Notice that in a graph without loops each column has two 1's and that the sum of a row gives the degree of the vertex identified with that row.

Section Review Exercises

1. What is an adjacency matrix?

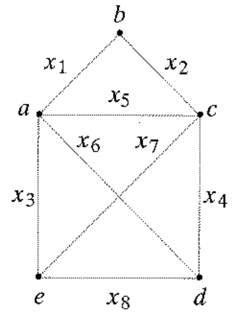
2. If A is the adjacency matrix of a simple graph, what are the values of the entries in A^n?

3. What is an incidence matrix?

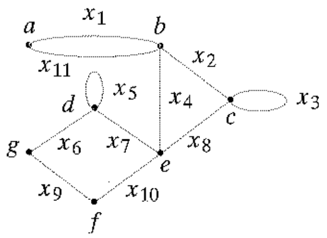
Exercises

In Exercises 1–6, write the adjacency matrix of each graph.

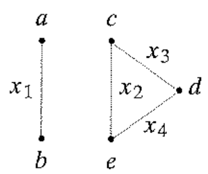
1.



2.



3.



4. The graph of Figure 8.2.2
5. The complete graph on five vertices K_5
6. The complete bipartite graph $K_{2,3}$

In Exercises 7–12, write the incidence matrix of each graph.

7. The graph of Exercise 1
8. The graph of Exercise 2
9. The graph of Exercise 3
10. The graph of Figure 8.2.1
11. The complete graph on five vertices K_5
12. The complete bipartite graph $K_{2,3}$

In Exercises 13–17, draw the graph represented by each adjacency matrix.

13.
$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

14.
$$\begin{array}{cc} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix} \end{array}$$

15.
$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

16.
$$\begin{array}{cc} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{pmatrix} 4 & 1 & 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 3 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 2 & 0 & 3 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

17. The 7×7 matrix whose ijth entry is 1 if $i+1$ divides $j+1$ or $j+1$ divides $i+1$, $i \neq j$; whose ijth entry is 2 if $i = j$; and whose ijth entry is 0 otherwise

18. Write the adjacency matrices of the components of the graphs given by the adjacency matrices of Exercises 13–17.

19. Compute the squares of the adjacency matrices of K_5 and the graphs of Exercises 1 and 3.

20. Let A be the adjacency matrix for the graph of Exercise 1. What is the entry in row a, column d of A^5?

21. Suppose that a graph has an adjacency matrix of the form

$$A = \left(\begin{array}{c|c} & A' \\ \hline A'' & \end{array} \right),$$

where all entries of the submatrices A' and A'' are 0. What must the graph look like?

22. Repeat Exercise 21 with "adjacency" replaced by "incidence."

23. Let A be an adjacency matrix of a graph. Why is A^n symmetric about the main diagonal for every positive integer n?

In Exercises 24 and 25, draw the graphs represented by the incidence matrices.

24.
$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

25.
$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

26. What must a graph look like if some row of its incidence matrix consists only of 0's?

27. Let A be the adjacency matrix of a graph G with n vertices. Let

$$Y = A + A^2 + \cdots + A^{n-1}.$$

If some off-diagonal entry in the matrix Y is zero, what can you say about the graph G?

Exercises 28–31 refer to the adjacency matrix A of K_5.

28. Let n be a positive integer. Explain why all the diagonal elements of A^n are equal and all the off-diagonal elements of A^n are equal.

Let d_n be the common value of the diagonal elements of A^n and let a_n be the common value of the off-diagonal elements of A^n.

★29. Show that

$$d_{n+1} = 4a_n; \quad a_{n+1} = d_n + 3a_n; \quad a_{n+1} = 3a_n + 4a_{n-1}.$$

★30. Show that

$$a_n = \frac{1}{5}[4^n + (-1)^{n+1}].$$

31. Show that

$$d_n = \frac{4}{5}[4^{n-1} + (-1)^n].$$

★32. Derive results similar to those of Exercises 29–31 for the adjacency matrix A of the graph K_m.

★33. Let A be the adjacency matrix of the graph $K_{m,n}$. Find a formula for the entries in A^j.

8.6 ➜ Isomorphisms of Graphs

The following instructions are given to two persons who cannot see each other's paper: "Draw and label five vertices a, b, c, d, and e. Connect a and b, b and c, c and d, d and e, and a and e." The graphs produced are shown in Figure 8.6.1. Surely these figures define the same graph even though they appear dissimilar. Such graphs are said to be **isomorphic**.

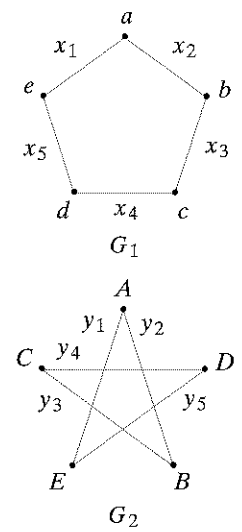


Figure 8.6.1
Isomorphic graphs.

Definition 8.6.1 ▶ Graphs G_1 and G_2 are *isomorphic* if there is a one-to-one, onto function f from the vertices of G_1 to the vertices of G_2 and a one-to-one, onto function g from the edges of G_1 to the edges of G_2, so that an edge e is incident on v and w in G_1 if and only if the edge $g(e)$ is incident on $f(v)$ and $f(w)$ in G_2. The pair of functions f and g is called an *isomorphism* of G_1 onto G_2. ◀

It would be easy to test whether a pair of graphs is isomorphic if we could find a small number of easily checked invariants that isomorphic graphs and *only* isomorphic graphs share. Unfortunately, no one has succeeded in finding such a set of invariants.

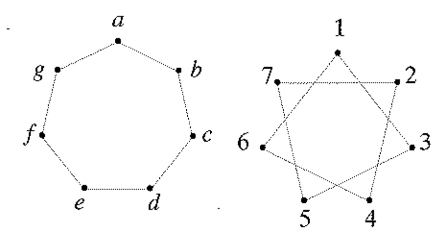
Section Review Exercises

1. Define what it means for two graphs to be isomorphic.
2. Give an example of isomorphic, nonidentical graphs. Explain why they are isomorphic.
3. Give an example of two graphs that are *not* isomorphic. Explain why they are not isomorphic.
4. What is an invariant in a graph?
5. How is "invariant" related to isomorphism?
6. How can one determine whether graphs are isomorphic from their adjacency matrices?
7. What is the mesh model for parallel computation?

Exercises

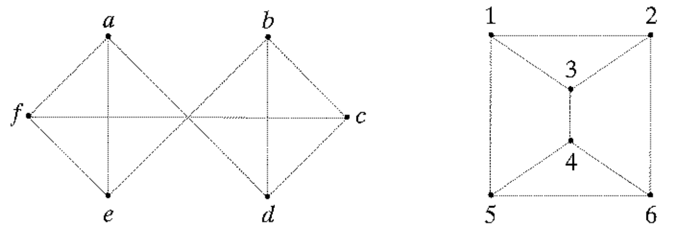
In Exercises 1–4, prove that the graphs G_1 and G_2 are isomorphic.

1.



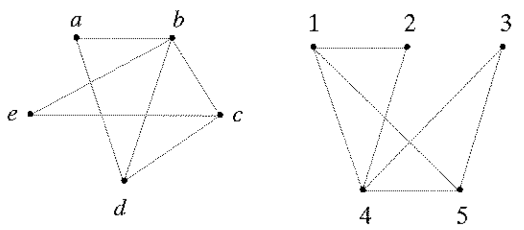
G_1 G_2

2.



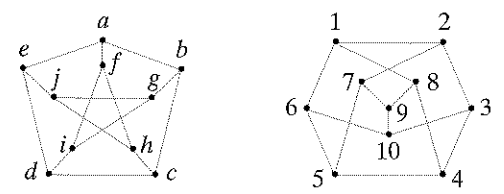
G_1 G_2

3.



G_1 G_2

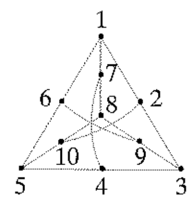
4.



G_1 G_2

Any graph isomorphic to G_1 and G_2 is called the **Petersen graph.** The Petersen graph is much used as an example; in fact, D. A. Holton and J. Sheehan wrote an entire book about it (see [Holton]).

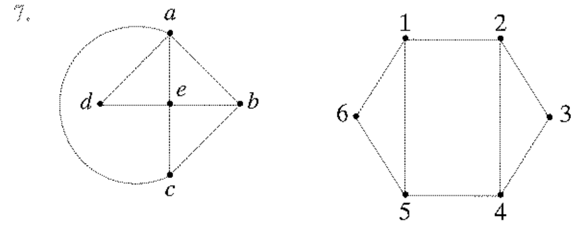
5. Prove that the following graph is the Petersen graph; that is, prove that it is isomorphic to the graphs in Exercise 4.



6. Draw a graph with 10 vertices. Label each vertex with one of the 10 distinct two-element subsets of $\{1, 2, 3, 4, 5\}$. Put an edge between two vertices if their labels (i.e., subsets) have no elements in common. Prove that your graph is the Petersen graph; that is, prove that it is isomorphic to the graphs in Exercise 4.

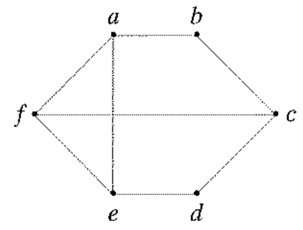
In Exercises 7–9, prove that the graphs G_1 and G_2 are not isomorphic.

7.



G_1 G_2

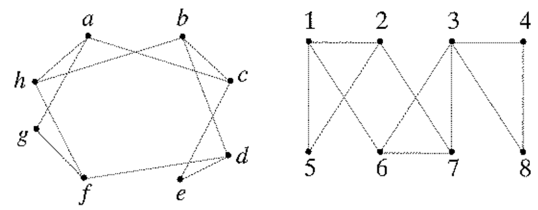
8.



G_1

The second graph is G_2 of Exercise 2.

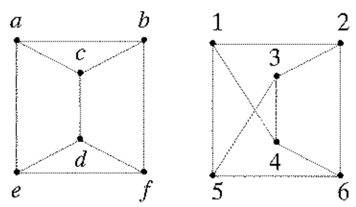
9.



G_1 G_2

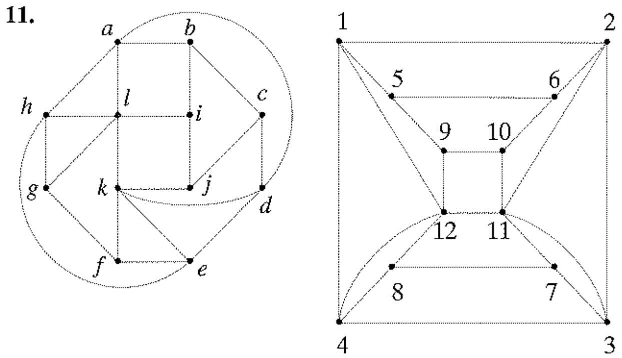
In Exercises 10–15, determine whether the graphs G_1 and G_2 are isomorphic. Prove your answer.

10.



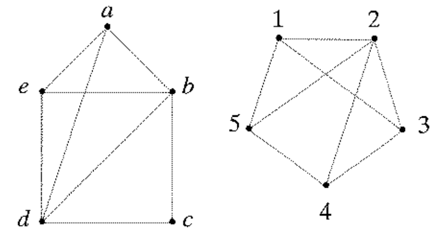
G_1 G_2

11.



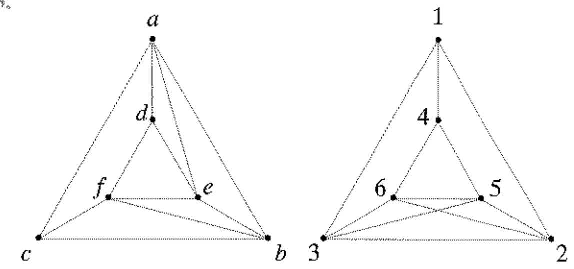
G_1 G_2

12.



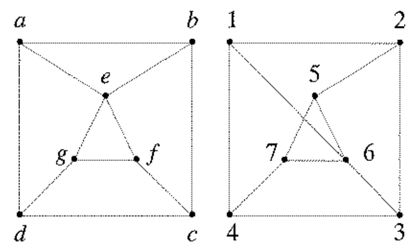
G_1 G_2

★13.



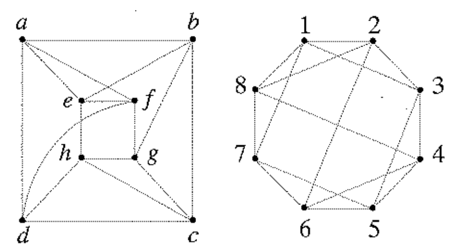
G_1 G_2

★14.



G_1 G_2

★15.



G_1 G_2

16. Show that if M is a $p_1 \times p_2 \times \cdots \times p_k$ mesh, where $p_i \leq 2^{t_i}$ for $i = 1, \ldots, k$, then the $(t_1 + t_2 + \cdots + t_k)$-cube contains a subgraph isomorphic to M.

In Exercises 17–21, show that the property given is an invariant.

17. Has a simple cycle of length k
18. Has n vertices of degree k
19. Is connected
20. Has n simple cycles of length k

21. Has an edge (v, w), where $\delta(v) = i$ and $\delta(w) = j$
22. Find an invariant not given in this section or in Exercises 17–21. Prove that your property is an invariant.

In Exercises 23–25, tell whether or not each property is an invariant. If the property is an invariant, prove that it is; otherwise, give a counterexample.

23. Has an Euler cycle
24. Has a vertex inside some simple cycle
25. Is bipartite
26. Draw all nonisomorphic simple graphs having three vertices.
27. Draw all nonisomorphic simple graphs having four vertices.
28. Draw all nonisomorphic, cycle-free, connected graphs having five vertices.
29. Draw all nonisomorphic, cycle-free, connected graphs having six vertices.
30. Show that graphs G_1 and G_2 are isomorphic if their vertices can be ordered so that their adjacency matrices are equal.

The complement of a simple graph G is the simple graph $\overline{G}$ with the same vertices as G. An edge exists in $\overline{G}$ if and only if it does not exist in G.

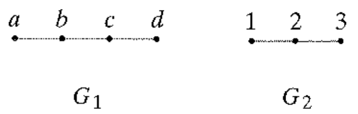
31. Draw the complement of the graph G_1 of Exercise 7.
32. Draw the complement of the graph G_2 of Exercise 7.
★33. Show that if G is a simple graph, either G or $\overline{G}$ is connected.
34. A simple graph G is **self-complementary** if G and $\overline{G}$ are isomorphic.
    (a) Find a self-complementary graph having five vertices.
    (b) Find another self-complementary graph.
35. Let G_1 and G_2 be simple graphs. Show that G_1 and G_2 are isomorphic if and only if $\overline{G_1}$ and $\overline{G_2}$ are isomorphic.
36. Given two graphs G_1 and G_2, suppose that there is a one-to-one, onto function f from the vertices of G_1 to the vertices of G_2 and a one-to-one, onto function g from the edges of G_1 to the edges of G_2, so that if an edge e is incident on v and w in G_1, the edge $g(e)$ is incident on $f(v)$ and $f(w)$ in G_2. Are G_1 and G_2 isomorphic?

A homomorphism from a graph G_1 to a graph G_2 is a function f from the vertex set of G_1 to the vertex set of G_2 with the property that if v and w are adjacent in G_1, then $f(v)$ and $f(w)$ are adjacent in G_2.

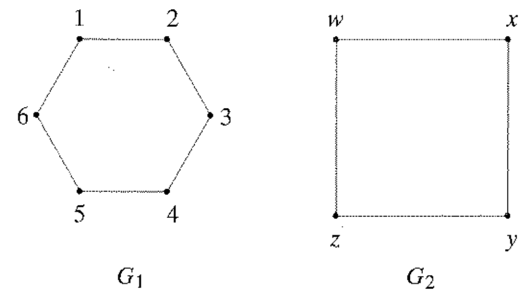
37. Suppose that G_1 and G_2 are simple graphs. Show that if f is a homomorphism of G_1 to G_2 and f is one-to-one and onto, G_1 and G_2 are isomorphic.

In Exercises 38–42, for each pair of graphs, give an example of a homomorphism from G_1 to G_2.

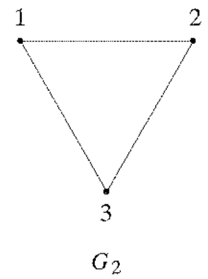
38.



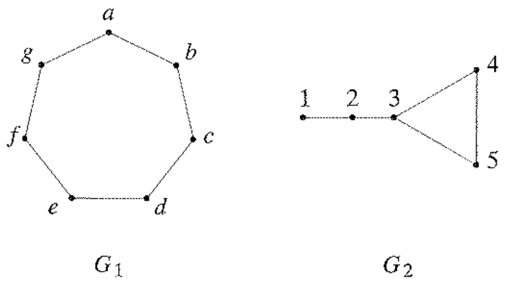
39.



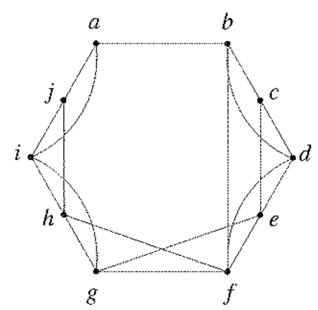
40. $G_1 = G_1$ of Exercise 39; $G_2 = G_1$ of Exercise 38
41. $G_1 = G_1$ of Exercise 38



42.



★43. [Hell] Show that the only homomorphism from the graph to itself is the identity function.



8.7 ➔ Planar Graphs

Three cities, C_1, C_2, and C_3, are to be directly connected by expressways to each of three other cities, C_4, C_5, and C_6. Can this road system be designed so that the expressways do not cross? A system in which the roads do cross is illustrated in Figure 8.7.1. If you try drawing a system in which the roads do not cross, you will soon be convinced that it cannot be done. Later in this section we explain carefully why it cannot be done.

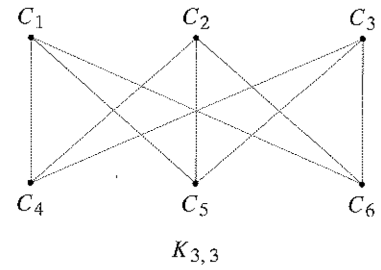


Figure 8.7.1 Cities connected by expressways.

Definition 8.7.1 ▶ A graph is *planar* if it can be drawn in the plane without its edges crossing. ◀

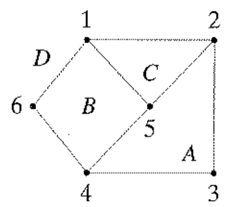


Figure 8.7.2 A connected, planar graph with $f = 4$ faces (A, B, C, D), $e = 8$ edges, and $v = 6$ vertices; $f = e - v + 2$.

In designing printed circuits it is desirable to have as few lines cross as possible; thus the designer of printed circuits faces the problem of planarity.

If a connected, planar graph is drawn in the plane, the plane is divided into contiguous regions called **faces.** A face is characterized by the cycle that forms its boundary. For example, in the graph of Figure 8.7.2, face A is bounded by the cycle (5, 2, 3, 4, 5) and face C is bounded by the cycle (1, 2, 5, 1). The outer face D is considered to be bounded by the cycle (1, 2, 3, 4, 6, 1). The graph of Figure 8.7.2 has $f = 4$ faces, $e = 8$ edges, and $v = 6$ vertices. Notice that f, e, and v satisfy the equation

$$f = e - v + 2. \tag{8.7.1}$$

In 1752, Euler proved that equation (8.7.1) holds for any connected, planar graph. At the end of this section we will show how to prove (8.7.1), but for now let us show how (8.7.1) can be used to show that certain graphs are not planar.

Example 8.7.2 ▶ Show that the graph $K_{3,3}$ of Figure 8.7.1 is not planar.

Suppose that $K_{3,3}$ is planar. Since every cycle has at least four edges, each face is bounded by at least four edges. Thus the number of edges that bound faces is at least $4f$. In a planar graph, each edge belongs to at most two bounding cycles. Therefore,

$$2e \geq 4f.$$

Using (8.7.1), we find that

$$2e \geq 4(e - v + 2). \tag{8.7.2}$$

For the graph of Figure 8.7.1, $e = 9$ and $v = 6$, so (8.7.2) becomes

$$18 = 2 \cdot 9 \geq 4(9 - 6 + 2) = 20,$$

which is a contradiction. Therefore, $K_{3,3}$ is not planar. ◀

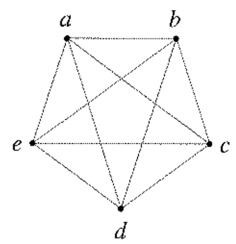


Figure 8.7.3 The nonplanar graph K_5.

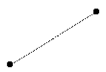
By a similar kind of argument (see Exercise 15), we can show that the graph K_5 of Figure 8.7.3 is not planar.

We will conclude this section by proving Euler's formula.

Theorem 8.7.9 **Euler's Formula for Graphs**

If G is a connected, planar graph with e edges, v vertices, and f faces, then

$$f = e - v + 2. \quad \textbf{(8.7.3)}$$



$f = 1, e = 1, v = 2$

$f = 2, e = 1, v = 1$

Figure 8.7.8 The Basis Step of Theorem 8.7.9.

Proof We will use induction on the number of edges.

Suppose that $e = 1$. Then G is one of the two graphs shown in Figure 8.7.8. In either case, the formula holds. We have verified the Basis Step.

Suppose that the formula holds for connected, planar graphs with n edges. Let G be a graph with $n + 1$ edges. First, suppose that G contains no cycles. Pick a vertex v and trace a path starting at v. Since G is cycle-free, every time we trace an edge, we arrive at a new vertex. Eventually, we will reach a vertex a, with degree 1, that we cannot leave (see Figure 8.7.9). We delete a and the edge x incident on a from the graph G. The resulting graph G' has n edges; hence, by the inductive assumption, (8.7.3) holds for G'. Since G has one more edge than G', one more vertex than G', and the same number of faces as G', it follows that (8.7.3) also holds for G.

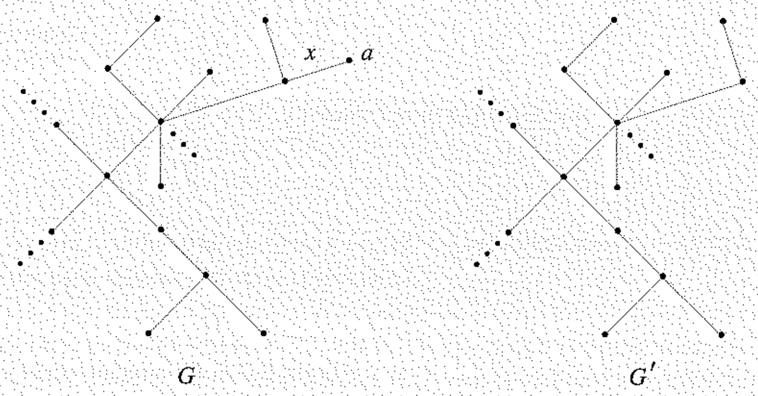


Figure 8.7.9 The proof of Theorem 8.7.9 for the case that G has no cycles. We find a vertex a of degree 1 and delete a and the edge x incident on it.

Now suppose that G contains a cycle. Let x be an edge in a cycle (see Figure 8.7.10). Now x is part of a boundary for two faces. This time we delete the edge x but no vertices to obtain the graph G' (see Figure 8.7.10). Again G' has n edges; hence,

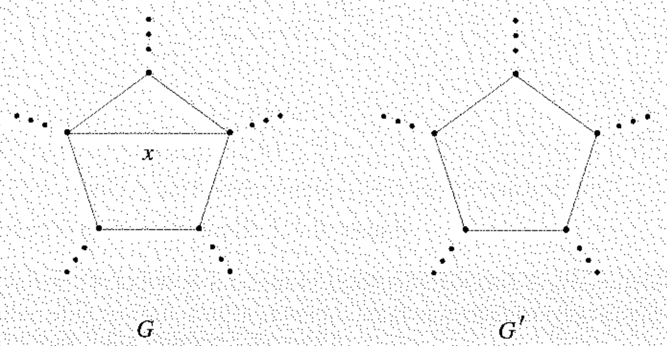


Figure 8.7.10 The proof of Theorem 8.7.9 for the case that G has a cycle. We delete edge x in a cycle.

by the inductive assumption, (8.7.3) holds for G'. Since G has one more face than G', one more edge than G', and the same number of vertices as G', it follows that (8.7.3) also holds for G.

Since we have verified the Inductive Step, by the Principle of Mathematical Induction, the theorem is proved.

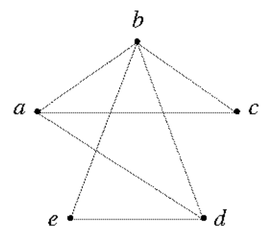
Section Review Exercises

1. What is a planar graph?
2. What is a face?
3. State Euler's equation for a connected, planar graph.
4. What are series edges?
5. What is a series reduction?
6. Define *homeomorphic graphs.*
7. State Kuratowski's theorem.

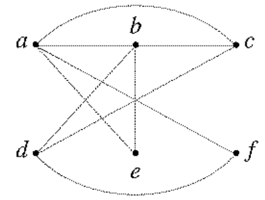
Exercises

In Exercises 1–3, show that each graph is planar by redrawing it so that no edges cross.

1.



2.



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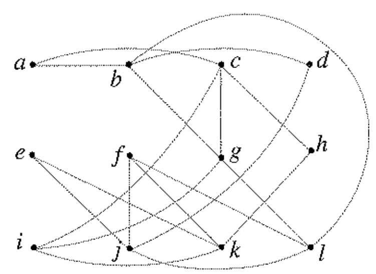
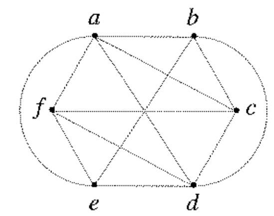


In Exercises 4 and 5, show that each graph is not planar by finding a subgraph homeomorphic to either K_5 or $K_{3,3}$.

4.



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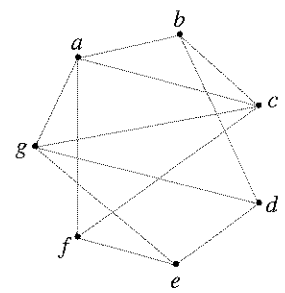
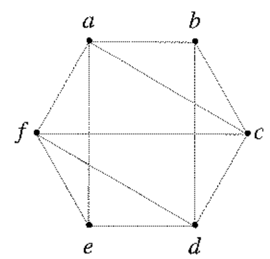


In Exercises 6–8, determine whether each graph is planar. If the graph is planar, redraw it so that no edges cross; otherwise, find a subgraph homeomorphic to either K_5 or $K_{3,3}$.

6.



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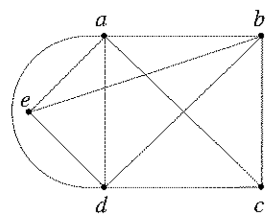
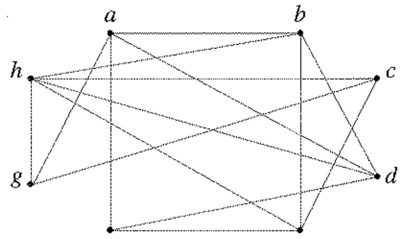


8.



9. A connected, planar graph has nine vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4, and 5. How many edges are there? How many faces are there?

10. Show that adding or deleting loops, parallel edges, or edges in series does not affect the planarity of a graph.

11. Show that any graph having four or fewer vertices is planar.

12. Show that any graph having five or fewer vertices and a vertex of degree 2 is planar.

13. Show that in any simple, connected, planar graph, $e \le 3v - 6$.

14. Give an example of a simple, connected, nonplanar graph for which $e \le 3v - 6$.

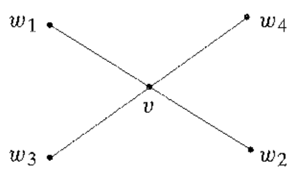
15. Use Exercise 13 to show that K_5 is not planar.

★16. Show that if a simple graph G has 11 or more vertices, then either G or its complement $\overline{G}$ is not planar.

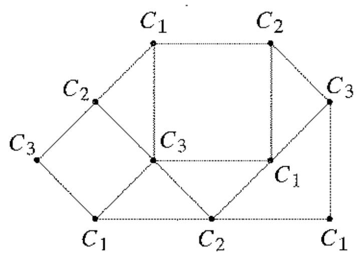
★17. Prove that if a planar graph has an Euler cycle, it has an Euler cycle with no crossings. A path P in a planar graph has a *crossing* if a vertex v appears at least twice in P and P crosses itself at v; that is,

$$P = (\ldots, w_1, v, w_2, \ldots, w_3, v, w_4, \ldots),$$

where the vertices are arranged so that w_1, v, w_2 crosses w_3, v, w_4 at v as in the following figure.

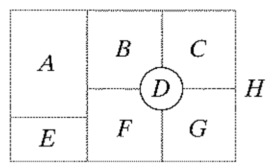


A coloring of a graph G by the colors $C_1, C_2, \ldots, C_n$ assigns to each vertex a color C_i so that any vertex has a color different from that of any adjacent vertex. For example, the following graph is colored with three colors. The rest of the exercises deal with coloring planar graphs.



A planar map is a planar graph where the faces are interpreted as countries, the edges are interpreted as borders between countries, and the vertices represent the intersections of borders. The problem of coloring a planar map G, so that no countries with adjoining boundaries have the same color, can be reduced to the problem of coloring a graph by first constructing the dual graph G′ of G in the following way. The vertices of the dual graph G′ consist of one point in each face of G, including the unbounded face. An edge in G′ connects two vertices if the corresponding faces in G are separated by a boundary. Coloring the map G is equivalent to coloring the vertices of the dual graph G′.

18. Find the dual of the following map.

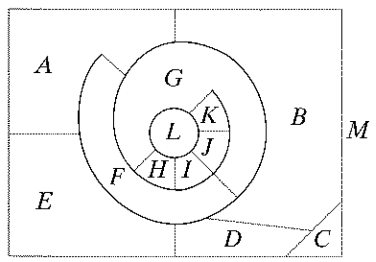


19. Show that the dual of a planar map is a planar graph.

20. Show that any coloring of the map of Exercise 18, excluding the unbounded region, requires at least three colors.

21. Color the map of Exercise 18, excluding the unbounded region, using three colors.

22. Find the dual of the following map.

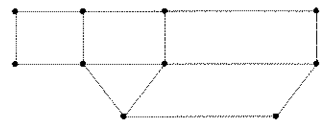


23. Show that any coloring of the map of Exercise 22, excluding the unbounded region, requires at least four colors.

24. Color the map of Exercise 22, excluding the unbounded region, using four colors.

A triangulation of a simple, planar graph G is obtained from G by connecting as many vertices as possible while maintaining planarity and not introducing loops or parallel edges.

25. Find a triangulation of the following graph.



26. Show that if a triangulation G' of a simple, planar graph G can be colored with n colors, so can G.

27. Show that in a triangulation of a simple, planar graph, $3f = 2e$.

Appel and Haken proved (see [Appel]) that every simple, planar graph can be colored with four colors. The problem had been posed in the mid-1800s and for years no one had succeeded in giving a proof. Those working on the four-color problem in recent years had one advantage their predecessors did not—the use of fast electronic computers. The following exercises show how the proof begins.

Suppose there is a simple, planar graph that requires more than four colors to color. Among all such graphs, there is one with the fewest number of vertices. Let G be a triangulation of this graph. Then G also has a minimal number of vertices and by Exercise 26, G requires more than four colors to color.

28. If the dual of a map has a vertex of degree 3, what must the original map look like?

29. Show that G cannot have a vertex of degree 3.

★30. Show that G cannot have a vertex of degree 4.

★31. Show that G has a vertex of degree 5.

The contribution of Appel and Haken was to show that only a finite number of cases involving the vertex of degree 5 needed to be considered and to analyze all of these cases and show that all could be colored using four colors. The reduction to a finite number of cases was facilitated by using the computer to help find the cases to be analyzed. The computer was then used again to analyze the resulting cases.

★32. Show that any simple, planar graph can be colored using five colors.

8.8 ➔ Instant Insanity†

Instant Insanity is a puzzle consisting of four cubes each of whose faces is painted one of four colors: red, white, blue, or green (see Figure 8.8.1). (There are different versions of the puzzle, depending on which faces are painted which colors.) The problem is to stack the cubes, one on top of the other, so that whether the cubes are viewed from front, back, left, or right, one sees all four colors (see Figure 8.8.2). Since 331,776 different stacks are possible (see Exercise 12), a solution by hand by trial and error is impractical. We present a solution, using a graph model, that makes it possible to discover a solution, if there is one, in a few minutes!

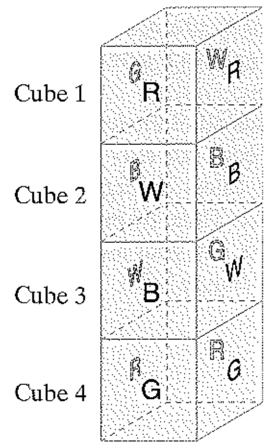


Figure 8.8.2 A solution to the Instant Insanity puzzle of Figure 8.8.1.

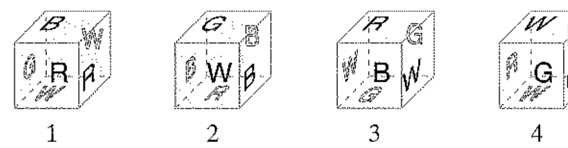


Figure 8.8.1 An Instant Insanity puzzle.

First, notice that any particular stacking can be represented by two graphs, one representing the front/back colors and the other representing the left/right colors. For example, in Figure 8.8.3 we represent the stacking of Figure 8.8.2. The vertices represent the colors, and an edge connects two vertices if the opposite faces have those colors. For example, in the front/back graph, the edge labeled 1 connects R and W, since the front and back faces of cube 1 are red and white. As another example, in the left/right graph, W has a loop, since both the left and right faces of cube 3 are white.

We can also construct a stacking from a pair of graphs such as those in Figure 8.8.3, which represent a solution of the Instant Insanity puzzle. Begin with the front/back graph. Cube 1 is to have red and white opposing faces. Arbitrarily assign one of these colors, say red, to the front. Then cube 1 has a white back face. The other edge incident on W is 2, so make cube 2's front face white. This gives cube 2 a blue back face. The other edge incident on B is 3, so make cube 3's front face blue. This gives cube 3 a green back face. The other edge incident on G is 4. Cube 4 then gets a green front face and a red back

†This section can be omitted without loss of continuity.