87. Find the area between $y = e^x$ and $y = e^{2x}$ over $[0, 1]$.

88. Find the area between $y = e^x$ and $y = e^{-x}$ over $[0, 2]$.

89. Find the area bounded by $y = e^2$, $y = e^x$, and $x = 0$.

90. Find the volume obtained by revolving $y = e^x$ about the x-axis for $0 \le x \le 1$.

91. Wind engineers have found that wind speed v (in m/s) at a given location follows a **Rayleigh distribution** of the type

$$W(v) = \frac{1}{32} v e^{-v^2/64}$$

This means that the probability that v lies between a and b is equal to the shaded area in Figure 8.

(a) Show that the probability that $v \in [0, b]$ is $1 - e^{-b^2/64}$.

(b) Calculate the probability that $v \in [2, 5]$.

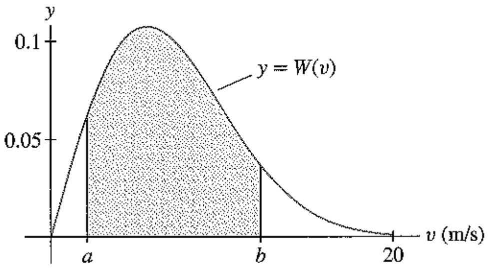


FIGURE 8 The shaded area is the probability that v lies between a and b.

92. The function $f(x) = e^x$ satisfies $f'(x) = f(x)$. Show that if $g(x)$ is another function satisfying $g'(x) = g(x)$, then $g(x) = Ce^x$ for some constant C. *Hint:* Compute the derivative of $g(x)e^{-x}$.

Further Insights and Challenges

93. Prove that $f(x) = e^x$ is not a polynomial function. *Hint:* Differentiation lowers the degree of a polynomial by 1.

94. Recall the following property of integrals: If $f(t) \ge g(t)$ for all $t \ge 0$, then for all $x \ge 0$,

$$\int_0^x f(t)\,dt \ge \int_0^x g(t)\,dt \qquad \boxed{4}$$

The inequality $e^t \ge 1$ holds for $t \ge 0$ because $e > 1$. Use (4) to prove that $e^x \ge 1 + x$ for $x \ge 0$. Then prove, by successive integration, the following inequalities (for $x \ge 0$):

$$e^x \ge 1 + x + \frac{1}{2}x^2, \qquad e^x \ge 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

95. Generalize Exercise 94; that is, use induction (if you are familiar with this method of proof) to prove that for all $n \ge 0$,

$$e^x \ge 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{n!}x^n \quad (x \ge 0)$$

96. Use Exercise 94 to show that $\dfrac{e^x}{x^2} \ge \dfrac{x}{6}$ and conclude that $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2} = \infty$. Then use Exercise 95 to prove more generally that $\lim\limits_{x\to\infty} \dfrac{e^x}{x^n} = \infty$ for all n.

97. Calculate the first three derivatives of $f(x) = xe^x$. Then guess the formula for $f^{(n)}(x)$ (use induction to prove it if you are familiar with this method of proof).

98. Consider the equation $e^x = \lambda x$, where λ is a constant.

(a) For which λ does it have a unique solution? For intuition, draw a graph of $y = e^x$ and the line $y = \lambda x$.

(b) For which λ does it have at least one solution?

99. Prove in two ways that the numbers $m(a)$ satisfy

$$m(ab) = m(a) + m(b)$$

(a) First method: Use the limit definition of m_b and

$$\frac{(ab)^h - 1}{h} = b^h\left(\frac{a^h - 1}{h}\right) + \frac{b^h - 1}{h}$$

(b) Second method: Apply the Product Rule to $a^x b^x = (ab)^x$.

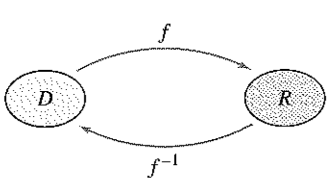


FIGURE 1 A function and its inverse.

7.2 Inverse Functions

In the next section, we will define logarithmic functions as inverses of exponential functions. But first, we review inverse functions and compute their derivatives.

The inverse of $f(x)$, denoted $f^{-1}(x)$, is the function that *reverses* the effect of $f(x)$ (Figure 1). For example, the inverse of $f(x) = x^3$ is the cube root function $f^{-1}(x) = x^{1/3}$. Given a table of function values for $f(x)$, we obtain a table for $f^{-1}(x)$ by interchanging the x and y columns:

Function			Inverse	
x	$y = x^3$		x	$y = x^{1/3}$
−2	−8	(Interchange columns) $\Longrightarrow$	−8	−2
−1	−1		−1	−1
0	0		0	0
1	1		1	1
2	8		8	2
3	27		27	3

If we apply both f and f^{-1} to a number x in either order, we get back x. For instance,

$$\text{Apply } f \text{ and then } f^{-1}: \quad 2 \xrightarrow{(\text{Apply } x^3)} 8 \xrightarrow{(\text{Apply } x^{1/3})} 2$$

$$\text{Apply } f^{-1} \text{ and then } f: \quad 8 \xrightarrow{(\text{Apply } x^{1/3})} 2 \xrightarrow{(\text{Apply } x^3)} 8$$

This property is used in the formal definition of the inverse function.

← REMINDER The "domain" is the set of numbers x such that $f(x)$ is defined (the set of allowable inputs), and the "range" is the set of all values $f(x)$ (the set of outputs).

DEFINITION Inverse Let $f(x)$ have domain D and range R. If there is a function $g(x)$ with domain R such that

$$g(f(x)) = x \quad \text{for } x \in D \qquad \text{and} \qquad f(g(x)) = x \quad \text{for } x \in R$$

then $f(x)$ is said to be **invertible.** The function $g(x)$ is called the **inverse function** and is denoted $f^{-1}(x)$.

EXAMPLE 1 Show that $f(x) = 2x - 18$ is invertible. What are the domain and range of $f^{-1}(x)$?

Solution We show that $f(x)$ is invertible by computing the inverse function in two steps.

***Step 1.* Solve the equation $y = f(x)$ for x in terms of y.**

$$y = 2x - 18$$
$$y + 18 = 2x$$
$$x = \frac{1}{2}y + 9$$

This gives us the inverse as a function of the variable y: $f^{-1}(y) = \frac{1}{2}y + 9$.

***Step 2.* Interchange variables.**

We usually prefer to write the inverse as a function of x, so we interchange the roles of x and y (Figure 2):

$$f^{-1}(x) = \frac{1}{2}x + 9$$

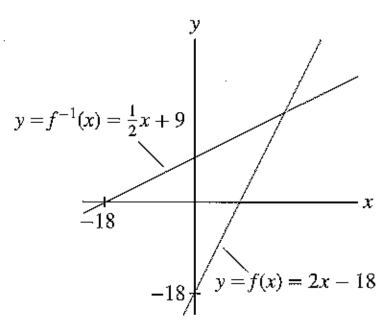


FIGURE 2

To check our calculation, let's verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$:

$$f^{-1}(f(x)) = f^{-1}(2x - 18) = \frac{1}{2}(2x - 18) + 9 = (x - 9) + 9 = x$$

$$f(f^{-1}(x)) = f\left(\frac{1}{2}x + 9\right) = 2\left(\frac{1}{2}x + 9\right) - 18 = (x + 18) - 18 = x$$

Because f^{-1} is a linear function, its domain and range are $\mathbf{R}$. ■

The inverse function, if it exists, is unique. However, some functions do not have an inverse. Consider $f(x) = x^2$. When we interchange the columns in a table of values (which should give us a table of values for f^{-1}), the resulting table does not define a function:

Function	
x	$y = x^2$
−2	4
−1	**1**
0	0
1	**1**
2	4

(Interchange columns) $\Longrightarrow$

Inverse (?)	
x	y
4	−2
1	−1
0	0
1	1
4	2

$f^{-1}(1)$ has two values: 1 and −1.

The problem is that every positive number occurs twice as an output of $f(x) = x^2$. For example, 1 occurs twice as an *output* in the first table and therefore occurs twice as an *input* in the second table. So the second table gives us two possible values for $f^{-1}(1)$, namely $f^{-1}(1) = 1$ and $f^{-1}(1) = -1$. Neither value satisfies the inverse property. For instance, if we set $f^{-1}(1) = 1$, then $f^{-1}(f(-1)) = f^{-1}(1) = 1$, but an inverse would have to satisfy $f^{-1}(f(-1)) = -1$.

*Another standard term for one-to-one is **injective**.*

So when does a function $f(x)$ have an inverse? The answer is: If $f(x)$ is **one-to-one**, which means that $f(x)$ takes on each value at most once (Figure 3). Here is the formal definition:

> **DEFINITION One-to-One Function** A function $f(x)$ is one-to-one on a domain D if, for every value c, the equation $f(x) = c$ has at most one solution for $x \in D$. Or, equivalently, if for all $a, b \in D$,
>
> $$f(a) \neq f(b) \qquad \text{unless} \qquad a = b$$

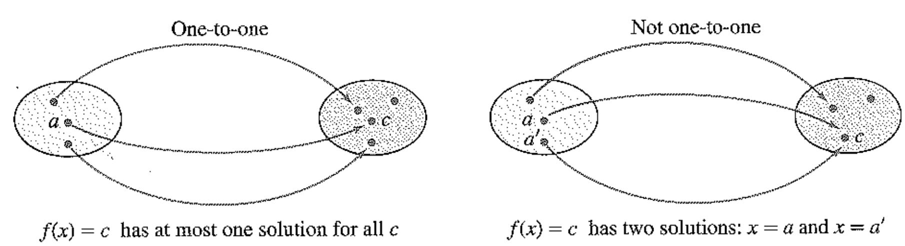


FIGURE 3 A one-to-one function takes on each value at most once.

Think of a function as a device for "labeling" members of the range by members of the domain. When f is one-to-one, this labeling is unique and f^{-1} maps each number in the range back to its label.

When $f(x)$ is one-to-one on its domain D, the inverse function $f^{-1}(x)$ exists and its domain is equal to the range R of f (Figure 4). Indeed, for every $c \in R$, there is precisely one element $a \in D$ such that $f(a) = c$ and we may define $f^{-1}(c) = a$. With this definition, $f(f^{-1}(c)) = f(a) = c$ and $f^{-1}(f(a)) = f^{-1}(c) = a$. This proves the following theorem.

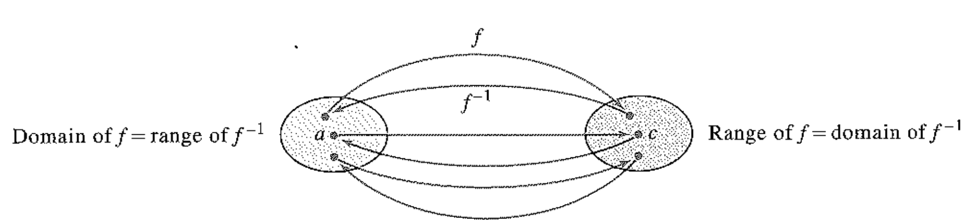


FIGURE 4 In passing from f to f^{-1}, the domain and range are interchanged.

THEOREM 1 Existence of Inverses The inverse function $f^{-1}(x)$ exists if and only if $f(x)$ is one-to-one on its domain D. Furthermore,

- Domain of f = range of f^{-1}.
- Range of f = domain of f^{-1}.

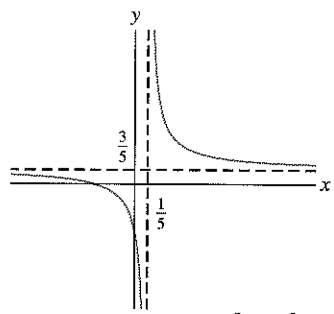


FIGURE 5 Graph of $f(x) = \dfrac{3x+2}{5x-1}$.

EXAMPLE 2 Show that $f(x) = \dfrac{3x+2}{5x-1}$ is invertible. Determine the domain and range of f and f^{-1}.

Solution The domain of $f(x)$ is $D = \left\{x : x \neq \frac{1}{5}\right\}$ (Figure 5). Assume that $x \in D$, and let's solve $y = f(x)$ for x in terms of y:

$$y = \frac{3x+2}{5x-1}$$

$$y(5x-1) = 3x+2$$

$$5xy - y = 3x + 2$$

$$5xy - 3x = y + 2 \qquad \text{(gather terms involving } x\text{)}$$

$$x(5y-3) = y+2 \qquad \text{(factor out } x \text{ in order to solve for } x\text{)} \qquad \boxed{1}$$

$$x = \frac{y+2}{5y-3} \qquad \text{(divide by } 5y-3\text{)} \qquad \boxed{2}$$

Often, it is impossible to find a formula for the inverse because we cannot solve for x explicitly in the equation $y = f(x)$. For example, the function $f(x) = x + e^x$ has an inverse, but we must make do without an explicit formula for it.

The last step is valid if $5y - 3 \neq 0$—that is, if $y \neq \frac{3}{5}$. But note that $y = \frac{3}{5}$ is not in the range of $f(x)$. For if it were, Eq. (1) would yield the false equation $0 = \frac{3}{5} + 2$. Now Eq. (2) shows that for all $y \neq \frac{3}{5}$ there is a unique value x such that $f(x) = y$. Therefore, $f(x)$ is one-to-one on its domain. By Theorem 1, $f(x)$ is invertible. The range of $f(x)$ is $R = \left\{x : x \neq \frac{3}{5}\right\}$ and

$$f^{-1}(x) = \frac{x+2}{5x-3}.$$

The inverse function has domain R and range D. ■

We can tell whether $f(x)$ is one-to-one from its graph. The horizontal line $y = c$ intersects the graph of $f(x)$ at points $(a, f(a))$, where $f(a) = c$ (Figure 6). There is at most one such point if $f(x) = c$ has at most one solution. This gives us the

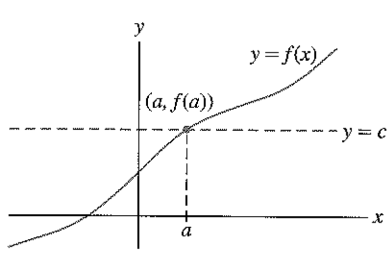


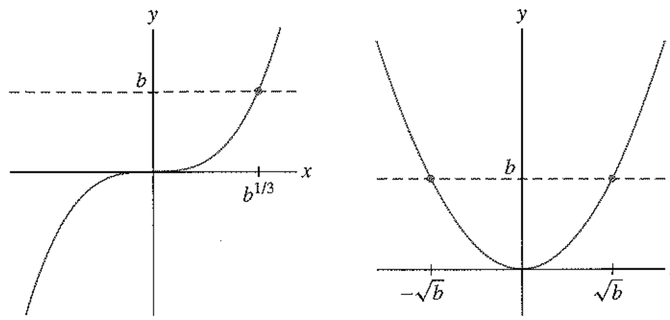
FIGURE 6 The line $y = c$ intersects the graph at points where $f(a) = c$.

Horizontal Line Test A function $f(x)$ is one-to-one if and only if every horizontal line intersects the graph of $f(x)$ in at most one point.

In Figure 7, we see that $f(x) = x^3$ passes the Horizontal Line Test and therefore is one-to-one, whereas $f(x) = x^2$ fails the test and is not one-to-one.

EXAMPLE 3 Increasing Functions Are One-to-One Show that increasing functions are one-to-one. Then show that $f(x) = x^5 + 4x + 3$ is one-to-one.

Solution An increasing function satisfies $f(a) < f(b)$ if $a < b$. Therefore f cannot take on any value more than once, and thus f is one-to-one. Note similarly that decreasing functions are one-to-one.



(A) $f(x) = x^3$ is one-to-one. (B) $f(x) = x^2$ is not one-to-one.

FIGURE 7

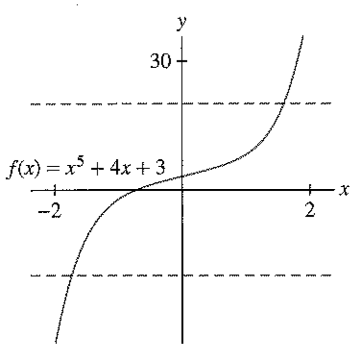


FIGURE 8 The increasing function $f(x) = x^5 + 4x + 3$ satisfies the Horizontal Line Test.

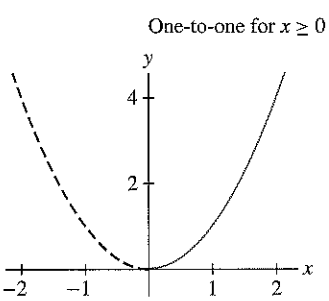


FIGURE 9 $f(x) = x^2$ satisfies the Horizontal Line Test on the domain $\{x : x \geq 0\}$.

Now observe that

- If n odd and $c > 0$, then cx^n is increasing.
- A sum of increasing functions is increasing.

Thus x^5, $4x$, and hence the sum $x^5 + 4x$ are increasing. It follows that $f(x) = x^5 + 4x + 3$ is increasing and therefore one-to-one (Figure 8). ■

We can make a function one-to-one by restricting its domain suitably.

■ **EXAMPLE 4 Restricting the Domain** Find a domain on which $f(x) = x^2$ is one-to-one and determine its inverse on this domain.

Solution The function $f(x) = x^2$ is one-to-one on the domain $D = \{x : x \geq 0\}$, for if $a^2 = b^2$ where a and b are both nonnegative, then $a = b$ (Figure 9). The inverse of $f(x)$ on D is the positive square root $f^{-1}(x) = \sqrt{x}$. Alternatively, we may restrict $f(x)$ to the domain $\{x : x \leq 0\}$, on which the inverse function is $f^{-1}(x) = -\sqrt{x}$. ■

Next we describe the graph of the inverse function. The **reflection** of a point (a, b) through the line $y = x$ is, by definition, the point (b, a) (Figure 10). Note that if the x- and y-axes are drawn to the same scale, then (a, b) and (b, a) are equidistant from the line $y = x$ and the segment joining them is perpendicular to $y = x$.

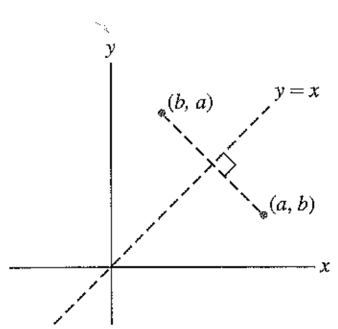


FIGURE 10 The reflection (a, b) through the line $y = x$ is the point (b, a).

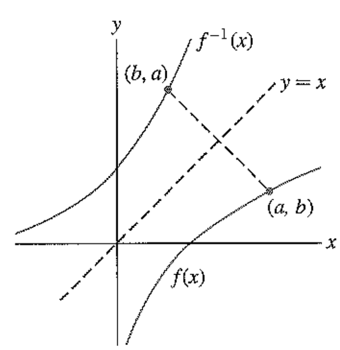


FIGURE 11 The graph of $f^{-1}(x)$ is the reflection of the graph of $f(x)$ through the line $y = x$.

The graph of f^{-1} is the reflection of the graph of f through $y = x$ (Figure 11). To check this, note that (a, b) lies on the graph of f if $f(a) = b$. But $f(a) = b$ if and only if $f^{-1}(b) = a$, and in this case, (b, a) lies on the graph of f^{-1}.

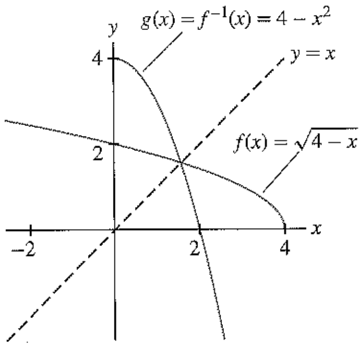


FIGURE 12 Graph of the inverse $g(x)$ of $f(x) = \sqrt{4-x}$.

EXAMPLE 5 Sketching the Graph of the Inverse Sketch the graph of the inverse of $f(x) = \sqrt{4-x}$.

Solution Let $g(x) = f^{-1}(x)$. Observe that $f(x)$ has domain $\{x : x \le 4\}$ and range $\{x : x \ge 0\}$. We do not need a formula for $g(x)$ to draw its graph. We simply reflect the graph of f through the line $y = x$ as in Figure 12. If desired, however, we can easily solve $y = \sqrt{4-x}$ to obtain $x = 4 - y^2$ and thus $g(x) = 4 - x^2$ with domain $\{x : x \ge 0\}$. ■

Derivatives of Inverse Functions

Next, we derive a formula for the derivative of the inverse $f^{-1}(x)$. We will use this formula to differentiate logarithmic functions in Section 7.3.

THEOREM 2 Derivative of the Inverse Assume that $f(x)$ is differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$. If b belongs to the domain of $g(x)$ and $f'(g(b)) \ne 0$, then $g'(b)$ exists and

$$g'(b) = \frac{1}{f'(g(b))} \qquad (3)$$

Proof The first claim, that $g(x)$ is differentiable if $f'(g(x)) \ne 0$, is verified in Appendix D (see Theorem 6). To prove Eq. (3), note that $f(g(x)) = x$ by definition of the inverse. Differentiate both sides of this equation, and apply the Chain Rule:

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx} x \quad \Rightarrow \quad f'(g(x))g'(x) = 1 \quad \Rightarrow \quad g'(x) = \frac{1}{f'(g(x))}$$

Set $x = b$ to obtain Eq. (3). ■

GRAPHICAL INSIGHT The formula for the derivative of the inverse function has a clear graphical interpretation. Consider a line L of slope m and let L' be its reflection through $y = x$ as in Figure 13(A). Then the slope of L' is $1/m$. Indeed, if (a, b) and (c, d) are any two points on L, then (b, a) and (d, c) lie on L' and

$$\underbrace{\text{Slope of } L = \frac{d-b}{c-a}, \qquad \text{Slope of } L' = \frac{c-a}{d-b}}_{\text{Reciprocal slopes}}$$

Now recall that the graph of the inverse $g(x)$ is obtained by reflecting the graph of $f(x)$ through the line $y = x$. As we see in Figure 13(B), the tangent line to $y = g(x)$ at $x = b$ is the reflection of the tangent line to $y = f(x)$ at $x = a$ [where $b = f(a)$ and $a = g(b)$]. These tangent lines have reciprocal slopes, and thus $g'(b) = 1/f'(a) = 1/f'(g(b))$, as claimed in Theorem 2.

EXAMPLE 6 Using Equation (3) Calculate $g'(x)$, where $g(x)$ is the inverse of the function $f(x) = x^4 + 10$ on the domain $\{x : x \ge 0\}$.

Solution Solve $y = x^4 + 10$ for x to obtain $x = (y-10)^{1/4}$. Thus $g(x) = (x-10)^{1/4}$. Since $f'(x) = 4x^3$, we have $f'(g(x)) = 4g(x)^3$, and by Eq. (3),

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{4g(x)^3} = \frac{1}{4(x-10)^{3/4}} = \frac{1}{4}(x-10)^{-3/4}$$

We obtain this same result by differentiating $g(x) = (x-10)^{1/4}$ directly. ■

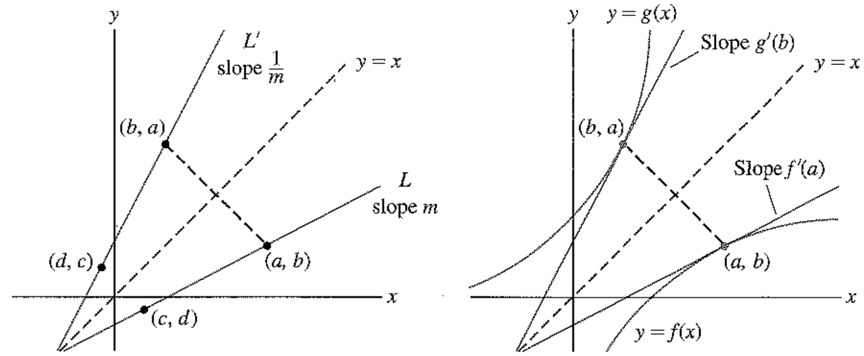


FIGURE 13 Graphical illustration of the formula $g'(b) = 1/f'(g(b))$.

(A) If L has slope m, then its reflection L' has slope $1/m$.

(B) The tangent line to the inverse $y = g(x)$ is the reflection of the tangent line to $y = f(x)$.

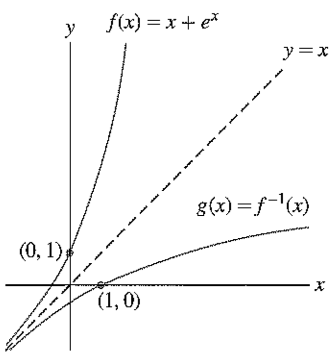


FIGURE 14 Graph of $f(x) = x + e^x$ and its inverse $g(x)$.

EXAMPLE 7 Calculating $g'(x)$ Without Solving for $g(x)$ Calculate $g'(1)$, where $g(x)$ is the inverse of $f(x) = x + e^x$.

Solution In this case, we cannot solve for $g(x)$ explicitly, but a formula for $g(x)$ is not needed (Figure 14). All we need is the particular value $g(1)$, which we can find by solving $f(x) = 1$. By inspection, $x + e^x = 1$ has solution $x = 0$. Therefore, $f(0) = 1$ and, by definition of the inverse, $g(1) = 0$. Since $f'(x) = 1 + e^x$,

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}$$

7.2 SUMMARY

- A function $f(x)$ is *one-to-one* on a domain D if for every value c, the equation $f(x) = c$ has at most one solution for $x \in D$, or, equivalently, if for all $a, b \in D$, $f(a) \neq f(b)$ unless $a = b$.
- Let $f(x)$ have domain D and range R. The *inverse* $f^{-1}(x)$ (if it exists) is the unique function with domain R and range D satisfying $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.
- The inverse of $f(x)$ exists if and only if $f(x)$ is one-to-one on its domain.
- To find the inverse function, solve $y = f(x)$ for x in terms of y to obtain $x = g(y)$. The inverse is the function $g(x)$.
- *Horizontal Line Test:* $f(x)$ is one-to-one if and only if every horizontal line intersects the graph of $f(x)$ in at most one point.
- The graph of $f^{-1}(x)$ is obtained by reflecting the graph of $f(x)$ through the line $y = x$.
- Derivative of the inverse: If $f(x)$ is differentiable and one-to-one with inverse $g(x)$, then for x such that $f'(g(x)) \neq 0$,

$$g(x) = \frac{1}{f'(g(x))}$$

7.2 EXERCISES

Preliminary Questions

1. Which of the following satisfy $f^{-1}(x) = f(x)$?

(a) $f(x) = x$ **(b)** $f(x) = 1 - x$

(c) $f(x) = 1$ **(d)** $f(x) = \sqrt{x}$

(e) $f(x) = |x|$ **(f)** $f(x) = x^{-1}$

2. The graph of a function looks like the track of a roller coaster. Is the function one-to-one?

3. The function f maps teenagers in the United States to their last names. Explain why the inverse function f^{-1} does not exist.

4. The following fragment of a train schedule for the New Jersey Transit System defines a function f from towns to times. Is f one-to-one? What is $f^{-1}(6{:}27)$?

Trenton	6:21
Hamilton Township	6:27
Princeton Junction	6:34
New Brunswick	6:38

5. A homework problem asks for a sketch of the graph of the *inverse* of $f(x) = x + \cos x$. Frank, after trying but failing to find a formula for $f^{-1}(x)$, says it's impossible to graph the inverse. Bianca hands in an accurate sketch without solving for f^{-1}. How did Bianca complete the problem?

6. What is the slope of the line obtained by reflecting the line $y = \frac{x}{2}$ through the line $y = x$?

7. Suppose that $P = (2, 4)$ lies on the graph of $f(x)$ and that the slope of the tangent line through P is $m = 3$. Assuming that $f^{-1}(x)$ exists, what is the slope of the tangent line to the graph of $f^{-1}(x)$ at the point $Q = (4, 2)$?

Exercises

1. Show that $f(x) = 7x - 4$ is invertible and find its inverse.

2. Is $f(x) = x^2 + 2$ one-to-one? If not, describe a domain on which it is one-to-one.

3. What is the largest interval containing zero on which $f(x) = \sin x$ is one-to-one?

4. Show that $f(x) = \dfrac{x-2}{x+3}$ is invertible and find its inverse.

(a) What is the domain of $f(x)$? The range of $f^{-1}(x)$?

(b) What is the domain of $f^{-1}(x)$? The range of $f(x)$?

5. Verify that $f(x) = x^3 + 3$ and $g(x) = (x-3)^{1/3}$ are inverses by showing that $f(g(x)) = x$ and $g(f(x)) = x$.

6. Repeat Exercise 5 for $f(t) = \dfrac{t+1}{t-1}$ and $g(t) = \dfrac{t+1}{t-1}$.

7. The escape velocity from a planet of radius R is $v(R) = \sqrt{\dfrac{2GM}{R}}$, where G is the universal gravitational constant and M is the mass. Find the inverse of $v(R)$ expressing R in terms of v.

In Exercises 8–15, find a domain on which f is one-to-one and a formula for the inverse of f restricted to this domain. Sketch the graphs of f and f^{-1}.

8. $f(x) = 3x - 2$

9. $f(x) = 4 - x$

10. $f(x) = \dfrac{1}{x+1}$

11. $f(x) = \dfrac{1}{7x-3}$

12. $f(s) = \dfrac{1}{s^2}$

13. $f(x) = \dfrac{1}{\sqrt{x^2+1}}$

14. $f(z) = z^3$

15. $f(x) = \sqrt{x^3 + 9}$

16. For each function shown in Figure 15, sketch the graph of the inverse (restrict the function's domain if necessary).

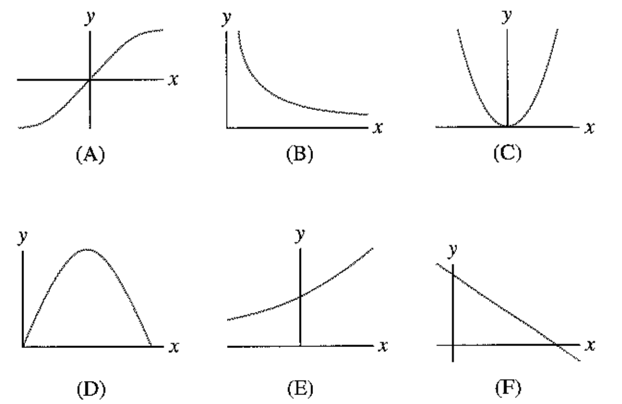


FIGURE 15

17. Which of the graphs in Figure 16 is the graph of a function satisfying $f^{-1} = f$?

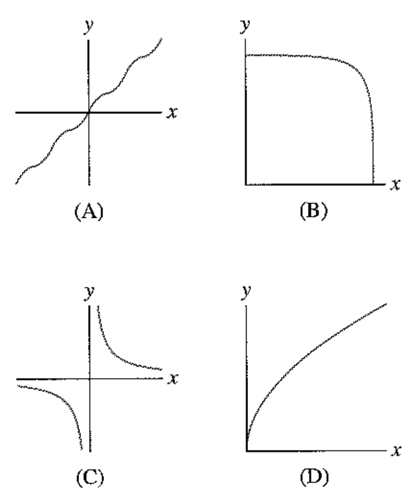


FIGURE 16

18. Let n be a nonzero integer. Find a domain on which $f(x) = (1 - x^n)^{1/n}$ coincides with its inverse. *Hint:* The answer depends on whether n is even or odd.

19. Let $f(x) = x^7 + x + 1$.

(a) Show that f^{-1} exists (but do not attempt to find it). *Hint:* Show that f is increasing.

(b) What is the domain of f^{-1}?

(c) Find $f^{-1}(3)$.

20. Show that $f(x) = (x^2 + 1)^{-1}$ is one-to-one on $(-\infty, 0]$, and find a formula for f^{-1} for this domain of f.

21. Let $f(x) = x^2 - 2x$. Determine a domain on which f^{-1} exists, and find a formula for f^{-1} for this domain of f.

22. Show that the inverse of $f(x) = e^{-x}$ exists (without finding it explicitly). What is the domain of f^{-1}?

23. Find the inverse $g(x)$ of $f(x) = \sqrt{x^2 + 9}$ with domain $x \geq 0$ and calculate $g'(x)$ in two ways: using Theorem 2 and by direct calculation.

24. Let $g(x)$ be the inverse of $f(x) = x^3 + 1$. Find a formula for $g(x)$ and calculate $g'(x)$ in two ways: using Theorem 2 and then by direct calculation.

In Exercises 25–30, use Theorem 2 to calculate $g'(x)$, where $g(x)$ is the inverse of $f(x)$.

25. $f(x) = 7x + 6$

26. $f(x) = \sqrt{3 - x}$

27. $f(x) = x^{-5}$

28. $f(x) = 4x^3 - 1$

29. $f(x) = \dfrac{x}{x + 1}$

30. $f(x) = 2 + x^{-1}$

31. Let $g(x)$ be the inverse of $f(x) = x^3 + 2x + 4$. Calculate $g(7)$ [without finding a formula for $g(x)$], and then calculate $g'(7)$.

32. Find $g'\left(-\frac{1}{2}\right)$, where $g(x)$ is the inverse of $f(x) = \dfrac{x^3}{x^2 + 1}$.

In Exercises 33–38, calculate $g(b)$ and $g'(b)$, where g is the inverse of f (in the given domain, if indicated).

33. $f(x) = x + \cos x, \quad b = 1$

34. $f(x) = 4x^3 - 2x, \quad b = -2$

35. $f(x) = \sqrt{x^2 + 6x}$ for $x \geq 0, \quad b = 4$

36. $f(x) = \sqrt{x^2 + 6x}$ for $x \leq -6, \quad b = 4$

37. $f(x) = \dfrac{1}{x + 1}, \quad b = \dfrac{1}{4}$

38. $f(x) = e^x, \quad b = e$

39. Let $f(x) = x^n$ and $g(x) = x^{1/n}$. Compute $g'(x)$ using Theorem 2 and check your answer using the Power Rule.

40. Show that $f(x) = \dfrac{1}{1 + x}$ and $g(x) = \dfrac{1 - x}{x}$ are inverses. Then compute $g'(x)$ directly and verify that $g'(x) = 1/f'(g(x))$.

41. Use graphical reasoning to determine if the following statements are true or false. If false, modify the statement to make it correct.

(a) If $f(x)$ is increasing, then $f^{-1}(x)$ is increasing.

(b) If $f(x)$ is decreasing, then $f^{-1}(x)$ is decreasing.

(c) If $f(x)$ is concave up, then $f^{-1}(x)$ is concave up.

(d) If $f(x)$ is concave down, then $f^{-1}(x)$ is concave down.

(e) Linear functions $f(x) = ax + b$ $(a \neq 0)$ are always one-to-one.

(f) Quadratic polynomials $f(x) = ax^2 + bx + c$ $(a \neq 0)$ are always one-to-one.

(g) $\sin x$ is not one-to-one.

Further Insights and Challenges

42. Show that if $f(x)$ is odd and $f^{-1}(x)$ exists, then $f^{-1}(x)$ is odd. Show, on the other hand, that an even function does not have an inverse.

43. Let g be the inverse of a function f satisfying $f'(x) = f(x)$. Show that $g'(x) = x^{-1}$. We will apply this in the next section to show that the inverse of $f(x) = e^x$ (the natural logarithm) is an antiderivative of x^{-1}.

FIGURE 1 Renato Solidum, director of the Philippine Institute of Volcanology and Seismology, checks the intensity of the October 8, 2004, Manila earthquake, which registered 6.2 on the Richter scale. The Richter scale is based on the logarithm (to base 10) of the amplitude of seismic waves. Each whole-number increase in Richter magnitude corresponds to a 10-fold increase in amplitude and around 31 times more energy.

7.3 Logarithms and Their Derivatives

Logarithm functions are inverses of exponential functions. More precisely, if $b > 0$ and $b \neq 1$, then the *logarithm to the base b*, denoted $\log_b x$, is the inverse of $f(x) = b^x$. By definition, $y = \log_b x$ if $b^y = x$.

$$b^{\log_b x} = x \qquad \text{and} \qquad \log_b(b^x) = x$$

Thus, $\log_b x$ is the number to which b must be raised in order to get x. For example,

$$\log_2(8) = 3 \quad \text{because} \quad 2^3 = 8$$

$$\log_{10}(1) = 0 \quad \text{because} \quad 10^0 = 1$$

$$\log_3\left(\frac{1}{9}\right) = -2 \quad \text{because} \quad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

The logarithm to the base e, denoted $\ln x$, plays a special role and is called the **natural logarithm**. We use a calculator to evaluate logarithms numerically. For example,

In this text, the natural logarithm is denoted $\ln x$*. Other common notations are* $\log x$ *or* $\operatorname{Log} x$*.*

$$\ln 17 \approx 2.83321 \quad \text{because} \quad e^{2.83321} \approx 17$$

Recall that the domain of b^x is $\mathbf{R}$ and its range is the set of positive real numbers $\{x : x > 0\}$. Since the domain and range are reversed in the inverse function,

- The *domain* of $\log_b x$ is $\{x : x > 0\}$.
- The *range* of $\log_b x$ is the set of all real numbers $\mathbf{R}$.

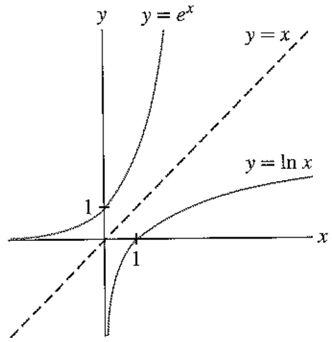


FIGURE 2 $y = \ln x$ is the inverse of $y = e^x$.

If $b > 1$, then $\log_b x$ is positive for $x > 1$ and negative for $0 < x < 1$, and

$$\lim_{x\to 0+} \ln x = -\infty, \qquad \lim_{x\to\infty} \ln x = \infty$$

Figure 2 illustrates these facts for the base $b = e$. Keep in mind that the logarithm of a negative number does not exist. For example, $\log_{10}(-2)$ does not exist because $10^y = -2$ has no solution.

For each law of exponents, there is a corresponding law for logarithms. The rule $b^{x+y} = b^x b^y$ corresponds to the rule

$$\log_b(xy) = \log_b x + \log_b y$$

In words: *The log of a product is the sum of the logs.* To verify this rule, observe:

$$b^{\log_b(xy)} = xy = b^{\log_b x} \cdot b^{\log_b y} = b^{\log_b x + \log_b y}$$

It follows that the exponents $\log_b(xy)$ and $\log_b x + \log_b y$ are equal as claimed. The remaining logarithm laws are collected in the following table.

Laws of Logarithms

	Law	Example
Log of 1	$\log_b(1) = 0$	
Log of b	$\log_b(b) = 1$	
Products	$\log_b(xy) = \log_b x + \log_b y$	$\log_5(2\cdot 3) = \log_5 2 + \log_5 3$
Quotients	$\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	$\log_2\left(\frac{3}{7}\right) = \log_2 3 - \log_2 7$
Reciprocals	$\log_b\left(\frac{1}{x}\right) = -\log_b x$	$\log_2\left(\frac{1}{7}\right) = -\log_2 7$
Powers (any n)	$\log_b(x^n) = n\log_b x$	$\log_{10}(8^2) = 2\cdot\log_{10} 8$

We note also that all logarithm functions are proportional. More precisely, the following **change of base** formula holds (see Exercise 119):

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \log_b x = \frac{\ln x}{\ln b} \qquad (1)$$

EXAMPLE 1 **Using the Logarithm Laws** Evaluate:

(a) $\log_6 9 + \log_6 4$ **(b)** $\ln\left(\frac{1}{\sqrt{e}}\right)$ **(c)** $10\log_b(b^3) - 4\log_b(\sqrt{b})$

Solution

(a) $\log_6 9 + \log_6 4 = \log_6(9 \cdot 4) = \log_6(36) = \log_6(6^2) = 2$

(b) $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln(e^{-1/2}) = -\frac{1}{2}\ln(e) = -\frac{1}{2}$

(c) $10\log_b(b^3) - 4\log_b(\sqrt{b}) = 10(3) - 4\log_b(b^{1/2}) = 30 - 4\left(\frac{1}{2}\right) = 28$ ■

■ **EXAMPLE 2 Solving an Exponential Equation** The bacteria population in a bottle at time t (in hours) has size $P(t) = 1000e^{0.35t}$. After how many hours will there be 5000 bacteria?

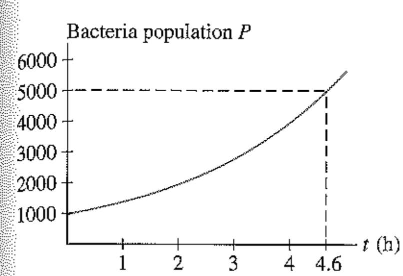


FIGURE 3 Bacteria population as a function of time.

Solution We must solve $P(t) = 1000e^{0.35t} = 5000$ for t (Figure 3):

$$e^{0.35t} = \frac{5000}{1000} = 5$$

$$\ln(e^{0.35t}) = \ln 5 \qquad \text{(take logarithms of both sides)}$$

$$0.35t = \ln 5 \approx 1.609 \qquad [\text{because } \ln(e^a) = a]$$

$$t \approx \frac{1.609}{0.35} \approx 4.6 \text{ hours}$$ ■

Calculus of Logarithms

In Section 7.1, we proved that for any base $b > 0$,

$$\frac{d}{dx}b^x = m(b)\,b^x, \qquad \text{where} \quad m(b) = \lim_{h\to 0}\frac{b^h - 1}{h}$$

← *REMINDER* $\ln x$ *is the natural logarithm, that is,* $\ln x = \log_e x$.

However, we were not able to identify the factor $m(b)$ (other than to say that e is the unique number for which $m(e) = 1$). Now we can use the Chain Rule to prove that $m(b) = \ln b$. The key point is that every exponential function can be written in terms of e, namely, $b^x = (e^{\ln(b)})^x = e^{(\ln b)x}$. By the Chain Rule,

$$\frac{d}{dx}b^x = \frac{d}{dx}e^{(\ln b)x} = (\ln b)e^{(\ln b)x} = (\ln b)b^x$$

THEOREM 1 Derivative of $f(x) = b^x$

$$\frac{d}{dx}b^x = (\ln b)b^x \qquad \text{for } b > 0 \qquad \boxed{2}$$

For example, $(10^x)' = (\ln 10)10^x$.

■ **EXAMPLE 3** Differentiate: **(a)** $f(x) = 4^{3x}$ and **(b)** $f(x) = 5^{x^2}$.

Solution

(a) The function $f(x) = 4^{3x}$ is a composite of 4^u and $u = 3x$:

$$\frac{d}{dx}4^{3x} = \left(\frac{d}{du}4^u\right)\frac{du}{dx} = (\ln 4)4^u(3x)' = (\ln 4)4^{3x}(3) = (3\ln 4)4^{3x}$$

(b) The function $f(x) = 5^{x^2}$ is a composite of 5^u and $u = x^2$:

$$\frac{d}{dx}5^{x^2} = \left(\frac{d}{du}5^u\right)\frac{du}{dx} = (\ln 5)5^u(x^2)' = (\ln 5)5^{x^2}(2x) = (2\ln 5)\, x\, 5^{x^2}$$ ■

Next, we'll find the derivative of $\ln x$. Let $f(x) = e^x$ and $g(x) = \ln x$. Then $f(g(x)) = x$ and $g'(x) = 1/f'(g(x))$ because $g(x)$ is the inverse of $f(x)$. However, $f'(x) = f(x)$, so

$$\frac{d}{dx}\ln x = g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f(g(x))} = \frac{1}{x}$$

The two most important calculus facts about exponentials and logs are

$$\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x}$$

THEOREM 2 Derivative of the Natural Logarithm

$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \text{for } x > 0 \tag{3}$$

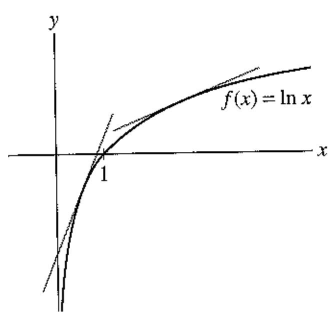


FIGURE 4 The tangent lines to $y = \ln x$ get flatter as $x \to \infty$.

■ **EXAMPLE 4** Describe the graph of $f(x) = \ln x$. Is $f(x)$ increasing or decreasing?

Solution The derivative $f'(x) = x^{-1}$ is positive on the domain $\{x : x > 0\}$, so $f(x) = \ln x$ is increasing. However, $f'(x) = x^{-1}$ is decreasing, so the graph of $f(x)$ is concave down and grows flatter as $x \to \infty$ (Figure 4). ■

■ **EXAMPLE 5** Differentiate: **(a)** $y = x\ln x$ and **(b)** $y = (\ln x)^2$.

Solution

(a) Use the Product Rule:

$$\frac{d}{dx}(x\ln x) = x\cdot(\ln x)' + (x)'\cdot\ln x = x\cdot\frac{1}{x} + \ln x = 1 + \ln x$$

(b) Use the General Power Rule,

$$\frac{d}{dx}(\ln x)^2 = 2\ln x\cdot\frac{d}{dx}\ln x = \frac{2\ln x}{x}$$ ■

In Section 3.2, we proved the Power Rule for whole number exponents. We can now prove it for all real exponents n and $x > 0$ by writing x^n as an exponential and using the Chain Rule:

$$x^n = (e^{\ln x})^n = e^{n\ln x}$$

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\ln x} = \left(\frac{d}{dx}n\ln x\right)e^{n\ln x}$$

$$= \left(\frac{n}{x}\right)x^n = nx^{n-1}$$

There is a useful formula for the derivative of a composite function of the form $\ln(f(x))$. Let $u = f(x)$ and apply the Chain Rule:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{du}\ln(u)\,\frac{du}{dx} = \frac{1}{u}\cdot u' = \frac{1}{f(x)}f'(x)$$

$$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)} \tag{4}$$

■ **EXAMPLE 6** Differentiate: **(a)** $y = \ln(x^3+1)$ and **(b)** $y = \ln(\sqrt{\sin x})$.

Solution Use Eq. (4):

(a) $\dfrac{d}{dx}\ln(x^3+1) = \dfrac{(x^3+1)'}{x^3+1} = \dfrac{3x^2}{x^3+1}$.

(b) The algebra is simpler if we write $\ln(\sqrt{\sin x}) = \ln\left((\sin x)^{1/2}\right) = \frac{1}{2}\ln(\sin x)$:

$$\frac{d}{dx}\ln(\sqrt{\sin x}) = \frac{1}{2}\frac{d}{dx}\ln(\sin x) = \frac{1}{2}\frac{(\sin x)'}{\sin x} = \frac{1}{2}\frac{\cos x}{\sin x} = \frac{1}{2}\cot x$$ ■

The "change of base" formula [Eq. (1)]

$$\log_b x = \frac{\ln x}{\ln b}$$

shows that for any base $b > 0$, $b \neq 1$:

$$\frac{d}{dx}\log_b x = \frac{1}{(\ln b)x}$$

EXAMPLE 7 Logarithm to Another Base Calculate $\frac{d}{dx}\log_{10} x$.

Solution By the change of base formula recalled in the margin, $\log_{10} x = \frac{\ln x}{\ln 10}$, and therefore

$$\frac{d}{dx}\log_{10} x = \frac{d}{dx}\left(\frac{\ln x}{\ln 10}\right) = \frac{1}{\ln 10}\frac{d}{dx}\ln x = \frac{1}{(\ln 10)x}$$

The next example illustrates **logarithmic differentiation.** This technique saves work when the function is a product or quotient with several factors.

EXAMPLE 8 Logarithmic Differentiation Find the derivative of

$$f(x) = \frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}$$

Solution In logarithmic differentiation, we differentiate $\ln(f(x))$ rather than $f(x)$ itself. First, expand $\ln(f(x))$ using the logarithm rules:

$$\ln(f(x)) = \ln\left((x+1)^2\right) + \ln\left(2x^2-3\right) - \ln\left(\sqrt{x^2+1}\right)$$
$$= 2\ln(x+1) + \ln(2x^2-3) - \frac{1}{2}\ln(x^2+1)$$

Then use Eq. (4):

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln(f(x)) = 2\frac{d}{dx}\ln\left(x+1\right) + \frac{d}{dx}\ln\left(2x^2-3\right) - \frac{1}{2}\frac{d}{dx}\ln\left(x^2+1\right)$$
$$\frac{f'(x)}{f(x)} = 2\frac{1}{x+1} + \frac{4x}{2x^2-3} - \frac{1}{2}\frac{2x}{x^2+1}$$

Finally, multiply through by $f(x)$:

$$f'(x) = \left(\frac{(x+1)^2(2x^2-3)}{\sqrt{x^2+1}}\right)\left(\frac{2}{x+1} + \frac{4x}{2x^2-3} - \frac{x}{x^2+1}\right)$$

EXAMPLE 9 Differentiate (for $x > 0$): **(a)** $f(x) = x^x$ and **(b)** $g(x) = x^{\sin x}$.

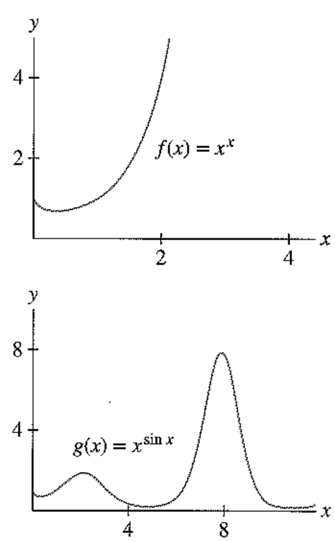


FIGURE 5 Graphs of $f(x) = x^x$ and $g(x) = x^{\sin x}$.

Solution The two problems are similar (Figure 5). We illustrate two different methods.

(a) Method 1: Use the identity $x = e^{\ln x}$ to write $f(x)$ as an exponential:

$$f(x) = x^x = (e^{\ln x})^x = e^{x\ln x}$$
$$f'(x) = (x\ln x)'e^{x\ln x} = (1+\ln x)e^{x\ln x} = (1+\ln x)x^x$$

(b) Method 2: Apply Eq. (4) to $\ln(g(x))$. Since $\ln(g(x)) = \ln(x^{\sin x}) = (\sin x)\ln x$,

$$\frac{g'(x)}{g(x)} = \frac{d}{dx}\ln(g(x)) = \frac{d}{dx}\Big((\sin x)\ln x\Big) = \frac{\sin x}{x} + (\cos x)\ln x$$
$$g'(x) = \left(\frac{\sin x}{x} + (\cos x)\ln x\right)g(x) = \left(\frac{\sin x}{x} + (\cos x)\ln x\right)x^{\sin x}$$

The Logarithm as an Integral

In Chapter 5, we noted that the Power Rule for Integrals is valid for all exponents $n \neq -1$:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$$

This formula is not valid (or meaningful) for $n = -1$, so the question remained: *What is the antiderivative of* $y = x^{-1}$? We can now give the answer: the natural logarithm. Indeed, the formula $(\ln x)' = x^{-1}$ tells us that $\ln x$ is an antiderivative of $y = x^{-1}$ for $x > 0$:

$$\int \frac{dx}{x} = \ln x + C$$

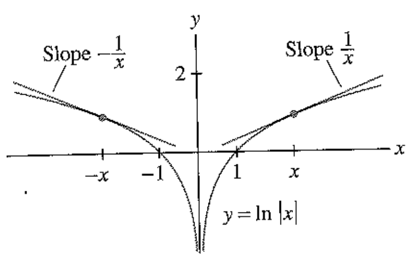


FIGURE 6

We would like to have an antiderivative of $y = \frac{1}{x}$ on its full domain, namely on the domain $\{x : x \neq 0\}$. To achieve this end, we extend $F(x)$ to an even function by setting $F(x) = \ln|x|$ (Figure 6). Then $F(x) = F(-x)$, and by the Chain Rule, $F'(x) = -F'(-x)$. For $x < 0$, we obtain

$$\frac{d}{dx}\ln|x| = F'(x) = -F'(-x) = -\frac{1}{-x} = \frac{1}{x}$$

This proves that $\frac{d}{dx}\ln|x| = \frac{1}{x}$ for all $x \neq 0$.

THEOREM 3 Antiderivative of $y = \frac{1}{x}$ The function $F(x) = \ln|x|$ is an antiderivative of $y = \frac{1}{x}$ in the domain $\{x : x \neq 0\}$, that is,

$$\int \frac{dx}{x} = \ln|x| + C \qquad \boxed{5}$$

By the Fundamental Theorem of Calculus, the following formula is valid if both a and b are either both positive or both negative (Figure 7):

$$\int_a^b \frac{dx}{x} = \ln|b| - \ln|a| = \ln\frac{b}{a} \qquad \boxed{6}$$

Setting $a = 1$ and $b = x$, we obtain a formula for the natural logarithm as an integral:

$$\ln x = \int_1^x \frac{dt}{t} \qquad \boxed{7}$$

EXAMPLE 10 The Logarithm as an Antiderivative Evaluate:

$$\textbf{(a)} \int_2^8 \frac{dx}{x} \quad \text{and} \quad \textbf{(b)} \int_{-4}^{-2} \frac{dx}{x}$$

Solution By Eq. (6),

$$\int_2^8 \frac{dx}{x} = \ln\frac{8}{2} = \ln 4 \approx 1.39 \quad \text{and} \quad \int_{-4}^{-2} \frac{dx}{x} = \ln\left(\frac{-2}{-4}\right) = \ln\frac{1}{2} \approx -0.69$$

The area represented by these integrals is shown in Figures 7(B) and (C).

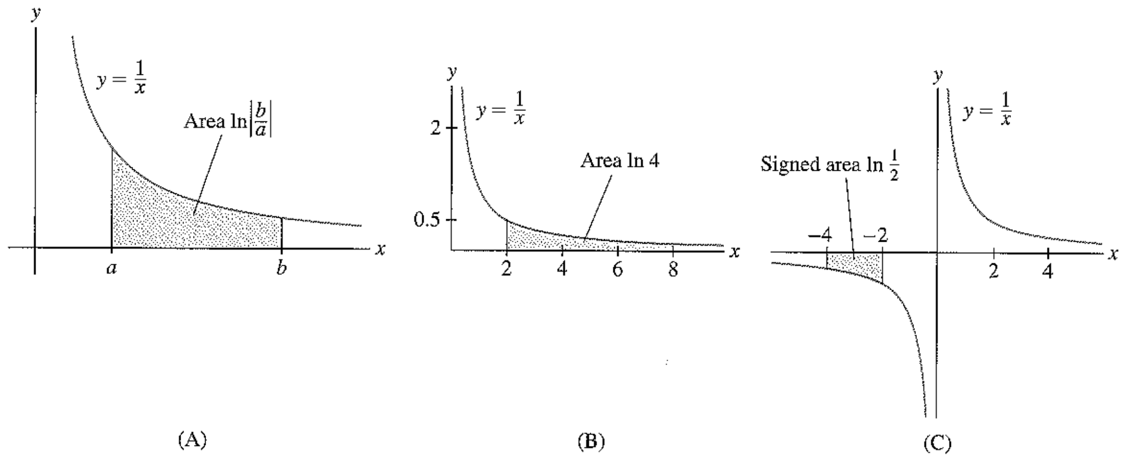


FIGURE 7

EXAMPLE 11 Evaluate:

(a) $\displaystyle\int_1^3 \frac{x}{x^2+1}\,dx$ **(b)** $\displaystyle\int \tan x\,dx$

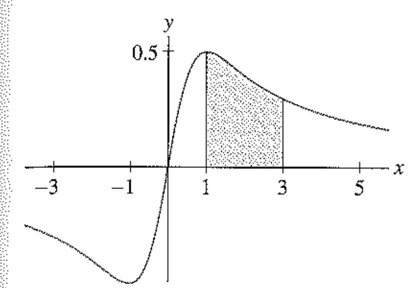


FIGURE 8 Area under the graph of $y = \dfrac{x}{x^2+1}$ over $[1, 3]$.

Solution

(a) Use the substitution $u = x^2 + 1$, $\frac{1}{2}du = x\,dx$. In the u-variable, the limits of the integral become $u(1) = 2$ and $u(3) = 10$. The integral is equal to the area shown in Figure 8:

$$\int_1^3 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_2^{10} \frac{du}{u} = \frac{1}{2}\ln|u|\,\bigg|_2^{10} = \frac{1}{2}\ln 10 - \frac{1}{2}\ln 2 \approx 0.805$$

(b) Use the substitution $u = \cos x$, $du = -\sin x$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

7.3 SUMMARY

- For $b > 0$ with $b \neq 1$, the *logarithm function* $\log_b x$ is the inverse of b^x;

$$x = b^y \quad \Leftrightarrow \quad y = \log_b x$$

- If $b > 1$, then $\log_b x$ is positive for $x > 1$ and negative for $0 < x < 1$, and

$$\lim_{x\to 0+} \ln x = -\infty, \qquad \lim_{x\to\infty} \ln x = \infty$$

- The *natural logarithm* is the logarithm to the base e and is denoted $\ln x$.
- Important logarithm laws:

(i) $\log_b(xy) = \log_b x + \log_b y$ **(ii)** $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

(iii) $\log_b(x^n) = n\log_b x$ **(iv)** $\log_b 1 = 0$ and $\log_b b = 1$

- Derivative formulas:

$$(e^x)' = e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x}, \qquad (b^x)' = (\ln b)b^x, \qquad \frac{d}{dx}\log_b x = \frac{1}{(\ln b)\,x}$$

- Integral formulas:

$$\ln x = \int_1^x \frac{dt}{t} \quad (x > 0), \qquad \int \frac{dx}{x} = \ln|x| + C$$

7.3 EXERCISES

Preliminary Questions

1. Compute $\log_{b^2}(b^4)$.
2. When is $\ln x$ negative?
3. What is $\ln(-3)$? Explain.
4. Explain the phrase "The logarithm converts multiplication into addition."
5. What are the domain and range of $\ln x$?
6. Does x^{-1} have an antiderivative for $x < 0$? If so, describe one.
7. What is the slope of the tangent line to $y = 4^x$ at $x = 0$?
8. What is the rate of change of $y = \ln x$ at $x = 10$?

Exercises

In Exercises 1–16, calculate without using a calculator.

1. $\log_3 27$
2. $\log_5 \frac{1}{25}$
3. $\ln 1$
4. $\log_5(5^4)$
5. $\log_2(2^{5/3})$
6. $\log_2(8^{5/3})$
7. $\log_{64} 4$
8. $\log_7(49^2)$
9. $\log_8 2 + \log_4 2$
10. $\log_{25} 30 + \log_{25} \frac{5}{6}$
11. $\log_4 48 - \log_4 12$
12. $\ln(\sqrt{e}\cdot e^{7/5})$
13. $\ln(e^3) + \ln(e^4)$
14. $\log_2 \frac{4}{3} + \log_2 24$
15. $7^{\log_7(29)}$
16. $8^{3\log_8(2)}$

17. Write as the natural log of a single expression:
(a) $2\ln 5 + 3\ln 4$ (b) $5\ln(x^{1/2}) + \ln(9x)$

18. Solve for x: $\ln(x^2+1) - 3\ln x = \ln(2)$.

In Exercises 19–24, solve for the unknown.

19. $7e^{5t} = 100$
20. $6e^{-4t} = 2$
21. $2^{x^2-2x} = 8$
22. $e^{2t+1} = 9e^{1-t}$
23. $\ln(x^4) - \ln(x^2) = 2$
24. $\log_3 y + 3\log_3(y^2) = 14$

25. Show, by producing a counterexample, that $\ln(ab)$ is not equal to $(\ln a)(\ln b)$.

26. What is b if $(\log_b x)' = \dfrac{1}{3x}$?

27. The population of a city (in millions) at time t (years) is $P(t) = 2.4e^{0.06t}$, where $t = 0$ is the year 2000. When will the population double from its size at $t = 0$?

28. The **Gutenberg–Richter Law** states that the number N of earthquakes per year worldwide of Richter magnitude at least M satisfies an approximate relation $\log_{10} N = a - M$ for some constant a. Find a, assuming that there is one earthquake of magnitude $M \geq 8$ per year. How many earthquakes of magnitude $M \geq 5$ occur per year?

In Exercises 29–48, find the derivative.

29. $y = x\ln x$
30. $y = t\ln t - t$
31. $y = (\ln x)^2$
32. $y = \ln(x^5)$
33. $y = \ln(9x^2 - 8)$
34. $y = \ln(t5^t)$
35. $y = \ln(\sin t + 1)$
36. $y = x^2 \ln x$
37. $y = \dfrac{\ln x}{x}$
38. $y = e^{(\ln x)^2}$
39. $y = \ln(\ln x)$
40. $y = \ln(\cot x)$
41. $y = \left(\ln(\ln x)\right)^3$
42. $y = \ln\left((\ln x)^3\right)$
43. $y = \ln\left((x+1)(2x+9)\right)$
44. $y = \ln\left(\dfrac{x+1}{x^3+1}\right)$
45. $y = 11^x$
46. $y = 7^{4x-x^2}$
47. $y = \dfrac{2^x - 3^{-x}}{x}$
48. $y = 16^{\sin x}$

In Exercises 49–52, compute the derivative.

49. $f'(x)$, $f(x) = \log_2 x$
50. $f'(3)$, $f(x) = \log_5 x$
51. $\dfrac{d}{dt}\log_3(\sin t)$
52. $\dfrac{d}{dt}\log_{10}(t + 2^t)$

In Exercises 53–64, find an equation of the tangent line at the point indicated.

53. $f(x) = 6^x$, $x = 2$
54. $y = (\sqrt{2})^x$, $x = 8$

55. $s(t) = 3^{9t}, \quad t = 2$

56. $y = \pi^{5x-2}, \quad x = 1$

57. $f(x) = 5^{x^2-2x}, \quad x = 1$

58. $s(t) = \ln t, \quad t = 5$

59. $s(t) = \ln(8 - 4t), \quad t = 1$

60. $f(x) = \ln(x^2), \quad x = 4$

61. $R(z) = \log_5(2z^2 + 7), \quad z = 3$

62. $y = \ln(\sin x), \quad x = \frac{\pi}{4}$

63. $f(w) = \log_2 w, \quad w = \frac{1}{8}$

64. $y = \log_2(1 + 4x^{-1}), \quad x = 4$

In Exercises 65–72, find the derivative using logarithmic differentiation as in Example 8.

65. $y = (x + 5)(x + 9)$

66. $y = (3x + 5)(4x + 9)$

67. $y = (x - 1)(x - 12)(x + 7)$

68. $y = \dfrac{x(x+1)^3}{(3x-1)^2}$

69. $y = \dfrac{x(x^2+1)}{\sqrt{x+1}}$

70. $y = (2x + 1)(4x^2)\sqrt{x - 9}$

71. $y = \sqrt{\dfrac{x(x+2)}{(2x+1)(3x+2)}}$

72. $y = (x^3 + 1)(x^4 + 2)(x^5 + 3)^2$

In Exercises 73–78, find the derivative using either method of Example 9.

73. $f(x) = x^{3x}$

74. $f(x) = x^{\cos x}$

75. $f(x) = x^{e^x}$

76. $f(x) = x^{x^2}$

77. $f(x) = x^{3^x}$

78. $f(x) = e^{x^x}$

In Exercises 79–82, find the local extreme values in the domain $\{x : x > 0\}$ and use the Second Derivative Test to determine whether these values are local minima or maxima.

79. $g(x) = \dfrac{\ln x}{x}$

80. $g(x) = x \ln x$

81. $g(x) = \dfrac{\ln x}{x^3}$

82. $g(x) = x - \ln x$

In Exercises 83 and 84, find the local extreme values and points of inflection, and sketch the graph of $y = f(x)$ over the interval $[1, 4]$.

83. $f(x) = \dfrac{10 \ln x}{x^2}$

84. $f(x) = x^2 - 8 \ln x$

In Exercises 85–105, evaluate the indefinite integral, using substitution if necessary.

85. $\displaystyle\int \frac{7\,dx}{x}$

86. $\displaystyle\int \frac{dx}{x+7}$

87. $\displaystyle\int \frac{dx}{2x+4}$

88. $\displaystyle\int \frac{dx}{9x-3}$

89. $\displaystyle\int \frac{t\,dt}{t^2+4}$

90. $\displaystyle\int \frac{x^2\,dx}{x^3+2}$

91. $\displaystyle\int \frac{(3x-1)\,dx}{9-2x+3x^2}$

92. $\displaystyle\int \tan(4x+1)\,dx$

93. $\displaystyle\int \cot x\,dx$

94. $\displaystyle\int \frac{\cos x}{2\sin x + 3}\,dx$

95. $\displaystyle\int \frac{\ln x}{x}\,dx$

96. $\displaystyle\int \frac{4\ln x + 5}{x}\,dx$

97. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

98. $\displaystyle\int \frac{dx}{x \ln x}$

99. $\displaystyle\int \frac{dx}{(4x-1)\ln(8x-2)}$

100. $\displaystyle\int \frac{\ln(\ln x)}{x \ln x}\,dx$

101. $\displaystyle\int \cot x \ln(\sin x)\,dx$

102. $\displaystyle\int 3^x\,dx$

103. $\displaystyle\int x 3^{x^2}\,dx$

104. $\displaystyle\int \cos x\, 3^{\sin x}\,dx$

105. $\displaystyle\int \left(\frac{1}{2}\right)^{3x+2} dx$

In Exercises 106–111, evaluate the definite integral.

106. $\displaystyle\int_1^2 \frac{1}{x}\,dx$

107. $\displaystyle\int_4^{12} \frac{1}{x}\,dx$

108. $\displaystyle\int_1^e \frac{1}{x}\,dx$

109. $\displaystyle\int_2^4 \frac{dt}{3t+4}$

110. $\displaystyle\int_{-e^2}^{-e} \frac{1}{t}\,dt$

111. $\displaystyle\int_e^{e^2} \frac{1}{t \ln t}\,dt$

112. CAS Find a good numerical approximation to the coordinates of the point on the graph of $y = \ln x - x$ closest to the origin (Figure 9).

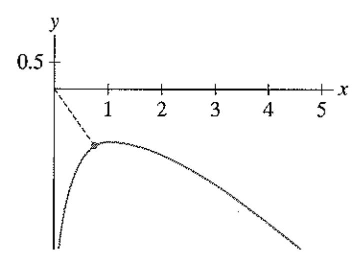


FIGURE 9 Graph of $y = \ln x - x$.

113. Find the minimum value of $f(x) = x^x$ for $x > 0$.

114. Use the formula $(\ln f(x))' = f'(x)/f(x)$ to show that $\ln x$ and $\ln(2x)$ have the same derivative. Is there a simpler explanation of this result?

115. According to one simplified model, the purchasing power of a dollar in the year $2000 + t$ is equal to $P(t) = 0.68(1.04)^{-t}$ (in 1983 dollars). Calculate the predicted rate of decline in purchasing power (in cents per year) in the year 2020.

116. The energy E (in joules) radiated as seismic waves by an earthquake of Richter magnitude M satisfies $\log_{10} E = 4.8 + 1.5M$.

(a) Show that when M increases by 1, the energy increases by a factor of approximately 31.5.

(b) Calculate dE/dM.

117. The Palermo Technical Impact Hazard Scale P is used to quantify the risk associated with the impact of an asteroid colliding with the earth:

$$P = \log_{10}\left(\frac{p_i E^{0.8}}{0.03T}\right)$$

where p_i is the probability of impact, T is the number of years until impact, and E is the energy of impact (in megatons of TNT). The risk is greater than a random event of similar magnitude if $P > 0$.

(a) Calculate dP/dT, assuming that $p_i = 2 \times 10^{-5}$ and $E = 2$ megatons.

(b) Use the derivative to estimate the change in P if T increases from 8 to 9 years.

Further Insights and Challenges

118. (a) Show that if f and g are differentiable, then

$$\frac{d}{dx}\ln(f(x)g(x)) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \qquad \boxed{8}$$

(b) Give a new proof of the Product Rule by observing that the left-hand side of Eq. (8) is equal to $\dfrac{(f(x)g(x))'}{f(x)g(x)}$.

119. Prove the formula

$$\log_b x = \frac{\log_a x}{\log_a b}$$

for all positive numbers a, b with $a \neq 1$ and $b \neq 1$.

120. Prove the formula $\log_a b \log_b a = 1$ for all positive numbers a, b with $a \neq 1$ and $b \neq 1$.

Exercises 121–123 develop an elegant approach to the exponential and logarithm functions. Define a function $G(x)$ for $x > 0$:

$$G(x) = \int_1^x \frac{1}{t}\,dt$$

121. Defining $\ln x$ as an Integral This exercise proceeds as if we didn't know that $G(x) = \ln x$ and shows directly that $G(x)$ has all the basic properties of the logarithm. Prove the following statements.

(a) $\displaystyle\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{t}\,dt$ for all $a, b > 0$. *Hint:* Use the substitution $u = t/a$.

(b) $G(ab) = G(a) + G(b)$. *Hint:* Break up the integral from 1 to ab into two integrals and use (a).

(c) $G(1) = 0$ and $G(a^{-1}) = -G(a)$ for $a > 0$.

(d) $G(a^n) = nG(a)$ for all $a > 0$ and integers n.

(e) $G(a^{1/n}) = \dfrac{1}{n}G(a)$ for all $a > 0$ and integers $n \neq 0$.

(f) $G(a^r) = rG(a)$ for all $a > 0$ and rational numbers r.

(g) $G(x)$ is increasing. *Hint:* Use FTC II.

(h) There exists a number a such that $G(a) > 1$. *Hint:* Show that $G(2) > 0$ and take $a = 2^m$ for $m > 1/G(2)$.

(i) $\displaystyle\lim_{x\to\infty} G(x) = \infty$ and $\displaystyle\lim_{x\to 0+} G(x) = -\infty$

(j) There exists a unique number E such that $G(E) = 1$.

(k) $G(E^r) = r$ for every rational number r.

122. Defining e^x Use Exercise 121 to prove the following statements.

(a) $G(x)$ has an inverse with domain $\mathbf{R}$ and range $\{x : x > 0\}$. Denote the inverse by $F(x)$.

(b) $F(x + y) = F(x)F(y)$ for all x, y. Hint: it suffices to show that $G(F(x)F(y)) = G(F(x + y))$.

(c) $F(r) = E^r$ for all numbers. In particular, $F(0) = 1$.

(d) $F'(x) = F(x)$. *Hint:* Use the formula for the derivative of an inverse function.

This shows that $E = e$ and that $F(x)$ is the function e^x as defined in the text.

123. Defining b^x Let $b > 0$ and let $f(x) = F(xG(b))$ with F as in Exercise 122. Use Exercise 121 (f) to prove that $f(r) = b^r$ for every rational number r. This gives us a way of defining b^x for irrational x, namely $b^x = f(x)$. With this definition, b^x is a differentiable function of x (because F is differentiable).

7.4 Exponential Growth and Decay

In this section, we explore some applications of the exponential function. Consider a quantity $P(t)$ that depends exponentially on time:

$$\boxed{P(t) = P_0 e^{kt}}$$

The constant k has units of "inverse time"; if t is measured in days, then k has units of $(\text{days})^{-1}$.

If $k > 0$, then $P(t)$ *grows exponentially* and k is called the *growth constant*. Note that P_0 is the initial size (the size at $t = 0$):

$$P(0) = P_0 e^{k\cdot 0} = P_0$$

We can also write $P(t) = P_0 b^t$ with $b = e^k$, because $b^t = (e^k)^t = e^{kt}$.

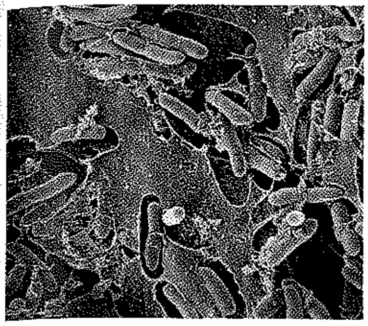


FIGURE 1 *E. coli* bacteria, found in the human intestine.

Exponential growth cannot continue over long periods of time. A colony starting with one E. coli cell would grow to 5×10^{89} cells after 3 weeks—much more than the estimated number of atoms in the observable universe. In actual cell growth, the exponential phase is followed by a period in which growth slows and may decline.

A quantity that decreases exponentially is said to have *exponential decay*. In this case, we write $P(t) = P_0 e^{-kt}$ with $k > 0$; k is then called the *decay constant*.

Population is a typical example of a quantity that grows exponentially, at least under suitable conditions. To understand why, consider a cell colony with initial population $P_0 = 100$ and assume that each cell divides into two cells after 1 hour. Then population $P(t)$ doubles with each passing hour:

$$\begin{aligned} P(0) &= 100 && \text{(initial population)} \\ P(1) = 2(100) &= 200 && \text{(population doubles)} \\ P(2) = 2(200) &= 400 && \text{(population doubles again)} \end{aligned}$$

After t hours, $P(t) = (100)2^t$.

EXAMPLE 1 In the laboratory, the number of *Escherichia coli* bacteria (Figure 1) grows exponentially with growth constant of $k = 0.41$ (hours)$^{-1}$. Assume that 1000 bacteria are present at time $t = 0$.

(a) Find the formula for the number of bacteria $P(t)$ at time t.

(b) How large is the population after 5 hours?

(c) When will the population reach 10,000?

Solution The growth is exponential, so $P(t) = P_0 e^{kt}$.

(a) The initial size is $P_0 = 1000$ and $k = 0.41$, so $P(t) = 1000e^{0.41t}$ (t in hours).

(b) After 5 hours, $P(5) = 1000e^{0.41 \cdot 5} = 1000e^{2.05} \approx 7767.9$. Because the number of bacteria is a whole number, we round off the answer to 7768.

(c) The problem asks for the time t such that $P(t) = 10{,}000$, so we solve

$$1000e^{0.41t} = 10{,}000 \quad \Rightarrow \quad e^{0.41t} = \frac{10{,}000}{1000} = 10$$

Taking the logarithm of both sides, we obtain $\ln\left(e^{0.41t}\right) = \ln 10$, or

$$0.41t = \ln 10 \quad \Rightarrow \quad t = \frac{\ln 10}{0.41} \approx 5.62$$

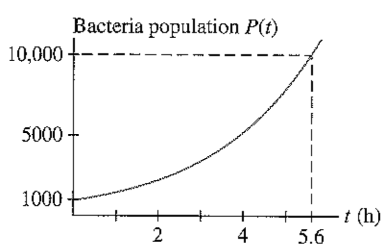


FIGURE 2 Growth of *E. coli* population.

Therefore, $P(t)$ reaches 10,000 after approximately 5 hours, 37 minutes (Figure 2). ■

The important role played by exponential functions is best understood in terms of the *differential equation* $y' = ky$. The function $y = P_0 e^{kt}$ satisfies this differential equation, as we can check directly:

$$y' = \frac{d}{dt}\left(P_0 e^{kt}\right) = kP_0 e^{kt} = ky$$

Theorem 1 goes further and asserts that the exponential functions are the *only* functions that satisfy this differential equation.

A differential equation is an equation relating a function $y = f(x)$ to its derivative y' (or higher derivatives y'', y''', ...).

THEOREM 1 If $y(t)$ is a differentiable function satisfying the differential equation

$$\boxed{y' = ky}$$

then $y(t) = P_0 e^{kt}$, where P_0 is the initial value $P_0 = y(0)$.

Proof Compute the derivative of ye^{-kt}. If $y' = ky$, then

$$\frac{d}{dt}(ye^{-kt}) = y'e^{-kt} - ke^{-kt}y = (ky)e^{-kt} - ke^{-kt}y = 0$$

Because the derivative is zero, $y(t)e^{-kt} = P_0$ for some constant P_0, and $y(t) = P_0e^{kt}$ as claimed. The initial value is $y(0) = P_0e^0 = P_0$. ■

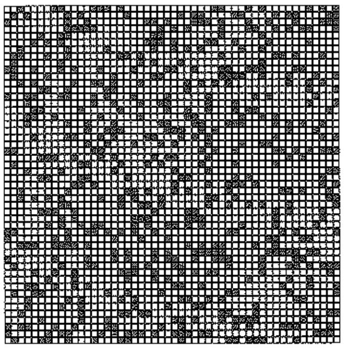


FIGURE 3 Computer simulation of radioactive decay as a random process. The red squares are atoms that have not yet decayed. A fixed fraction of red squares turns white in each unit of time.

CONCEPTUAL INSIGHT Theorem 1 tells us that a process obeys an exponential law precisely when *its rate of change is proportional to the amount present.* This helps us understand why certain quantities grow or decay exponentially.

A population grows exponentially because each organism contributes to growth through reproduction, and thus the growth rate is proportional to the population size. However, this is true only under certain conditions. If the organisms interact—say, by competing for food or mates—then the growth rate may not be proportional to population size and we cannot expect exponential growth.

Similarly, experiments show that radioactive substances decay exponentially. This suggests that radioactive decay is a random process in which a fixed fraction of atoms, randomly chosen, decays per unit time (Figure 3). If exponential decay were not observed, we might suspect that the decay was influenced by some interaction between the atoms.

■ **EXAMPLE 2** Find all solutions of $y' = 3y$. Which solution satisfies $y(0) = 9$?

Solution The solutions to $y' = 3y$ are the functions $y(t) = Ce^{3t}$, where C is the initial value $C = y(0)$. The particular solution satisfying $y(0) = 9$ is $y(t) = 9e^{3t}$. ■

■ **EXAMPLE 3 Modeling Penicillin** Pharmacologists have shown that penicillin leaves a person's bloodstream at a rate proportional to the amount present.

(a) Express this statement as a differential equation.

(b) Find the decay constant if 50 mg of penicillin remains in the bloodstream 7 hours after an initial injection of 450 mg.

(c) Under the hypothesis of (b), at what time was 200 mg of penicillin present?

Solution

(a) Let $A(t)$ be the quantity of penicillin present in the bloodstream at time t. Since the rate at which penicillin leaves the bloodstream is proportional to $A(t)$,

$$A'(t) = -kA(t) \qquad \boxed{1}$$

where $k > 0$ because $A(t)$ is decreasing.

(b) Eq. (1) and the condition $A(0) = 450$ tell us that $A(t) = 450e^{-kt}$. The additional condition $A(7) = 50$ enables us to solve for k:

$$A(7) = 450e^{-7k} = 50 \quad \Rightarrow \quad e^{-7k} = \frac{1}{9} \quad \Rightarrow \quad -7k = \ln\frac{1}{9}$$

Thus, $k = -\frac{1}{7}\ln\frac{1}{9} \approx 0.31$.

(c) To find the time t at which 200 mg was present, we solve

$$A(t) = 450e^{-0.31t} = 200 \quad \Rightarrow \quad e^{-0.31t} = \frac{4}{9}$$

Therefore, $t = -\frac{1}{0.31}\ln\left(\frac{4}{9}\right) \approx 2.62$ hours (Figure 4). ■

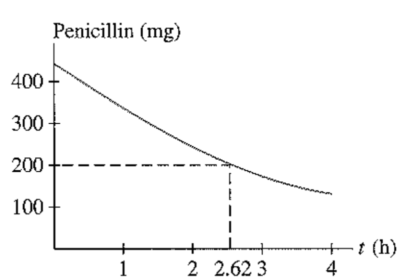


FIGURE 4 The quantity of penicillin in the bloodstream decays exponentially.

The constant k has units of time^{-1}, so the doubling time $T = (\ln 2)/k$ has units of time, as we should expect. A similar calculation shows that the tripling time is $(\ln 3)/k$, the quadrupling time is $(\ln 4)/k$, and, in general, the time to n-fold increase is $(\ln n)/k$.

Number of hosts infected with Sapphire: 74855

FIGURE 5 Spread of the Sapphire computer virus 30 minutes after release. The infected hosts spewed billions of copies of the virus into cyberspace, significantly slowing Internet traffic and interfering with businesses, flight schedules, and automated teller machines.

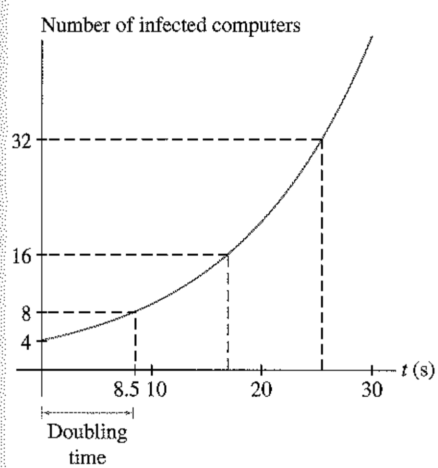


FIGURE 6 Doubling (from 4 to 8 to 16, etc.) occurs at equal time intervals.

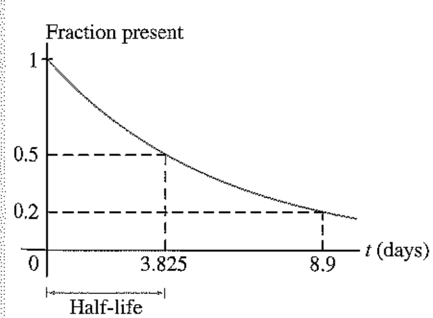


FIGURE 7 Fraction of radon-222 present at time t.

Quantities that grow exponentially possess an important property: There is a doubling time T such that $P(t)$ doubles in size over every time interval of length T. To prove this, let $P(t) = P_0e^{kt}$ and solve for T in the equation $P(t+T) = 2P(t)$.

$$P_0e^{k(t+T)} = 2P_0e^{kt}$$
$$e^{kt}e^{kT} = 2e^{kt}$$
$$e^{kT} = 2$$

We obtain $kT = \ln 2$ or $T = (\ln 2)/k$.

Doubling Time If $P(t) = P_0e^{kt}$ with $k > 0$, then the doubling time of P is

$$\text{Doubling time} = \frac{\ln 2}{k}$$

EXAMPLE 4 Spread of the Sapphire Worm A computer virus nicknamed the *Sapphire Worm* spread throughout the Internet on January 25, 2003 (Figure 5). Studies suggest that during the first few minutes, the population of infected computer hosts increased exponentially with growth constant $k = 0.0815\ \text{s}^{-1}$.

(a) What was the doubling time of the virus?

(b) If the virus began in four computers, how many hosts were infected after 2 minutes? After 3 minutes?

Solution

(a) The doubling time is $(\ln 2)/0.0815 \approx 8.5$ seconds (Figure 6).

(b) If $P_0 = 4$, the number of infected hosts after t seconds is $P(t) = 4e^{0.0815t}$. After 2 minutes (120 seconds), the number of infected hosts is

$$P(120) = 4e^{0.0815(120)} \approx 70{,}700$$

After 3 minutes, the number would have been $P(180) = 4e^{0.0815(180)} \approx 9.4$ million. However, it is estimated that a total of around 75,000 hosts were infected, so the exponential phase of the virus could not have lasted much more than 2 minutes.

In the situation of exponential decay $P(t) = P_0e^{-kt}$, the **half-life** is the time it takes for the quantity to decrease by a factor of $\frac{1}{2}$. The calculation similar to that of doubling time above shows that

$$\text{Half-life} = \frac{\ln 2}{k}$$

EXAMPLE 5 The isotope radon-222 decays exponentially with a half-life of 3.825 days. How long will it take for 80% of the isotope to decay?

Solution By the equation for half-life, k equals $\ln 2$ divided by half-life:

$$k = \frac{\ln 2}{3.825} \approx 0.181$$

Therefore, the quantity of radon-222 at time t is $R(t) = R_0e^{-0.181t}$, where R_0 is the amount present at $t = 0$ (Figure 7). When 80% has decayed, 20% remains, so we solve for t in the equation $R_0 e^{-0.181t} = 0.2R_0$:

$$e^{-0.181t} = 0.2$$

$$-0.181t = \ln(0.2) \quad \Rightarrow \quad t = \frac{\ln(0.2)}{-0.181} \approx 8.9 \text{ days}$$

The quantity of radon-222 decreases by 80% after 8.9 days. ■

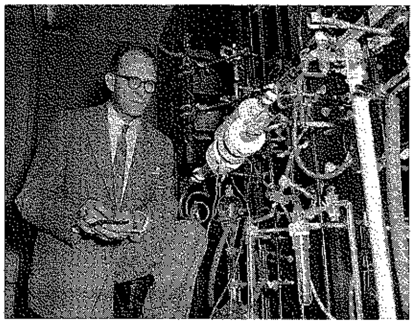


FIGURE 8 American chemist Willard Libby (1908–1980) developed the technique of carbon dating in 1946 to determine the age of fossils and was awarded the Nobel Prize in Chemistry for this work in 1960. Since then the technique has been refined considerably.

Carbon Dating

Carbon dating (Figure 8) relies on the fact that all living organisms contain carbon that enters the food chain through the carbon dioxide absorbed by plants from the atmosphere. Carbon in the atmosphere is made up of nonradioactive C^{12} and a minute amount of radioactive C^{14} that decays into nitrogen. The ratio of C^{14} to C^{12} is approximately $R_{atm} = 10^{-12}$.

The carbon in a living organism has the same ratio R_{atm} because this carbon originates in the atmosphere, but when the organism dies, its carbon is no longer replenished. The C^{14} begins to decay exponentially while the C^{12} remains unchanged. Therefore, the ratio of C^{14} to C^{12} in the organism decreases exponentially. By measuring this ratio, we can determine when the death occurred. The decay constant for C^{14} is $k = 0.000121$ yr^{-1}, so

$$\text{Ratio of } C^{14} \text{ to } C^{12} \text{ after } t \text{ years} = R_{atm} e^{-0.000121t}$$

FIGURE 9 Replica of Lascaux cave painting of a bull and horse.

■ **EXAMPLE 6** Cave Paintings In 1940, a remarkable gallery of prehistoric animal paintings was discovered in the Lascaux cave in Dordogne, France (Figure 9). A charcoal sample from the cave walls had a C^{14}-to-C^{12} ratio equal to 15% of that found in the atmosphere. Approximately how old are the paintings?

Solution The C^{14}-to-C^{12} ratio in the charcoal is now equal to $0.15R_{atm}$, so

$$R_{atm} e^{-0.000121t} = 0.15R_{atm}$$

where t is the age of the paintings. We solve for t:

$$e^{-0.000121t} = 0.15$$

$$-0.000121t = \ln(0.15) \quad \Rightarrow \quad t = \frac{\ln(0.15)}{0.000121} \approx 15{,}700$$

The cave paintings are approximately 16,000 years old (Figure 10). ■

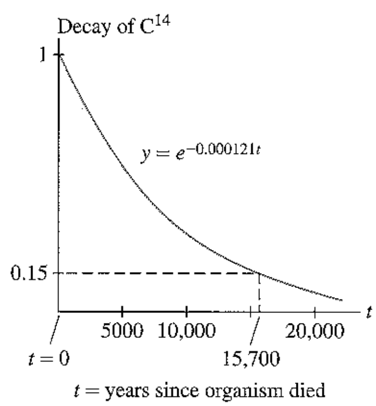


FIGURE 10 If only 15% of the C^{14} remains, the object is approximately 16,000 years old.

7.4 SUMMARY

- *Exponential growth* with growth constant $k > 0$: $P(t) = P_0 e^{kt}$.
- *Exponential decay* with decay constant $k > 0$: $P(t) = P_0 e^{-kt}$.
- The solutions of the differential equation $y' = ky$ are the exponential functions $y = Ce^{kt}$, where C is a constant.
- A quantity $P(t)$ grows exponentially if it grows at a rate proportional to its size—that is, if $P'(t) = kP(t)$.
- The *doubling time* for exponential growth and the *half-life* for exponential decay are both equal to $(\ln 2)/k$.
- For use in carbon dating: the decay constant of C^{14} is $k = 0.000121$.

7.4 EXERCISES

Preliminary Questions

1. Two quantities increase exponentially with growth constants $k = 1.2$ and $k = 3.4$, respectively. Which quantity doubles more rapidly?

2. A cell population grows exponentially beginning with one cell. Which takes longer: increasing from one to two cells or increasing from 15 million to 20 million cells?

3. Referring to his popular book *A Brief History of Time*, the renowned physicist Stephen Hawking said, "Someone told me that each equation I included in the book would halve its sales." Find a differential equation satisfied by the function $S(n)$, the number of copies sold if the book has n equations.

4. Carbon dating is based on the assumption that the ratio R of C^{14} to C^{12} in the atmosphere has been constant over the past 50,000 years. If R were actually smaller in the past than it is today, would the age estimates produced by carbon dating be too ancient or too recent?

Exercises

1. A certain population P of bacteria obeys the exponential growth law $P(t) = 2000e^{1.3t}$ (t in hours).

(a) How many bacteria are present initially?

(b) At what time will there be 10,000 bacteria?

2. A quantity P obeys the exponential growth law $P(t) = e^{5t}$ (t in years).

(a) At what time t is $P = 10$?

(b) What is the doubling time for P?

3. Write $f(t) = 5(7)^t$ in the form $f(t) = P_0 e^{kt}$ for some P_0 and k.

4. Write $f(t) = 9e^{1.4t}$ in the form $f(t) = P_0 b^t$ for some P_0 and b.

5. A certain RNA molecule replicates every 3 minutes. Find the differential equation for the number $N(t)$ of molecules present at time t (in minutes). How many molecules will be present after one hour if there is one molecule at $t = 0$?

6. A quantity P obeys the exponential growth law $P(t) = Ce^{kt}$ (t in years). Find the formula for $P(t)$, assuming that the doubling time is 7 years and $P(0) = 100$.

7. Find all solutions to the differential equation $y' = -5y$. Which solution satisfies the initial condition $y(0) = 3.4$?

8. Find the solution to $y' = \sqrt{2}y$ satisfying $y(0) = 20$.

9. Find the solution to $y' = 3y$ satisfying $y(2) = 1000$.

10. Find the function $y = f(t)$ that satisfies the differential equation $y' = -0.7y$ and the initial condition $y(0) = 10$.

11. The decay constant of cobalt-60 is 0.13 year^{-1}. Find its half-life.

12. The half-life radium-226 is 1622 years. Find its decay constant.

13. One of the world's smallest flowering plants, *Wolffia globosa* (Figure 11), has a doubling time of approximately 30 hours. Find the growth constant k and determine the initial population if the population grew to 1000 after 48 hours.

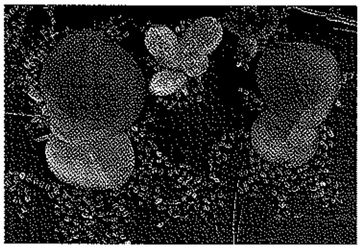


FIGURE 11 The tiny plants are *Wolffia*, with plant bodies smaller than the head of a pin.

14. A 10-kg quantity of a radioactive isotope decays to 3 kg after 17 years. Find the decay constant of the isotope.

15. The population of a city is $P(t) = 2 \cdot e^{0.06t}$ (in millions), where t is measured in years. Calculate the time it takes for the population to double, to triple, and to increase seven-fold.

16. What is the differential equation satisfied by $P(t)$, the number of infected computer hosts in Example 4? Over which time interval would $P(t)$ increase one hundred-fold?

17. The decay constant for a certain drug is $k = 0.35 \text{ day}^{-1}$. Calculate the time it takes for the quantity present in the bloodstream to decrease by half, by one-third, and by one-tenth.

18. **Light Intensity** The intensity of light passing through an absorbing medium decreases exponentially with the distance traveled. Suppose the decay constant for a certain plastic block is $k = 4 \text{ m}^{-1}$. How thick must the block be to reduce the intensity by a factor of one-third?

19. Assuming that population growth is approximately exponential, which of the following two sets of data is most likely to represent the population (in millions) of a city over a 5-year period?

Year	2000	2001	2002	2003	2004
Set I	3.14	3.36	3.60	3.85	4.11
Set II	3.14	3.24	3.54	4.04	4.74

20. The **atmospheric pressure** $P(h)$ (in kilopascals) at a height h meters above sea level satisfies a differential equation $P' = -kP$ for some positive constant k.

(a) Barometric measurements show that $P(0) = 101.3$ and $P(30,900) = 1.013$. What is the decay constant k?

(b) Determine the atmospheric pressure at $h = 500$.

21. Degrees in Physics One study suggests that from 1955 to 1970, the number of bachelor's degrees in physics awarded per year by U.S. universities grew exponentially, with growth constant $k = 0.1$.

(a) If exponential growth continues, how long will it take for the number of degrees awarded per year to increase 14-fold?

(b) If 2500 degrees were awarded in 1955, in which year were 10,000 degrees awarded?

22. The **Beer–Lambert Law** is used in spectroscopy to determine the molar absorptivity α or the concentration c of a compound dissolved in a solution at low concentrations (Figure 12). The law states that the intensity I of light as it passes through the solution satisfies $\ln(I/I_0) = \alpha c x$, where I_0 is the initial intensity and x is the distance traveled by the light. Show that I satisfies a differential equation $dI/dx = -kI$ for some constant k.

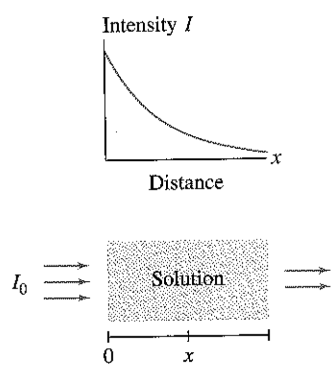


FIGURE 12 Light of intensity I_0 passing through a solution.

23. A sample of sheepskin parchment discovered by archaeologists had a C^{14}-to-C^{12} ratio equal to 40% of that found in the atmosphere. Approximately how old is the parchment?

24. Chauvet Caves In 1994, three French speleologists (geologists specializing in caves) discovered a cave in southern France containing prehistoric cave paintings. A C^{14} analysis carried out by archeologist Helene Valladas showed the paintings to be between 29,700 and 32,400 years old, much older than any previously known human art. Given that the C^{14}-to-C^{12} ratio of the atmosphere is $R = 10^{-12}$, what range of C^{14}-to-C^{12} ratios did Valladas find in the charcoal specimens?

25. A paleontologist discovers remains of animals that appear to have died at the onset of the Holocene ice age, between 10,000 and 12,000 years ago. What range of C^{14}-to-C^{12} ratio would the scientist expect to find in the animal remains?

26. Inversion of Sugar When cane sugar is dissolved in water, it converts to invert sugar over a period of several hours. The percentage $f(t)$ of unconverted cane sugar at time t (in hours) satisfies $f' = -0.2f$. What percentage of cane sugar remains after 5 hours? After 10 hours?

27. Continuing with Exercise 26, suppose that 50 grams of sugar are dissolved in a container of water. After how many hours will 20 grams of invert sugar be present?

28. Two bacteria colonies are cultivated in a laboratory. The first colony has a doubling time of 2 hours and the second a doubling time of 3 hours. Initially, the first colony contains 1000 bacteria and the second colony 3000 bacteria. At what time t will the sizes of the colonies be equal?

29. Moore's Law In 1965, Gordon Moore predicted that the number N of transistors on a microchip would increase exponentially.

(a) Does the table of data below confirm Moore's prediction for the period from 1971 to 2000? If so, estimate the growth constant k.

(b) CAS Plot the data in the table.

(c) Let $N(t)$ be the number of transistors t years after 1971. Find an approximate formula $N(t) \approx Ce^{kt}$, where t is the number of years after 1971.

(d) Estimate the doubling time in Moore's Law for the period from 1971 to 2000.

(e) How many transistors will a chip contain in 2015 if Moore's Law continues to hold?

(f) Can Moore have expected his prediction to hold indefinitely?

Processor	Year	No. Transistors
4004	1971	2250
8008	1972	2500
8080	1974	5000
8086	1978	29,000
286	1982	120,000
386 processor	1985	275,000
486 DX processor	1989	1,180,000
Pentium processor	1993	3,100,000
Pentium II processor	1997	7,500,000
Pentium III processor	1999	24,000,000
Pentium 4 processor	2000	42,000,000
Xeon processor	2008	1,900,000,000

30. Assume that in a certain country, the rate at which jobs are created is proportional to the number of people who already have jobs. If there are 15 million jobs at $t = 0$ and 15.1 million jobs 3 months later, how many jobs will there be after 2 years?

31. The only functions with a *constant* doubling time are the exponential functions P_0e^{kt} with $k > 0$. Show that the doubling time of linear function $f(t) = at + b$ at time t_0 is $t_0 + b/a$ (which increases with t_0). Compute the doubling times of $f(t) = 3t + 12$ at $t_0 = 10$ and $t_0 = 20$.

32. Verify that the half-life of a quantity that decays exponentially with decay constant k is equal to $(\ln 2)/k$.

33. **Drug Dosing Interval** Let $y(t)$ be the drug concentration (in mg/kg) in a patient's body at time t. The initial concentration is $y(0) = L$. Additional doses that increase the concentration by an amount d are administered at regular time intervals of length T. In between doses, $y(t)$ decays exponentially—that is, $y' = -ky$. Find the value of T (in terms of k and d) for which the the concentration varies between L and $L - d$ as in Figure 13.

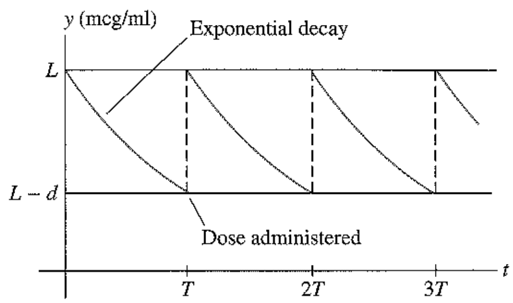


FIGURE 13 Drug concentration with periodic doses.

Exercises 34 and 35: The ***Gompertz differential equation***

$$\frac{dy}{dt} = ky \ln\left(\frac{y}{M}\right) \qquad \boxed{2}$$

(where M and k are constants) was introduced in 1825 by the English mathematician Benjamin Gompertz and is still used today to model aging and mortality.

34. Show that $y = Me^{ae^{kt}}$ satisfies Eq. (2) for any constant a.

35. To model mortality in a population of 200 laboratory rats, a scientist assumes that the number $P(t)$ of rats alive at time t (in months) satisfies Eq. (2) with $M = 204$ and $k = 0.15$ month^{-1} (Figure 14). Find $P(t)$ [note that $P(0) = 200$] and determine the population after 20 months.

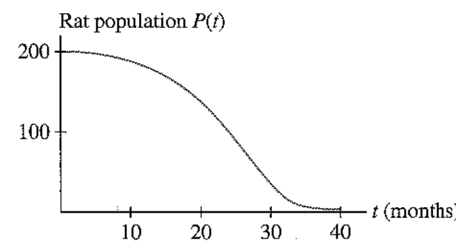


FIGURE 14

36. **Isotopes for Dating** Which of the following would be most suitable for dating extremely old rocks: carbon-14 (half-life 5570 years), lead-210 (half-life 22.26 years), or potassium-49 (half-life 1.3 billion years)? Explain why.

Further Insights and Challenges

37. Let $P = P(t)$ be a quantity that obeys an exponential growth law with growth constant k. Show that P increases m-fold after an interval of $(\ln m)/k$ years.

38. **Average Time of Decay** Physicists use the radioactive decay law $R = R_0 e^{-kt}$ to compute the average or *mean time* M until an atom decays. Let $F(t) = R/R_0 = e^{-kt}$ be the fraction of atoms that have survived to time t without decaying.

(a) Find the inverse function $t(F)$.

(b) By definition of $t(F)$, a fraction $1/N$ of atoms decays in the time interval

$$\left[t\left(\frac{j}{N}\right), t\left(\frac{j-1}{N}\right)\right]$$

Use this to justify the approximation $M \approx \frac{1}{N}\sum_{j=1}^{N} t\left(\frac{j}{N}\right)$. Then argue, by passing to the limit as $N \to \infty$, that $M = \int_0^1 t(F)\,dF$. Strictly speaking, this is an *improper integral* because $t(0)$ is infinite (it takes an infinite amount of time for all atoms to decay). Therefore, we define M as a limit

$$M = \lim_{c \to 0} \int_c^1 t(F)\,dF$$

(c) Verify the formula $\int \ln x\,dx = x\ln x - x$ by differentiation and use it to show that for $c > 0$,

$$M = \lim_{c \to 0}\left(\frac{1}{k} + \frac{1}{k}(c\ln c - c)\right)$$

(d) Verify numerically that $\lim_{c \to 0}(c - \ln c) = 0$ (we will prove this using L'Hôpital's Rule in Section 7.7). Use this to show that $M = 1/k$.

(e) What is the mean time to decay for radon (with a half-life of 3.825 days)?

7.5 Compound Interest and Present Value

Exponential functions are used extensively in financial calculations. Two basic applications are compound interest and present value.

Convention: Time t is measured in years and interest rates are given as yearly rates, either as a decimal or as a percentage. Thus, $r = 0.05$ corresponds to an interest rate of 5% per year.

When a sum of money P_0, called the **principal**, is deposited into an interest-bearing account, the amount or **balance** in the account at time t depends on two factors: the **interest rate** r and frequency with which interest is **compounded**. Interest paid out once a year at the end of the year is said to be *compounded annually*. The balance increases by the factor $(1 + r)$ after each year, leading to exponential growth: