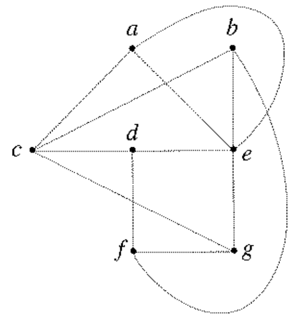
6.



7.



8.



9. Give an example of a graph that has an Euler cycle but contains no Hamiltonian cycle.

10. Give an example of a graph that has an Euler cycle that is also a Hamiltonian cycle.

11. Give an example of a graph that has an Euler cycle and a Hamiltonian cycle that are not identical.

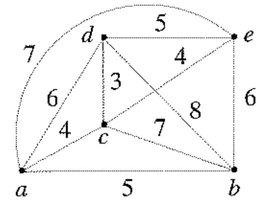
⋆12. For which values of m and n does the graph of Exercise 37, Section 8.2, contain a Hamiltonian cycle?

13. Modify the graph of Exercise 37, Section 8.2, by inserting an edge between the vertex in row i, column 1, and the vertex in row i, column m, for $i = 1, \ldots, n$. Show that the resulting graph always has a Hamiltonian cycle.

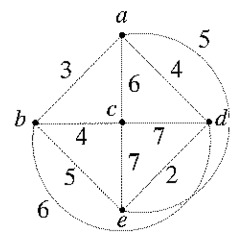
14. Show that if $n \geq 3$, the complete graph on n vertices K_n contains a Hamiltonian cycle.

15. When does the complete bipartite graph $K_{m,n}$ contain a Hamiltonian cycle?

16. Show that the cycle (e, b, a, c, d, e) provides a solution to the traveling salesperson problem for the graph shown.



17. Solve the traveling salesperson problem for the graph.



⋆18. Let m and n be integers satisfying $1 \leq m \leq 2^n$. Prove that the n-cube has a simple cycle of length m if and only if $m \geq 4$ and m is even.

19. Use Theorem 8.3.6 to compute the Gray code G_4.

20. Let G be a bipartite graph with disjoint vertex sets V_1 and V_2, as in Definition 8.1.11. Show that if G has a Hamiltonian cycle, V_1 and V_2 have the same number of elements.

21. Find a Hamiltonian cycle in GK_6 (see Example 8.3.9).

22. Describe a graph model appropriate for solving the following problem: Can the permutations of $\{1, 2, \ldots, n\}$ be arranged in a sequence so that adjacent permutations

$$p: \; p_1, \ldots, p_n \quad \text{and} \quad q: \; q_1, \ldots, q_n$$

satisfy $p_i \neq q_i$ for $i = 1, \ldots, n$?

23. Solve the problem of Exercise 22 for $n = 1, 2, 3, 4$. (The answer to the question is "yes" for $n \geq 5$; see [Problem 1186] in the References.)

24. Show that the consecutive labels of the vertices on the unit circle in Bain's depiction of the n-cube (see Exercises 43–45, Section 8.1) give a Gray code.

A Hamiltonian path in a graph G is a simple path that contains every vertex in G exactly once. (A Hamiltonian path begins and ends at different vertices.)

25. If a graph has a Hamiltonian cycle, must it have a Hamiltonian path? Explain.

26. If a graph has a Hamiltonian path, must it have a Hamiltonian cycle? Explain.

27. Does the graph of Figure 8.3.5 have a Hamiltonian path?

28. Does the graph of Figure 8.3.7 have a Hamiltonian path?

29. Does the graph of Exercise 3 have a Hamiltonian path?

30. Does the graph of Exercise 4 have a Hamiltonian path?

31. Does the graph of Exercise 5 have a Hamiltonian path?

32. Does the graph of Exercise 6 have a Hamiltonian path?

33. Does the graph of Exercise 7 have a Hamiltonian path?

34. Does the graph of Exercise 8 have a Hamiltonian path?

35. For which values of m and n does the graph of Exercise 37, Section 8.2, have a Hamiltonian path?

36. For which n does the complete graph on n vertices have a Hamiltonian path?

8.4 ➜ A Shortest-Path Algorithm

Recall (see Section 8.1) that a weighted graph is a graph in which values are assigned to the edges and that the length of a path in a weighted graph is the sum of the weights of the edges in the path. We let $w(i, j)$ denote the weight of edge (i, j). In weighted graphs, we often want to find a **shortest path** (i.e., a path having minimum length) between two given vertices. Algorithm 8.4.1, due to E. W. Dijkstra, which efficiently solves this problem, is the topic of this section.

Edsger W. Dijkstra (1930–2002) was born in The Netherlands. He was an early proponent of programming as a science. So dedicated to programming was he that when he was married in 1957, he listed his profession as a programmer. However, the Dutch authorities said that there was no such profession, and he had to change the entry to "theoretical physicist." He won the prestigious Turing Award from the Association for Computing Machinery in 1972. He was appointed to the Schlumberger Centennial Chair in Computer Science at the University of Texas at Austin in 1984 and retired as Professor Emeritus in 1999.

Throughout this section, G denotes a connected, weighted graph. We assume that the weights are positive numbers and that we want to find a shortest path from vertex a to vertex z. The assumption that G is connected can be dropped (see Exercise 9).

while loop, which takes time $O(n)$, the total time for lines 9 and 12 is $O(n^2)$. Thus Dijkstra's algorithm runs in time $O(n^2)$.

In fact, for an appropriate choice of z, the time is $\Omega(n^2)$ for K_n, the complete graph on n vertices, because every vertex is adjacent to every other. Thus the worst-case run time is $\Theta(n^2)$.

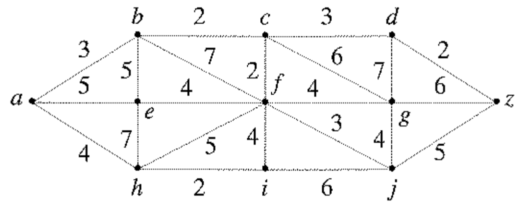
Any shortest-path algorithm that receives as input K_n, the complete graph on n vertices, must examine all of the edges of K_n at least once. Since K_n has $n(n-1)/2$ edges (see Exercise 15, Section 8.1), its worst-case run time must be at least $n(n-1)/2 = \Omega(n^2)$. It follows from Theorem 8.4.5 that Algorithm 8.4.1 is optimal.

Section Review Exercises

1. Describe Dijkstra's shortest-path algorithm.
2. Give an example to show how Dijkstra's shortest-path algorithm finds a shortest path.
3. Prove that Dijkstra's shortest-path algorithm correctly finds a shortest path.

Exercises

In Exercises 1–5, find the length of a shortest path and a shortest path between each pair of vertices in the weighted graph.



1. a, f
2. a, g
3. a, z
4. b, j
5. h, d
6. Write an algorithm that finds the length of a shortest path between two given vertices in a connected, weighted graph and also finds a shortest path.
7. Write an algorithm that finds the lengths of the shortest paths from a given vertex to every other vertex in a connected, weighted graph G.
★8. Write an algorithm that finds the lengths of the shortest paths between all vertex pairs in a simple, connected, weighted graph having n vertices in time $O(n^3)$.
9. Modify Algorithm 8.4.1 so that it accepts a weighted graph that is not necessarily connected. At termination, what is $L(z)$ if there is no path from a to z?
10. True or false? When a connected, weighted graph and vertices a and z are input to the following algorithm, it returns the length of a shortest path from a to z. If the algorithm is correct, prove it; otherwise, give an example of a connected, weighted graph and vertices a and z for which it fails.

Algorithm 8.4.6

```
algor(w, a, z) {
  length = 0
  v = a
  T = set of all vertices
  while (v ¬= z) {
    T = T − {v}
    choose x ∈ T with minimum w(v, x)
    length = length + w(v, x)
    v = x
  }
  return length
}
```

11. True or false? Algorithm 8.4.1 finds the length of a shortest path in a connected, weighted graph even if some weights are negative. If true, prove it; otherwise, provide a counterexample.

8.5 ➔ Representations of Graphs

In the preceding sections we represented a graph by drawing it. Sometimes, as for example in using a computer to analyze a graph, we need a more formal representation. Our first method of representing a graph uses the **adjacency matrix.**

Example 8.5.1 ▶ **Adjacency Matrix**

Consider the graph of Figure 8.5.1. To obtain the adjacency matrix of this graph, we first select an ordering of the vertices, say a, b, c, d, e. Next, we label the rows and columns of a matrix with the ordered vertices. The entry in this matrix in row i, column j, $i \neq j$, is the number of edges incident on i and j. If $i = j$, the entry is twice the number of loops incident on i. The adjacency matrix for this graph is

$$\begin{array}{c} \\ a\\ b\\ c\\ d\\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 1\\ 1 & 0 & 1 & 0 & 1\\ 0 & 1 & 2 & 0 & 1\\ 0 & 0 & 0 & 0 & 2\\ 1 & 1 & 1 & 2 & 0 \end{pmatrix}\end{array}.$$

◀

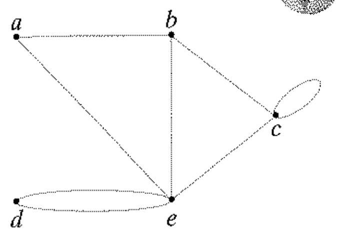


Figure 8.5.1 The graph for Example 8.5.1.

Notice that we can obtain the degree of a vertex v in a graph G by summing row v or column v in G's adjacency matrix.

The adjacency matrix is not a very efficient way to represent a graph. Since the matrix is symmetric about the main diagonal (the elements on a line from the upper-left corner to the lower-right corner), the information, except that on the main diagonal, appears twice.

Example 8.5.2 ▶ The adjacency matrix of the simple graph of Figure 8.5.2 is

$$A = \begin{array}{c} \\ a\\ b\\ c\\ d\\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}\end{array}.$$

◀

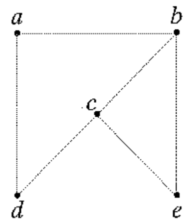


Figure 8.5.2 The graph for Example 8.5.2.

We will show that if A is the adjacency matrix of a simple graph G, the powers of A,

$$A, A^2, A^3, \ldots,$$

count the number of paths of various lengths. More precisely, if the vertices of G are labeled $1, 2, \ldots$, the ijth entry in the matrix A^n is equal to the number of paths from i to j of length n. For example, suppose that we square the matrix A of Example 8.5.2 to obtain

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 1\\ 0 & 1 & 0 & 1 & 1\\ 1 & 0 & 1 & 0 & 0\\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} = \begin{array}{c} \\ a\\ b\\ c\\ d\\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 2 & 0 & 2 & 0 & 1\\ 0 & 3 & 1 & 2 & 1\\ 2 & 1 & 3 & 0 & 1\\ 0 & 2 & 0 & 2 & 1\\ 1 & 1 & 1 & 1 & 2 \end{pmatrix}\end{array}.$$

Consider the entry for row a, column c in A^2, obtained by multiplying pairwise the entries in row a by the entries in column c of the matrix A and summing:

$$a\,\overset{\quad b \quad\; d}{(0\;\;1\;\;0\;\;1\;\;0)}\overset{c}{\begin{pmatrix} 0\\ 1\\ 0\\ 1\\ 1 \end{pmatrix}}\begin{array}{c} \\ b\\ \\ d\\ \\ \end{array} = 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 1\cdot 1 + 0\cdot 1 = 2.$$