061 - Midterm 2

20 May 2011

1. (20 points). Solve the recurrence relation

$$a_n = 7a_{n-1} - 10a_{n-2}$$

subject to the initial conditions $a_0 = 1$ and $a_1 = 5$.

Solution First, solve the quadratic equation

$$t^2 - 7t + 10 = 0.$$

The roots of this are 2 and 5. Now, we need to solve the system of linear equations

$$\begin{aligned} a + b &= 1 \\ 2a + 5b &= 5. \end{aligned}$$

The unique solution to this equation is $b = 1$ and $a = 0$. So, the solution satisfying these initial conditions is $a_n = 5^n$.

2. (20 points). Let K_5 be the complete simple graph on 5 vertices $v_1, \ldots, v_5$. How many paths of length 3 are there from v_1 to v_2?

Solution The adjacency matrix A of K_5 is

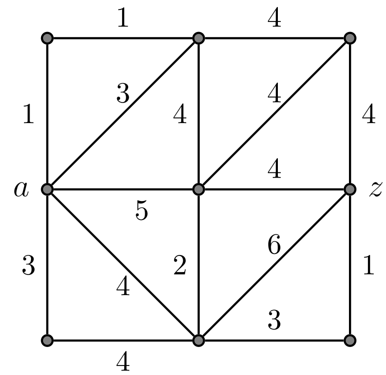
$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

We need to compute the $(1,2)$-entry of A^3. This is

$$A^3 = \begin{pmatrix} 12 & 13 & 13 & 13 & 13 \\ 13 & 12 & 13 & 13 & 13 \\ 13 & 13 & 12 & 13 & 13 \\ 13 & 13 & 13 & 12 & 13 \\ 13 & 13 & 13 & 13 & 12 \end{pmatrix}.$$

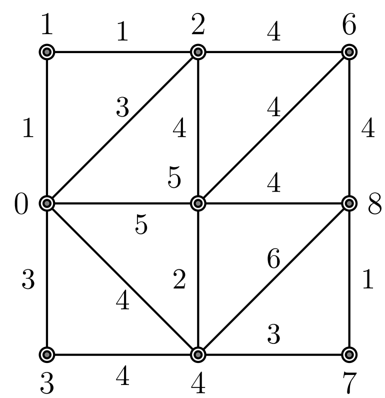
So, there are 13 paths of length 3 from v_1 to v_2.

3. (20 points). Consider the weighted graph G

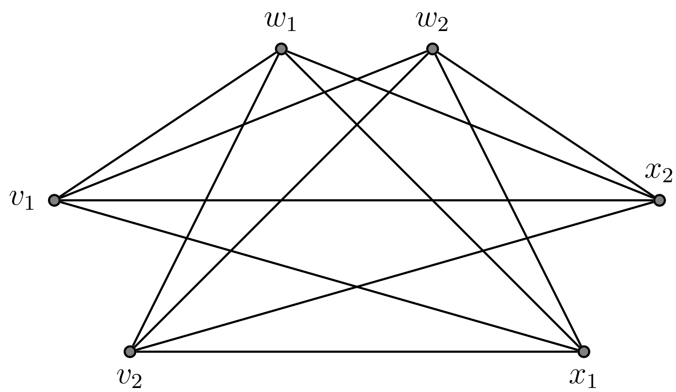


Run Dijkstra's algorithm to find the length of the shortest path from a to z. Draw the state of the graph when the algorithm finishes together with all labels permanent or not on all vertices. Circle the vertices whose labels are permanent. This picture of the final state of the graph is the only part of the solution that counts towards the grade.

Solution



4. (20 points). Let l, m, n be positive integers, let $X = \{v_1, \ldots, v_l\}$, $Y = \{w_1, \ldots, w_m\}$, and $Z = \{x_1, \ldots, x_n\}$, and let the graph $K_{l,m,n}$ be the graph on the vertex set $X \cup Y \cup Z$ where there is a unique edge between two vertices a and b if a and b are in different sets X, Y, Z. Thus, the v_i are connected to the w_i and x_i, but there are no edges from v_i to v_j. Similarly, the w_i are connected to the v_i and the x_i, but not to themselves, and the x_i are connected to the v_i and the w_i, but not to themselves. For example, $K_{2,2,2}$ is the graph

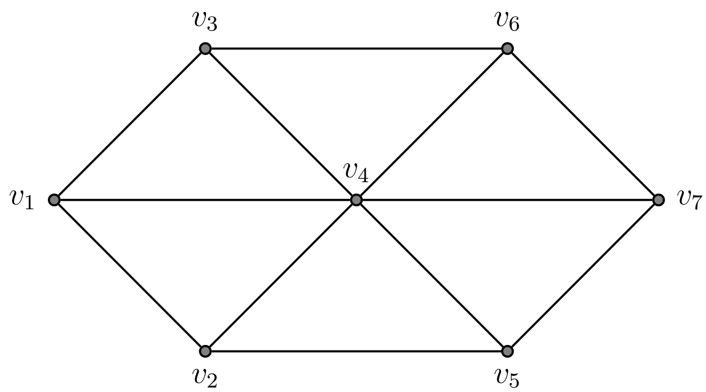


Determine all pairs of positive integers l, m, n such that $K_{l,m,n}$ has an Euler cycle?

Solution A vertex v_i has degree $m + n$, a vertex w_i has degree $l + n$, and a vertex x_i has degree $l + m$. By the theorem proved in class, it is necessary and sufficient for $K_{l,m,n}$ to have an Euler cycle that $m + n$, $l + n$, and $l + m$ all be even. Thus either l, m, n are all even or all odd.

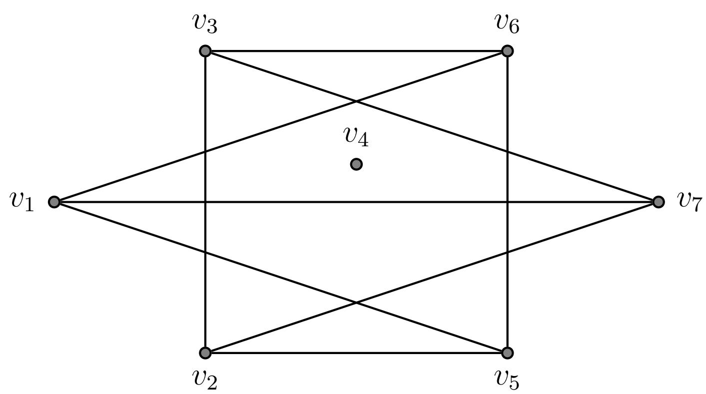
5. (20 points). Recall that the complement of a simple graph G is a graph $\overline{G}$ on the same vertex set of G but where the edge (v, w) is in $\overline{G}$ if and only if it is not in G. Draw a simple graph G on 7 vertices such that G and $\overline{G}$ are both planar. Your answer needs to include both G and $\overline{G}$ with labeled vertices drawn as planar graphs.

Solution An example graph G is



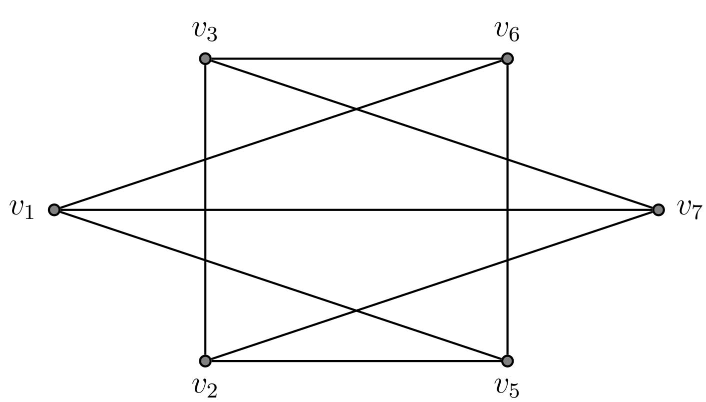
.

Its complement $\overline{G}$ is



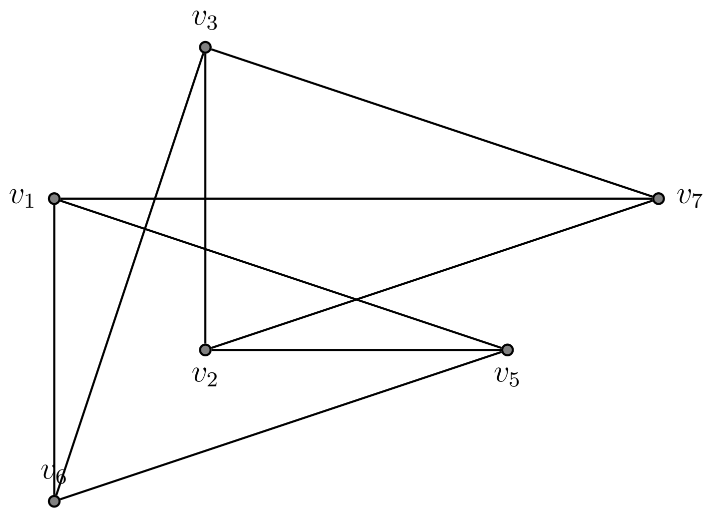
.

Since v_4 is not connected to the rest, we can ignore it for the rest of the discussion. Let G' be $\overline{G}$ minus v_4. So, G' is



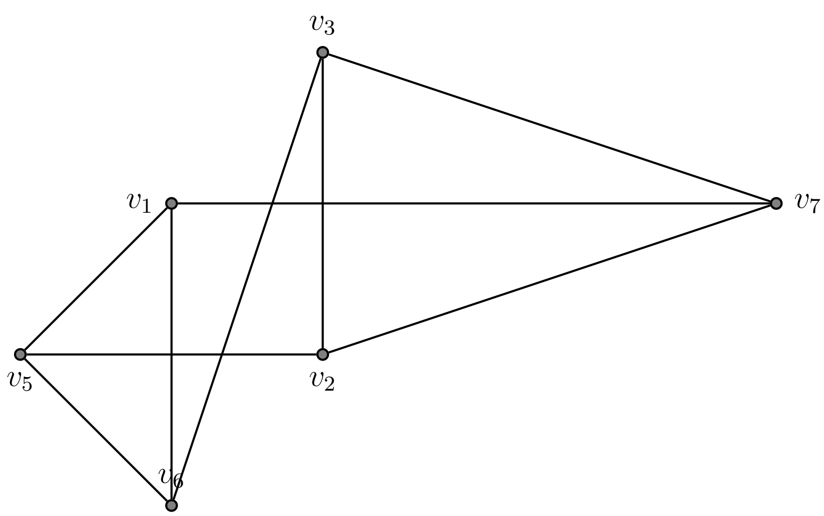
.

This can be re-drawn as



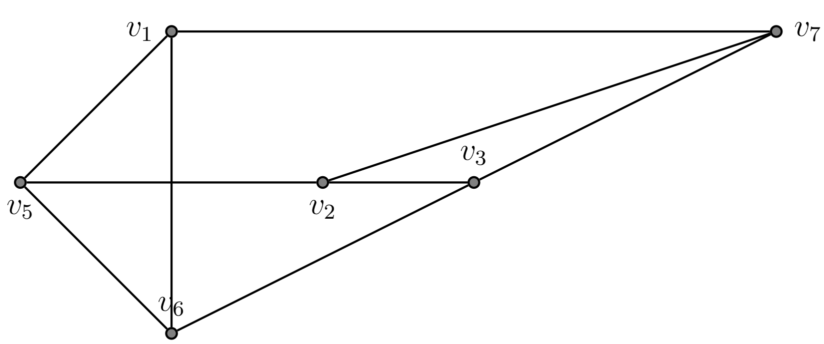
.

And then,



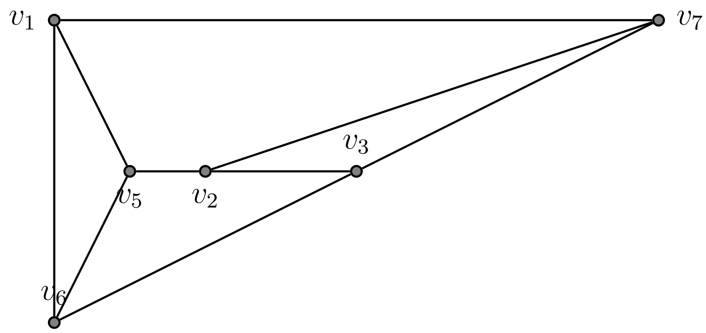
.

Then,



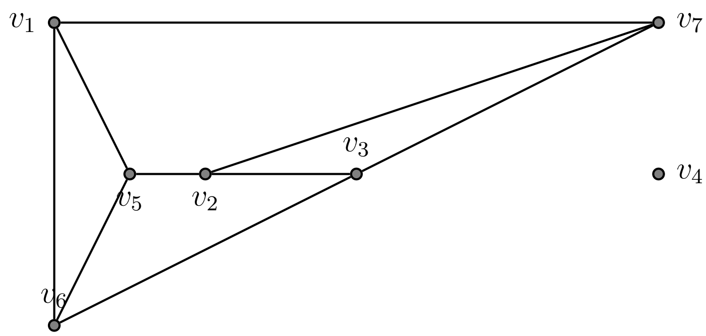
.

Then,



.

So, the final picture of $\overline{G}$ is



.

Planar!