31B/1 - Midterm 1 - Solutions

14 October 2011

Name:
Student ID #:

This is a closed-book, closed-notes exam. Calculators are not allowed.

Show all work.

If you need more room, write on the back, and make a note on the front.

There are 5 problems of 20 points each for a total of 100 points.

POINTS:

1.

2.

3.

4.

5.

TOTAL:

1.a. (20 points). Compute the derivative of $\sec^{-1}(x)$ at $x = \frac{2}{\sqrt{2}}$.

Solution If $x = \frac{\pi}{4}$, then $\sec(x) = \frac{2}{\sqrt{2}}$. So, $\sec^{-1}(\frac{2}{\sqrt{2}}) = \frac{\pi}{4}$. Let $f(x) = \sec(x)$ and $g(x) = \sec^{-1}(x)$. Then,

$$g'\left(\frac{2}{\sqrt{2}}\right) = \frac{1}{f'g(x)} = \frac{1}{\tan\pi/4\sec\pi/4} = \frac{1}{\frac{\sqrt{2}/2}{\sqrt{2}2}\cdot\frac{1}{\sqrt{2}/2}} = \frac{1}{\frac{2}{\sqrt{2}}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}.$$

1.b. (10 points). Let $f(x) = \sqrt{x^2+6x}$ for $x \geq 0$. Let $g(x)$ be the inverse of $f(x)$. Compute $g'(4)$.

Solutions First let's find $g(4)$. We know that $g(4) = x$ where $f(x) = 4$. So, we solve

$$\sqrt{x^2+6x} = 4.$$

Squaring both sides, we find

$$x^2 + 6x = 16.$$

Then, we solve

$$x^2 + 6x - 16 = (x+8)(x-2) = 0.$$

In the given interval, we thus have $g(4) = 2$. Also,

$$f'(x) = \frac{2x+6}{2\sqrt{x^2+6x}} = \frac{x+3}{\sqrt{x^2+6x}}.$$

Therefore,

$$g'(4) = \frac{1}{\frac{2+3}{\sqrt{4+12}}} = \frac{4}{5}.$$

2. (20 points). Compute the present value of an annuity that pays out 5000 per year continuously and then ends in 20 years with a lump sum additional payment of 10000, assuming you can invest at $r = .1$.

Solution The present value is simply the present value of the income stream $R(t) = 5000$ over 20 years together with the present value of receiving 10000 in 20 years. So,

$$\begin{aligned} PV &= 10000e^{-.1\cdot 20} + \int_0^{20} 5000e^{-.1t}\,dt \\ &= 10000e^{-2} + \frac{5000}{-.1}(e^{-.1\cdot 20} - 1) \\ &= (10000 - 50000)e^{-2} + 50000 \\ &= 50000 - 40000e^{-2}. \end{aligned}$$

3. (20 points). Compute

$$\lim_{x\to 0}\left(\frac{e^x}{e^x-1} - \frac{1}{x}\right).$$

Solution By putting this over a common denominator, we get

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{e^x}{e^x-1}-\frac{1}{x}\right) &= \lim_{x\to 0}\frac{xe^x-e^x+1}{xe^x-x}\\
&= \lim_{x\to 0}\frac{e^x+xe^x-e^x}{xe^x+e^x-1}\\
&= \lim_{x\to 0}\frac{xe^x}{xe^x+e^x-1}\\
&= \lim_{x\to 0}\frac{xe^x+e^x}{xe^x+e^x+e^x}\\
&= \frac{1}{2}
\end{aligned}$$

by two applications of L'Hôpital's rule.

4. (20 points) Compute

$$\int x^4\cos x\,dx.$$

Solution Using integration by parts (with $u = x^k$ at each stage), we get the following

$$\begin{aligned}
\int x^4\cos x\,dx &= x^4\sin x - 4\int x^3\sin x\,dx\\
&= x^4\sin x - 4(-x^3\cos x + 3\int x^2\cos x\,dx)\\
&= x^4\sin x - 4(-x^3\cos x + 3(x^2\sin x - 2\int x\sin x\,dx))\\
&= x^4\sin x - 4(-x^3\cos x + 3(x^2\sin x - 2(-x\cos x + \int \cos x\,dx)))\\
&= x^4\sin x - 4(-x^3\cos x + 3(x^2\sin x - 2(-x\cos x + \sin x))) + C\\
&= x^4\sin x + 4x^3\cos x - 12x^2\sin x - 24x\cos x + 24\sin x + C.
\end{aligned}$$

5. (20 points). Compute

$$\int \frac{dx}{(4-x^2)^{3/2}}.$$

Solution Do the substitution $x = 2\sin\theta$. Then, $dx = 2\cos x d\theta$. We get

$$\begin{aligned}
\int \frac{dx}{(4-x^2)^{3/2}} &= \int \frac{2\cos x d\theta}{(4-4\sin^2 x)^{3/2}} \\
&= \int \frac{2\cos x d\theta}{(4\cos^2 x)^{3/2}} \\
&= \int \frac{2\cos x d\theta}{8\cos^3 x} \\
&= \frac{1}{4}\int \sec^2\theta d\theta \\
&= \frac{1}{4}\tan\theta + C \\
&= \frac{x}{4\sqrt{4-x^2}}
\end{aligned}$$

By using a triangle to compute $\tan\theta$.