31/B - Practice Final - Solutions

December 3, 2011

1. (20 points) Calculate $g(1)$ and $g'(1)$, where $g(x)$ is the inverse of $f(x) = x + \cos x$.

Solution Note that $f(0) = 0 + \cos 0 = 1$. Thus, $g(1) = 0$. Now, $f'(x) = 1 - \sin x$, and

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{1 - \sin 0} = 1.$$

2. (20 points) Evaluate

$$\int \frac{dx}{x^2\sqrt{5 - x^2}}$$

using trigonometric substitution.

Solution We substitute $x = \sqrt{5}\cos\theta$. Then, $dx = -\sqrt{5}\sin\theta\, d\theta$, and

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{5 - x^2}} &= \int \frac{-\sqrt{5}\sin\theta\, d\theta}{5\cos^2\theta\sqrt{5 - 5\cos^2\theta}} \\
&= -\frac{1}{5}\int \frac{\sin\theta\, d\theta}{\cos^2\theta \sin\theta} \\
&= -\frac{1}{5}\int \frac{d\theta}{\cos^2\theta} \\
&= -\frac{1}{5}\int \sec^2\theta\, d\theta \\
&= -\frac{1}{5}\tan\theta + C.
\end{aligned}$$

Now, using triangles, one sees that $\sin\theta = \sqrt{5 - x^2}$, so that

$$\tan\theta = \frac{\sqrt{5 - x^2}}{x}.$$

Thus, the final answer is

$$\int \frac{dx}{x^2\sqrt{5 - x^2}} = -\frac{\sqrt{5 - x^2}}{5x} + C.$$

3. (20 points) Evaluate the integral

$$\int \frac{x^4+1}{x(x+1)^2}\,dx.$$

Solution First, doing long division, we see that

$$\frac{x^4+1}{x(x+1)^2} = x - 2 + \frac{3x^2+2x+1}{x(x+1)^2}.$$

Thus,

$$\int \frac{x^4+1}{x(x+1)^2}\,dx = \int \left(x - 2 + \frac{3x^2+2x+1}{x(x+1)^2} \right) dx = \frac{x^2}{2} - 2x + \int \frac{3x^2+2x+1}{x(x+1)^2}\,dx.$$

Now, we use partial fractions. Define A, B, and C by

$$\frac{3x^2+2x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

We solve for A, B, and C. Multiplying across by $x(x+1)^2$, we get

$$\begin{aligned} 3x^2+2x+1 &= A(x+1)^2 + Bx(x+1) + Cx \\ &= Ax^2 + 2Ax + A + Bx^2 + Bx + Cx \\ &= (A+B)x^2 + (2A+B+C)x + A. \end{aligned}$$

Equating coefficients, we see that $A = 1$, $B = 2$, and $C = -2$. Thus, we get

$$\begin{aligned} \int \frac{x^4+1}{x(x+1)^2}\,dx &= \frac{x^2}{2} - 2x + \int \frac{3x^2+2x+1}{x(x+1)^2}\,dx \\ &= \frac{x^2}{2} - 2x + \int \left(\frac{1}{x} + \frac{2}{x+1} - \frac{2}{(x+1)^2} \right) dx \\ &= \frac{x^2}{2} - 2x + \ln|x| + 2\ln|x+1| + \frac{2}{x+1} + C. \end{aligned}$$

4. (20 points) Use the error bound for Simpson's Rule to find an integer N for which $error(S_N) \le 10^{-15}$ in the integral

$$\int_1^5 \frac{dx}{x}.$$

Solution Let $f(x) = \frac{1}{x}$. Then, the nth derivative of $f(x)$ is

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}}.$$

Thus, on the interval $[1,5]$, $|f^{(n)}(x)|$ is a decreasing function. Therefore, we may take

$$K_4 = |f^{(4)}(1)| = 4! = 24.$$

The error is bounded

$$error(S_N) \leq \frac{K_4(5-1)^4}{180N^4} = \frac{24 \cdot 4^4}{180N^4} = \frac{2}{15}\left(\frac{4}{N}\right)^4.$$

Setting this less than or equal to 10^{-15}, we find the inequality

$$\frac{2 \cdot 4^4 \cdot 10^{15}}{15} = \frac{2 \cdot 10^3}{15}(4 \cdot 10^3)^4 = \frac{400}{3} \cdot (4000)^4 \leq 625 \cdot (4000)^4 = 5^4 \cdot (4000)^4 \leq N^4.$$

So, we can take $N \geq 20\,000$.

5. (20 points) Calculate the arc length of $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ over the interval $[1, 2e]$.

Solution Set $f(x) = \frac{1}{4}x^2 - \frac{1}{2}\ln x$. Then, $f'(x) = \frac{1}{2}x - \frac{1}{2x}$. So,

$$\begin{aligned}
s &= \int_1^{2e} \sqrt{1 + f'(x)^2}\, dx \\
&= \int_1^{2e} \sqrt{1 + \left(\frac{1}{2}x - \frac{1}{2x}\right)^2}\, dx \\
&= \int_1^{2e} \sqrt{\frac{1}{4}x^2 + \frac{1}{2} + \frac{1}{4x^2}}\, dx \\
&= \int_1^{2e} \sqrt{\left(\frac{1}{2}x + \frac{1}{2x}\right)^2}\, dx \\
&= \int_1^{2e} \left(\frac{1}{2}x + \frac{1}{2x}\right) dx \\
&= \left(\frac{1}{4}x^2 + \frac{1}{2}\ln x\right) \Big|_1^{2e} \\
&= e^2 + \frac{\ln 2 + 1}{2} - \frac{1}{4} \\
&= e^2 + \frac{\ln 2}{2} + \frac{1}{4}.
\end{aligned}$$

6. (20 points) Find the limit

$$\lim_{n \to \infty} \frac{(\ln n)^2}{n}.$$

Solution We use that

$$L = \lim_{n \to \infty} \frac{(\ln n)^2}{n} = \lim_{x \to \infty} \frac{(\ln x)^2}{x},$$

where the latter can be computed using L'Hôpital's Rule. So, applying the rule twice, we get

$$\begin{aligned} L &= \lim_{x\to\infty} \frac{(\ln x)^2}{x} \\ &= \lim_{x\to\infty} \frac{\frac{2\ln x}{x}}{1} \\ &= \lim_{x\to\infty} \frac{\frac{2}{x}}{1} \\ &= 0. \end{aligned}$$

7. (20 points) Use the error bound to find a value of n for which

$$|e^{-0.1} - T_n(-0.1)| \le 10^{-6},$$

where T_n is the nth Taylor polynomial for $f(x) = e^x$ with center 0.

Solution The nth derivative of $f(x)$ is just e^x. This is an increasing function, so that

$$|f^{(n)}(x)| \le e^0 = 1$$

for all x in the interval $[-0.1, 0]$. Thus, set $K_n = 1$. Then,

$$|e^{-0.1} - T_n(-0.1)| \le \frac{K_n|-0.1-0|^{n+1}}{(n+1)!} = \frac{1}{10^{n+1}(n+1)!}.$$

We must solve the inequality

$$10^{n+1}(n+1)! \ge 10^6 = 1\,000\,000.$$

Obviously, $n = 5$ works. So, in fact, does $n = 4$, since $5! = 120 \ge 10$.

8. (20 points) For which real numbers a does

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^a}$$

converge?

Solution By the integral test, the series converges if and only if the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^a}$$

does. We can evaluate this integral by substituting $u = \ln x$. Then, $du = \frac{dx}{x}$, and the integral becomes

$$\int_{\ln 2}^{\infty} \frac{du}{u^a},$$

which converges if and only if $a > 1$.

9. (20 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n3^n}.$$

Solution Let

$$\rho(x) = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)3^{n+1}}}{\frac{x^n}{n3^n}} \right|.$$

Then,

$$\begin{aligned} \rho(x) &= \lim_{n\to\infty} \left|\frac{x}{3}\right| \frac{n}{n+1} \\ &= \frac{|x|}{3} \lim_{n\to\infty} \frac{n}{n+1} \\ &= \frac{|x|}{3}. \end{aligned}$$

Therefore, the radius is $R = 3$. When $x = -3$, the series converges by the Leibniz test. When $x = 3$, we have the harmonic series, which diverges. Therefore, the interval of convergence is $[-3, 3)$.

10. (20 points) Find the terms through degree 5 of the Taylor series $T(x)$ centered at $c = 0$ of $f(x) = e^x \tan^{-1} x$.

Solution Let $T_0(x)$ be the Taylor series for e^x at 0, and let $T_1(x)$ be the Taylor series of $\tan^{-1} x$ at 0. We found via integrating $\frac{1}{1+x^2}$ that

$$T_1(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Thus,

$$\begin{aligned} T(x) = T_0(x)T_1(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\ &= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \cdots\right)\left(x - \frac{x^3}{3} + \frac{x^5}{5} \cdots\right) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + x^2 - \frac{x^4}{3} + \frac{x^3}{2} - \frac{x^5}{6} + \frac{x^4}{6} + \frac{x^5}{24} + \cdots \\ &= x + x^2 + \frac{x^3}{6} - \frac{x^4}{6} + \frac{3x^5}{40} + \cdots \end{aligned}$$