Multiply through by n, using the rule $n \ln a = \ln a^n$:

$$\frac{n}{n+1} \le \ln\left(\left(1+\frac{1}{n}\right)^n\right) \le 1$$

Since $\lim_{n\to\infty} \frac{n}{n+1} = 1$, the middle quantity must approach 1 by the Squeeze Theorem:

$$\lim_{n\to\infty} \ln\left(\left(1+\frac{1}{n}\right)^n\right) = 1$$

Now we can apply e^x (because it is continuous) to obtain the desired result:

$$e^1 = e^{\lim_{n\to\infty} \ln\left(\left(1+\frac{1}{n}\right)^n\right)} = \lim_{n\to\infty} e^{\ln\left(\left(1+\frac{1}{n}\right)^n\right)} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

See Exercise 27 for a proof of the more general formula $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

7.5 SUMMARY

- Interest rate r, compounded M times per year:

$$P(t) = P_0(1 + r/M)^{Mt}$$

- Interest rate r, compounded continuously: $P(t) = P_0 e^{rt}$.
- The *present value* (PV) of P dollars (or other currency), to be paid t years in the future, is Pe^{-rt}.
- Present value of an income stream paying $R(t)$ dollars per year continuously for T years:

$$\text{PV} = \int_0^T R(t)e^{-rt}\,dt$$

7.5 EXERCISES

Preliminary Questions

1. Which is preferable: an interest rate of 12% compounded quarterly, or an interest rate of 11% compounded continuously?

2. Find the yearly multiplier if $r = 9\%$ and interest is compounded (a) continuously and (b) quarterly.

3. The PV of N dollars received at time T is (choose the correct answer):

(a) The value at time T of N dollars invested today

(b) The amount you would have to invest today in order to receive N dollars at time T

4. In one year, you will be paid \$1. Will the PV increase or decrease if the interest rate goes up?

5. Xavier expects to receive a check for \$1000 one year from today. Explain using the concept of PV, whether he will be happy or sad to learn that the interest rate has just increased from 6% to 7%.

Exercises

1. Compute the balance after 10 years if \$2000 is deposited in an account paying 9% interest and interest is compounded (a) quarterly, (b) monthly, and (c) continuously.

2. Suppose \$500 is deposited into an account paying interest at a rate of 7%, continuously compounded. Find a formula for the value of the account at time t. What is the value of the account after 3 years?

3. A bank pays interest at a rate of 5%. What is the yearly multiplier if interest is compounded

(a) three times a year? **(b)** continuously?

4. How long will it take for \$4000 to double in value if it is deposited in an account bearing 7% interest, continuously compounded?

5. How much must one invest today in order to receive \$20,000 after 5 years if interest is compounded continuously at the rate $r = 9\%$?

6. An investment increases in value at a continuously compounded rate of 9%. How large must the initial investment be in order to build up a value of \$50,000 over a 7-year period?

7. Compute the PV of \$5000 received in 3 years if the interest rate is (a) 6% and (b) 11%. What is the PV in these two cases if the sum is instead received in 5 years?

8. Is it better to receive \$1000 today or \$1300 in 4 years? Consider $r = 0.08$ and $r = 0.03$.

9. Find the interest rate r if the PV of \$8000 to be received in 1 year is \$7300.

10. A company can earn additional profits of \$500,000/year for 5 years by investing \$2 million to upgrade its factory. Is the investment worthwhile if the interest rate is 6%? (Assume the savings are received as a lump sum at the end of each year.)

11. A new computer system costing \$25,000 will reduce labor costs by \$7,000/year for 5 years.

(a) Is it a good investment if $r = 8\%$?

(b) How much money will the company actually save?

12. After winning \$25 million in the state lottery, Jessica learns that she will receive five yearly payments of \$5 million beginning immediately.

(a) What is the PV of Jessica's prize if $r = 6\%$?

(b) How much more would the prize be worth if the entire amount were paid today?

13. Use Eq. (2) to compute the PV of an income stream paying out $R(t) = \$5000$/year continuously for 10 years, assuming $r = 0.05$.

14. Find the PV of an investment that pays out continuously at a rate of \$800/year for 5 years, assuming $r = 0.08$.

15. Find the PV of an income stream that pays out continuously at a rate $R(t) = \$5000e^{0.1t}$/year for 7 years, assuming $r = 0.05$.

16. A commercial property generates income at the rate $R(t)$. Suppose that $R(0) = \$70,000$/year and that $R(t)$ increases at a continuously compounded rate of 5%. Find the PV of the income generated in the first 4 years if $r = 6\%$.

17. Show that an investment that pays out R dollars per year continuously for T years has a PV of $R(1 - e^{-rT})/r$.

18. Explain this statement: If T is very large, then the PV of the income stream described in Exercise 17 is approximately R/r.

19. Suppose that $r = 0.06$. Use the result of Exercise 18 to estimate the payout rate R needed to produce an income stream whose PV is \$20,000, assuming that the stream continues for a large number of years.

20. Verify by differentiation:

$$\int te^{-rt}\,dt = -\frac{e^{-rt}(1 + rt)}{r^2} + C \qquad \boxed{4}$$

Use Eq. (4) to compute the PV of an investment that pays out income continuously at a rate $R(t) = (5000 + 1000t)$ dollars per year for 5 years, assuming $r = 0.05$.

21. Use Eq. (4) to compute the PV of an investment that pays out income continuously at a rate $R(t) = (5000 + 1000t)e^{0.02t}$ dollars per year for 10 years, assuming $r = 0.08$.

22. **Banker's Rule of 70** If you earn an interest rate of R percent, continuously compounded, your money doubles after approximately $70/R$ years. For example, at $R = 5\%$, your money doubles after $70/5$ or 14 years. Use the concept of doubling time to justify the Banker's Rule. (*Note:* Sometimes, the rule $72/R$ is used. It is less accurate but easier to apply because 72 is divisible by more numbers than 70.)

In Exercises 23–26, calculate the limit.

23. $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{6n}$

24. $\displaystyle\lim_{n\to\infty}\left(1 + \frac{3}{n}\right)^{n}$

25. $\displaystyle\lim_{n\to\infty}\left(1 + \frac{3}{n}\right)^{2n}$

26. $\displaystyle\lim_{n\to\infty}\left(1 + \frac{1}{4n}\right)^{12n}$

Further Insights and Challenges

27. Modify the proof of the relation $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ given in the text to prove $e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$. *Hint:* Express $\ln(1 + xn^{-1})$ as an integral and estimate above and below by rectangles.

28. Prove that, for $n > 0$,

$$\left(1 + \frac{1}{n}\right)^n \le e \le \left(1 + \frac{1}{n}\right)^{n+1}$$

Hint: Take logarithms and use Eq. (3).

29. A bank pays interest at the rate r, compounded M times yearly. The **effective interest rate** r_e is the rate at which interest, if compounded annually, would have to be paid to produce the same yearly return.

(a) Find r_e if $r = 9\%$ compounded monthly.

(b) Show that $r_e = (1 + r/M)^M - 1$ and that $r_e = e^r - 1$ if interest is compounded continuously.

(c) Find r_e if $r = 11\%$ compounded continuously.

(d) Find the rate r that, compounded weekly, would yield an effective rate of 20%.

7.6 SUMMARY

- The general solution of $y' = k(y - b)$ is $y = b + Ce^{kt}$, where C is a constant.
- The following tables describe the solutions to $y' = k(y - b)$ (see Figure 5).

Equation ($k > 0$)	Solution	Behavior as $t \to \infty$
$y' = k(y - b)$	$y(t) = b + Ce^{kt}$	$\lim_{t\to\infty} y(t) = \begin{cases} \infty & \text{if } C > 0 \\ -\infty & \text{if } C < 0 \end{cases}$
$y' = -k(y - b)$	$y(t) = b + Ce^{-kt}$	$\lim_{t\to\infty} y(t) = b$

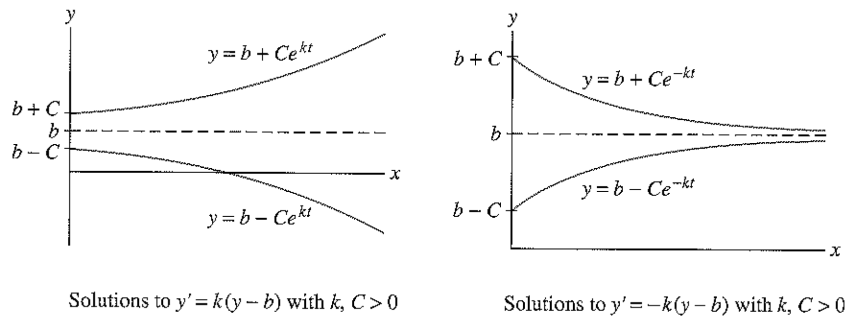

FIGURE 5

- Three applications:
  - Newton's law of cooling: $y' = -k(y - T_0)$, $y(t)$ = temperature of the object, T_0 = ambient temperature, k = cooling constant
  - Free-fall with air resistance: $v' = -\frac{k}{m}\left(v + \frac{mg}{k}\right)$, $v(t)$ = velocity, m = mass, k = air resistance constant, g = acceleration due to gravity
  - Continuous annuity: $P' = r\left(P - \frac{N}{r}\right)$, $P(t)$ = balance in the annuity, r = interest rate, N = withdrawal rate

7.6 EXERCISES

Preliminary Questions

1. Write down a solution to $y' = 4(y - 5)$ that tends to $-\infty$ as $t \to \infty$.

2. Does $y' = -4(y - 5)$ have a solution that tends to ∞ as $t \to \infty$?

3. True or false? If $k > 0$, then all solutions of $y' = -k(y - b)$ approach the same limit as $t \to \infty$.

4. As an object cools, its rate of cooling slows. Explain how this follows from Newton's Law of Cooling.

Exercises

1. Find the general solution of

$$y' = 2(y - 10)$$

Then find the two solutions satisfying $y(0) = 25$ and $y(0) = 5$, and sketch their graphs.

2. Verify directly that $y = 12 + Ce^{-3t}$ satisfies

$$y' = -3(y - 12) \quad \text{for all } C$$

Then find the two solutions satisfying $y(0) = 20$ and $y(0) = 0$, and sketch their graphs.

3. Solve $y' = 4y + 24$ subject to $y(0) = 5$.

4. Solve $y' + 6y = 12$ subject to $y(2) = 10$.

In Exercises 5–12, use Newton's Law of Cooling.

5. A hot anvil with cooling constant $k = 0.02\ \text{s}^{-1}$ is submerged in a large pool of water whose temperature is 10°C. Let $y(t)$ be the anvil's temperature t seconds later.

(a) What is the differential equation satisfied by $y(t)$?

(b) Find a formula for $y(t)$, assuming the object's initial temperature is 100°C.

(c) How long does it take the object to cool down to 20°?

6. Frank's automobile engine runs at 100°C. On a day when the outside temperature is 21°C, he turns off the ignition and notes that five minutes later, the engine has cooled to 70°C.

(a) Determine the engine's cooling constant k.

(b) What is the formula for $y(t)$?

(c) When will the engine cool to 40°C?

7. At 10:30 AM, detectives discover a dead body in a room and measure its temperature at 26°C. One hour later, the body's temperature had dropped to 24.8°C. Determine the time of death (when the body temperature was a normal 37°C), assuming that the temperature in the room was held constant at 20°C.

8. A cup of coffee with cooling constant $k = 0.09\ \text{min}^{-1}$ is placed in a room at temperature 20°C.

(a) How fast is the coffee cooling (in degrees per minute) when its temperature is $T = 80$°C?

(b) Use the Linear Approximation to estimate the change in temperature over the next 6 s when $T = 80$°C.

(c) If the coffee is served at 90°C, how long will it take to reach an optimal drinking temperature of 65°C?

9. A cold metal bar at −30°C is submerged in a pool maintained at a temperature of 40°C. Half a minute later, the temperature of the bar is 20°C. How long will it take for the bar to attain a temperature of 30°C?

10. When a hot object is placed in a water bath whose temperature is 25°C, it cools from 100 to 50°C in 150 s. In another bath, the same cooling occurs in 120 s. Find the temperature of the second bath.

11. GU Objects A and B are placed in a warm bath at temperature $T_0 = 40$°C. Object A has initial temperature −20°C and cooling constant $k = 0.004\ \text{s}^{-1}$. Object B has initial temperature 0°C and cooling constant $k = 0.002\ \text{s}^{-1}$. Plot the temperatures of A and B for $0 \le t \le 1000$. After how many seconds will the objects have the same temperature?

12. In Newton's Law of Cooling, the constant $\tau = 1/k$ is called the "characteristic time." Show that τ is the time required for the temperature difference $(y - T_0)$ to decrease by the factor $e^{-1} \approx 0.37$. For example, if $y(0) = 100$°C and $T_0 = 0$°C, then the object cools to $100/e \approx 37$°C in time τ, to $100/e^2 \approx 13.5$°C in time 2τ, and so on.

In Exercises 13–16, use Eq. (3) as a model for free-fall with air resistance.

13. A 60-kg skydiver jumps out of an airplane. What is her terminal velocity, in meters per second, assuming that $k = 10$ kg/s for free-fall (no parachute)?

14. Find the terminal velocity of a skydiver of weight $w = 192$ lb if $k = 1.2$ lb-s/ft. How long does it take him to reach half of his terminal velocity if his initial velocity is zero? Mass and weight are related by $w = mg$, and Eq. (3) becomes $v' = -(kg/w)(v + w/k)$ with $g = 32$ ft/s^2.

15. A 80-kg skydiver jumps out of an airplane (with zero initial velocity). Assume that $k = 12$ kg/s with a closed parachute and $k = 70$ kg/s with an open parachute. What is the skydiver's velocity at $t = 25$ s if the parachute opens after 20 s of free fall?

16. Does a heavier or a lighter skydiver reach terminal velocity faster?

17. A continuous annuity with withdrawal rate $N = \$5000$/year and interest rate $r = 5\%$ is funded by an initial deposit of $P_0 = \$50{,}000$.

(a) What is the balance in the annuity after 10 years?

(b) When will the annuity run out of funds?

18. Show that a continuous annuity with withdrawal rate $N = \$5000$/year and interest rate $r = 8\%$, funded by an initial deposit of $P_0 = \$75{,}000$, never runs out of money.

19. Find the minimum initial deposit P_0 that will allow an annuity to pay out \$6000/year indefinitely if it earns interest at a rate of 5%.

20. Find the minimum initial deposit P_0 necessary to fund an annuity for 20 years if withdrawals are made at a rate of \$10,000/year and interest is earned at a rate of 7%.

21. An initial deposit of 100,000 euros are placed in an annuity with a French bank. What is the minimum interest rate the annuity must earn to allow withdrawals at a rate of 8000 euros/year to continue indefinitely?

22. Show that a continuous annuity never runs out of money if the initial balance is greater than or equal to N/r, where N is the withdrawal rate and r the interest rate.

23. Sam borrows \$10,000 from a bank at an interest rate of 9% and pays back the loan continuously at a rate of N dollars per year. Let $P(t)$ denote the amount still owed at time t.

(a) Explain why $P(t)$ satisfies the differential equation

$$y' = 0.09y - N$$

(b) How long will it take Sam to pay back the loan if $N = \$1200$?

(c) Will the loan ever be paid back if $N = \$800$?

24. April borrows \$18,000 at an interest rate of 5% to purchase a new automobile. At what rate (in dollars per year) must she pay back the loan, if the loan must be paid off in 5 years? *Hint:* Set up the differential equation as in Exercise 23).

25. Let $N(t)$ be the fraction of the population who have heard a given piece of news t hours after its initial release. According to one model, the rate $N'(t)$ at which the news spreads is equal to k times the fraction of the population that has not yet heard the news, for some constant $k > 0$.

(a) Determine the differential equation satisfied by $N(t)$.

(b) Find the solution of this differential equation with the initial condition $N(0) = 0$ in terms of k.

(c) Suppose that half of the population is aware of an earthquake 8 hours after it occurs. Use the model to calculate k and estimate the percentage that will know about the earthquake 12 hours after it occurs.

26. Current in a Circuit When the circuit in Figure 6 (which consists of a battery of V volts, a resistor of R ohms, and an inductor of L henries) is connected, the current $I(t)$ flowing in the circuit satisfies

$$L\frac{dI}{dt} + RI = V$$

with the initial condition $I(0) = 0$.

(a) Find a formula for $I(t)$ in terms of L, V, and R.

(b) Show that $\lim_{t\to\infty} I(t) = V/R$.

(c) Show that $I(t)$ reaches approximately 63% of its maximum value at the "characteristic time" $\tau = L/R$.

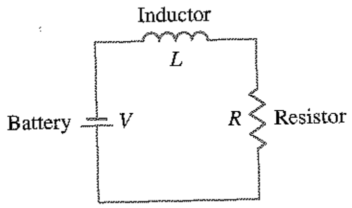

FIGURE 6 Current flow approaches the level $I_{\max} = V/R$.

Further Insights and Challenges

27. Show that the cooling constant of an object can be determined from two temperature readings $y(t_1)$ and $y(t_2)$ at times $t_1 \neq t_2$ by the formula

$$k = \frac{1}{t_1 - t_2}\ln\left(\frac{y(t_2) - T_0}{y(t_1) - T_0}\right)$$

28. Show that by Newton's Law of Cooling, the time required to cool an object from temperature A to temperature B is

$$t = \frac{1}{k}\ln\left(\frac{A - T_0}{B - T_0}\right)$$

where T_0 is the ambient temperature.

29. Air Resistance A projectile of mass $m = 1$ travels straight up from ground level with initial velocity v_0. Suppose that the velocity v satisfies $v' = -g - kv$.

(a) Find a formula for $v(t)$.

(b) Show that the projectile's height $h(t)$ is given by

$$h(t) = C(1 - e^{-kt}) - \frac{g}{k}t$$

where $C = k^{-2}(g + kv_0)$.

(c) Show that the projectile reaches its maximum height at time $t_{\max} = k^{-1}\ln(1 + kv_0/g)$.

(d) In the absence of air resistance, the maximum height is reached at time $t = v_0/g$. In view of this, explain why we should expect that

$$\lim_{k\to 0}\frac{\ln(1 + \frac{kv_0}{g})}{k} = \frac{v_0}{g} \qquad \boxed{8}$$

(e) Verify Eq. (8). *Hint:* Use Theorem 1 in Section 7.5 to show that $\lim_{k\to 0}\left(1 + \frac{kv_0}{g}\right)^{1/k} = e^{v_0/g}$.

7.7 L'Hôpital's Rule

L'Hôpital's Rule is named for the French mathematician Guillaume François Antoine Marquis de L'Hôpital (1661–1704), who wrote the first textbook on calculus in 1696. The name L'Hôpital is pronounced "Lo-pee-tal."

L'Hôpital's Rule is a valuable tool for computing certain limits that are otherwise difficult to evaluate, and also for determining "asymptotic behavior" (limits at infinity).

Consider the limit of a quotient

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

Roughly speaking, L'Hôpital's Rule states that *when $f(x)/g(x)$ has an indeterminate form of type $0/0$ or ∞/∞ at $x = a$, then we can replace $f(x)/g(x)$ by the quotient of the derivatives $f'(x)/g'(x)$.*

Exercises

In Exercises 1–10, use L'Hôpital's Rule to evaluate the limit, or state that L'Hôpital's Rule does not apply.

1. $\lim_{x\to 3} \frac{2x^2-5x-3}{x-4}$

2. $\lim_{x\to -5} \frac{x^2-25}{5-4x-x^2}$

3. $\lim_{x\to 4} \frac{x^3-64}{x^2+16}$

4. $\lim_{x\to -1} \frac{x^4+2x+1}{x^5-2x-1}$

5. $\lim_{x\to 9} \frac{x^{1/2}+x-6}{x^{3/2}-27}$

6. $\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x^3-7x-6}$

7. $\lim_{x\to 0} \frac{\sin 4x}{x^2+3x+1}$

8. $\lim_{x\to 0} \frac{x^3}{\sin x - x}$

9. $\lim_{x\to 0} \frac{\cos 2x - 1}{\sin 5x}$

10. $\lim_{x\to 0} \frac{\cos x - \sin^2 x}{\sin x}$

In Exercises 11–16, show that L'Hôpital's Rule is applicable to the limit as $x \to \pm\infty$ and evaluate.

11. $\lim_{x\to\infty} \frac{9x+4}{3-2x}$

12. $\lim_{x\to -\infty} x \sin \frac{1}{x}$

13. $\lim_{x\to\infty} \frac{\ln x}{x^{1/2}}$

14. $\lim_{x\to\infty} \frac{x}{e^x}$

15. $\lim_{x\to -\infty} \frac{\ln(x^4+1)}{x}$

16. $\lim_{x\to\infty} \frac{x^2}{e^x}$

In Exercises 17–50, evaluate the limit.

17. $\lim_{x\to 1} \frac{\sqrt{8+x}-3x^{1/3}}{x^2-3x+2}$

18. $\lim_{x\to 4} \left[\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right]$

19. $\lim_{x\to -\infty} \frac{3x-2}{1-5x}$

20. $\lim_{x\to\infty} \frac{x^{2/3}+3x}{x^{5/3}-x}$

21. $\lim_{x\to -\infty} \frac{7x^2+4x}{9-3x^2}$

22. $\lim_{x\to\infty} \frac{3x^3+4x^2}{4x^3-7}$

23. $\lim_{x\to 1} \frac{(1+3x)^{1/2}-2}{(1+7x)^{1/3}-2}$

24. $\lim_{x\to 8} \frac{x^{5/3}-2x-16}{x^{1/3}-2}$

25. $\lim_{x\to 0} \frac{\sin 2x}{\sin 7x}$

26. $\lim_{x\to \pi/2} \frac{\tan 4x}{\tan 5x}$

27. $\lim_{x\to 0} \frac{\tan x}{x}$

28. $\lim_{x\to 0} \left(\cot x - \frac{1}{x}\right)$

29. $\lim_{x\to 0} \frac{\sin x - x\cos x}{x - \sin x}$

30. $\lim_{x\to \pi/2} \left(x - \frac{\pi}{2}\right) \tan x$

31. $\lim_{x\to 0} \frac{\cos(x+\frac{\pi}{2})}{\sin x}$

32. $\lim_{x\to 0} \frac{x^2}{1-\cos x}$

33. $\lim_{x\to \pi/2} \frac{\cos x}{\sin(2x)}$

34. $\lim_{x\to 0} \left(\frac{1}{x^2} - \csc^2 x\right)$

35. $\lim_{x\to \pi/2} (\sec x - \tan x)$

36. $\lim_{x\to 2} \frac{e^{x^2}-e^4}{x-2}$

37. $\lim_{x\to 1} \tan\left(\frac{\pi x}{2}\right) \ln x$

38. $\lim_{x\to 1} \frac{x(\ln x - 1)+1}{(x-1)\ln x}$

39. $\lim_{x\to 0} \frac{e^x-1}{\sin x}$

40. $\lim_{x\to 1} \frac{e^x-e}{\ln x}$

41. $\lim_{x\to 0} \frac{e^{2x}-1-x}{x^2}$

42. $\lim_{x\to\infty} \frac{e^{2x}-1-x}{x^2}$

43. $\lim_{t\to 0+} (\sin t)(\ln t)$

44. $\lim_{x\to\infty} e^{-x}(x^3-x^2+9)$

45. $\lim_{x\to 0} \frac{a^x-1}{x}$ $(a > 0)$

46. $\lim_{x\to\infty} x^{1/x^2}$

47. $\lim_{x\to 1} (1+\ln x)^{1/(x-1)}$

48. $\lim_{x\to 0+} x^{\sin x}$

49. $\lim_{x\to 0} (\cos x)^{3/x^2}$

50. $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$

51. Evaluate $\lim_{x\to \pi/2} \frac{\cos mx}{\cos nx}$, where $m, n \neq 0$ are integers.

52. Evaluate $\lim_{x\to 1} \frac{x^m-1}{x^n-1}$ for any numbers $m, n \neq 0$.

53. Prove the following limit formula for e:

$$e = \lim_{x\to 0} (1+x)^{1/x}$$

Then find a value of x such that $|(1+x)^{1/x} - e| \leq 0.001$.

54. GU Can L'Hôpital's Rule be applied to $\lim_{x\to 0+} x^{\sin(1/x)}$? Does a graphical or numerical investigation suggest that the limit exists?

55. Let $f(x) = x^{1/x}$ for $x > 0$.

(a) Calculate $\lim_{x\to 0+} f(x)$ and $\lim_{x\to\infty} f(x)$.

(b) Find the maximum value of $f(x)$, and determine the intervals on which $f(x)$ is increasing or decreasing.

56. (a) Use the results of Exercise 55 to prove that $x^{1/x} = c$ has a unique solution if $0 < c \leq 1$ or $c = e^{1/e}$, two solutions if $1 < c < e^{1/e}$, and no solutions if $c > e^{1/e}$.

(b) GU Plot the graph of $f(x) = x^{1/x}$ and verify that it confirms the conclusions of (a).

57. Determine whether $f << g$ or $g << f$ (or neither) for the functions $f(x) = \log_{10} x$ and $g(x) = \ln x$.

58. Show that $(\ln x)^2 << \sqrt{x}$ and $(\ln x)^4 << x^{1/10}$.

59. Just as exponential functions are distinguished by their rapid rate of increase, the logarithm functions grow particularly slowly. Show that $\ln x << x^a$ for all $a > 0$.

60. Show that $(\ln x)^N << x^a$ for all N and all $a > 0$.

61. Determine whether $\sqrt{x} << e^{\sqrt{\ln x}}$ or $e^{\sqrt{\ln x}} << \sqrt{x}$. *Hint:* Use the substitution $u = \ln x$ instead of L'Hôpital's Rule.

62. Show that $\lim_{x\to\infty} x^n e^{-x} = 0$ for all whole numbers $n > 0$.

63. Assumptions Matter Let $f(x) = x(2 + \sin x)$ and $g(x) = x^2 + 1$.

(a) Show directly that $\lim_{x\to\infty} f(x)/g(x) = 0$.

(b) Show that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, but $\lim_{x\to\infty} f'(x)/g'(x)$ does not exist.

Do (a) and (b) contradict L'Hôpital's Rule? Explain.

64. Let $H(b) = \lim_{x\to\infty} \dfrac{\ln(1 + b^x)}{x}$ for $b > 0$.

(a) Show that $H(b) = \ln b$ if $b \geq 1$

(b) Determine $H(b)$ for $0 < b \leq 1$.

65. Let $G(b) = \lim_{x\to\infty} (1 + b^x)^{1/x}$.

(a) Use the result of Exercise 64 to evaluate $G(b)$ for all $b > 0$.

(b) GU Verify your result graphically by plotting $y = (1 + b^x)^{1/x}$ together with the horizontal line $y = G(b)$ for the values $b = 0.25, 0.5, 2, 3$.

66. Show that $\lim_{t\to\infty} t^k e^{-t^2} = 0$ for all k. *Hint:* Compare with $\lim_{t\to\infty} t^k e^{-t} = 0$.

In Exercises 67–69, let

$$f(x) = \begin{cases} e^{-1/x^2} & \textit{for } x \neq 0 \\ 0 & \textit{for } x = 0 \end{cases}$$

These exercises show that $f(x)$ has an unusual property: All of its derivatives at $x = 0$ exist and are equal to zero.

67. Show that $\lim_{x\to 0} \dfrac{f(x)}{x^k} = 0$ for all k. *Hint:* Let $t = x^{-1}$ and apply the result of Exercise 66.

68. Show that $f'(0)$ exists and is equal to zero. Also, verify that $f''(0)$ exists and is equal to zero.

69. Show that for $k \geq 1$ and $x \neq 0$,

$$f^{(k)}(x) = \frac{P(x)e^{-1/x^2}}{x^r}$$

for some polynomial $P(x)$ and some exponent $r \geq 1$. Use the result of Exercise 67 to show that $f^{(k)}(0)$ exists and is equal to zero for all $k \geq 1$.

70. (a) Verify for $r \neq 0$:

$$\int_0^T te^{rt}\,dt = \frac{e^{rT}(rT - 1) + 1}{r^2} \qquad \boxed{1}$$

Hint: For fixed r, let $F(T)$ be the value of the integral on the left. By FTC II, $F'(t) = te^{rt}$ and $F(0) = 0$. Show that the same is true of the function on the right.

(b) Use L'Hôpital's Rule to show that for fixed T, the limit as $r \to 0$ of the right-hand side of Eq. (1) is equal to the value of the integral for $r = 0$.

71. The formula $\displaystyle\int_1^x t^n\,dt = \frac{x^{n+1} - 1}{n + 1}$ is valid for $n \neq -1$. Use L'Hôpital's Rule to prove that

$$\lim_{n\to -1} \frac{x^{n+1} - 1}{n + 1} = \ln x$$

Use this to show that

$$\lim_{n\to -1} \int_1^x t^n\,dt = \int_1^x t^{-1}\,dt$$

Thus, the definite integral of x^{-1} is a limit of the definite integrals of x^n as n approaches -1.

Further Insights and Challenges

72. Show that L'Hôpital's Rule applies to $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2 + 1}}$ but that it does not help. Then evaluate the limit directly.

73. The Second Derivative Test for critical points fails if $f''(c) = 0$. This exercise develops a **Higher Derivative Test** based on the sign of the first nonzero derivative. Suppose that

$$f'(c) = f''(c) = \cdots = f^{(n-1)}(c) = 0, \quad \text{but} \quad f^{(n)}(c) \neq 0$$

(a) Show, by applying L'Hôpital's Rule n times, that

$$\lim_{x\to c} \frac{f(x) - f(c)}{(x - c)^n} = \frac{1}{n!} f^{(n)}(c)$$

where $n! = n(n - 1)(n - 2) \cdots (2)(1)$.

(b) Use (a) to show that if n is even, then $f(c)$ is a local minimum if $f^{(n)}(c) > 0$ and is a local maximum if $f^{(n)}(c) < 0$. *Hint:* If n is even, then $(x - c)^n > 0$ for $x \neq a$, so $f(x) - f(c)$ must be positive for x near c if $f^{(n)}(c) > 0$.

(c) Use (a) to show that if n is odd, then $f(c)$ is neither a local minimum nor a local maximum.

74. When a spring with natural frequency $\lambda/2\pi$ is driven with a sinusoidal force $\sin(\omega t)$ with $\omega \neq \lambda$, it oscillates according to

$$y(t) = \frac{1}{\lambda^2 - \omega^2}\big(\lambda \sin(\omega t) - \omega \sin(\lambda t)\big)$$

Let $y_0(t) = \lim_{\omega\to\lambda} y(t)$.

(a) Use L'Hôpital's Rule to determine $y_0(t)$.

(b) Show that $y_0(t)$ ceases to be periodic and that its amplitude $|y_0(t)|$ tends to ∞ as $t \to \infty$ (the system is said to be in **resonance**; eventually, the spring is stretched beyond its limits).

(c) CAS Plot $y(t)$ for $\lambda = 1$ and $\omega = 0.8, 0.9, 0.99$, and 0.999. Do the graphs confirm your conclusion in (b)?

75. We expended a lot of effort to evaluate $\lim_{x\to 0} \dfrac{\sin x}{x}$ in Chapter 2. Show that we could have evaluated it easily using L'Hôpital's Rule. Then explain why this method would involve *circular reasoning*.

76. By a fact from algebra, if f, g are polynomials such that $f(a) = g(a) = 0$, then there are polynomials f_1, g_1 such that

$$f(x) = (x-a)f_1(x), \qquad g(x) = (x-a)g_1(x)$$

Use this to verify L'Hôpital's Rule directly for $\lim_{x\to a} f(x)/g(x)$.

77. Patience Required Use L'Hôpital's Rule to evaluate and check your answers numerically:

(a) $\lim_{x\to 0+} \left(\frac{\sin x}{x}\right)^{1/x^2}$ **(b)** $\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$

78. In the following cases, check that $x = c$ is a critical point and use Exercise 73 to determine whether $f(c)$ is a local minimum or a local maximum.

(a) $f(x) = x^5 - 6x^4 + 14x^3 - 16x^2 + 9x + 12 \quad (c = 1)$

(b) $f(x) = x^6 - x^3 \quad (c = 0)$

7.8 Inverse Trigonometric Functions

In this section, we discuss the inverse trigonometric functions and their derivatives. Recall that an inverse $f^{-1}(x)$ exists if and only if the function $f(x)$ is one-to-one on its domain. However, the trigonometric functions are not one-to-one (because they are periodic). Therefore, to define their inverses, we shall restrict their domains so that the resulting functions are one-to-one.

First consider the sine function. Figure 1 shows that $f(\theta) = \sin\theta$ is one-to-one on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. With this interval as domain, the inverse is called the **arcsine function** and is denoted $\theta = \sin^{-1} x$ or $\theta = \arcsin x$. By definition,

Do not confuse the inverse $\sin^{-1} x$ *with the reciprocal*

$$(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$$

The inverse functions $\sin^{-1} x$, $\cos^{-1} x, \ldots$ *are often denoted* $\arcsin x$, $\arccos x$, *etc.*

$$\theta = \sin^{-1} x \text{ is the unique angle in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ such that } \sin\theta = x$$

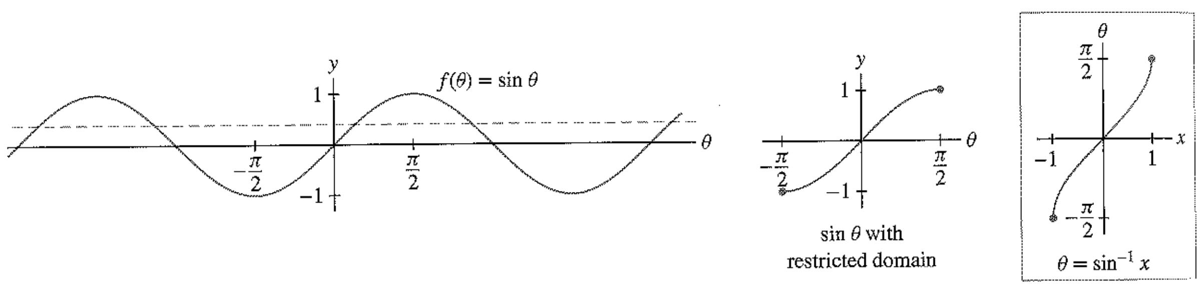

FIGURE 1

The range of $\sin x$ is $[-1, 1]$ and therefore $\sin^{-1} x$ has domain $[-1, 1]$. A table of values for the arcsine (Table 1) is obtained by reversing the columns in a table of values for $\sin x$.

Summary of inverse relation between the sine and arcsine functions:

$\sin(\sin^{-1} x) = x \quad$ for $-1 \le x \le 1$

$\sin^{-1}(\sin\theta) = \theta \quad$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

TABLE 1

x	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\theta = \sin^{-1} x$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$

EXAMPLE 1 **(a)** Show that $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$.

(b) Explain why $\sin^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) \ne \frac{5\pi}{4}$.

Solution The equation $\sin^{-1}(\sin\theta) = \theta$ is valid only if θ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(a) Since $\frac{\pi}{4}$ lies in the required interval, $\sin^{-1}\left(\sin\left(\frac{\pi}{4}\right)\right) = \frac{\pi}{4}$.

$$\int x^3 e^x\,dx = x^3e^x - 3\int x^2e^x\,dx$$
$$= x^3e^x - 3\left(x^2e^x - 2\int xe^x\,dx\right)$$
$$= x^3e^x - 3x^2e^x + 6\int xe^x\,dx$$
$$= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x\,dx\right)$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$
$$= (x^3 - 3x^2 + 6x - 6)e^x + C$$

8.1 SUMMARY

- *Integration by Parts* formula: $\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$.
- The key step is deciding how to write the integrand as a product uv'. Keep in mind that Integration by Parts is useful when $u'v$ is easier (or, at least, not more difficult) to integrate than uv'. Here are some guidelines:
  - Choose u so that u' is simpler than u itself.
  - Choose v' so that $v = \int v'\,dx$ can be evaluated.
  - Sometimes, $v' = 1$ is a good choice.

8.1 EXERCISES

Preliminary Questions

1. Which derivative rule is used to derive the Integration by Parts formula?

2. For each of the following integrals, state whether substitution or Integration by Parts should be used:

$$\int x\cos(x^2)\,dx, \quad \int x\cos x\,dx, \quad \int x^2e^x\,dx, \quad \int xe^{x^2}\,dx$$

3. Why is $u = \cos x$, $v' = x$ a poor choice for evaluating $\int x\cos x\,dx$?

Exercises

In Exercises 1–6, evaluate the integral using the Integration by Parts formula with the given choice of u and v'.

1. $\int x\sin x\,dx$; $u = x$, $v' = \sin x$

2. $\int xe^{2x}\,dx$; $u = x$, $v' = e^{2x}$

3. $\int (2x+9)e^x\,dx$; $u = 2x+9$, $v' = e^x$

4. $\int x\cos 4x\,dx$; $u = x$, $v' = \cos 4x$

5. $\int x^3\ln x\,dx$; $u = \ln x$, $v' = x^3$

6. $\int \tan^{-1}x\,dx$; $u = \tan^{-1}x$, $v' = 1$

In Exercises 7–36, evaluate using Integration by Parts.

7. $\int (4x-3)e^{-x}\,dx$

8. $\int (2x+1)e^x\,dx$

9. $\int x\,e^{5x+2}\,dx$

10. $\int x^2e^x\,dx$

11. $\int x\cos 2x\,dx$

12. $\int x\sin(3-x)\,dx$

13. $\int x^2\sin x\,dx$

14. $\int x^2\cos 3x\,dx$

15. $\int e^{-x}\sin x\,dx$

16. $\int e^x\sin 2x\,dx$

17. $\int e^{-5x} \sin x \, dx$

18. $\int e^{3x} \cos 4x \, dx$

19. $\int x \ln x \, dx$

20. $\int \frac{\ln x}{x^2} \, dx$

21. $\int x^2 \ln x \, dx$

22. $\int x^{-5} \ln x \, dx$

23. $\int (\ln x)^2 \, dx$

24. $\int x(\ln x)^2 \, dx$

25. $\int x \sec^2 x \, dx$

26. $\int x \tan x \sec x \, dx$

27. $\int \cos^{-1} x \, dx$

28. $\int \sin^{-1} x \, dx$

29. $\int \sec^{-1} x \, dx$

30. $\int x 5^x \, dx$

31. $\int 3^x \cos x \, dx$

32. $\int x \sinh x \, dx$

33. $\int x^2 \cosh x \, dx$

34. $\int \cos x \cosh x \, dx$

35. $\int \tanh^{-1} 4x \, dx$

36. $\int \sinh^{-1} x \, dx$

In Exercises 37–38, evaluate using substitution and then Integration by Parts.

37. $\int e^{\sqrt{x}} \, dx$ *Hint:* Let $u = x^{1/2}$

38. $\int x^3 e^{x^2} \, dx$

In Exercises 39–48, evaluate using Integration by Parts, substitution, or both if necessary.

39. $\int x \cos 4x \, dx$

40. $\int \frac{\ln(\ln x) \, dx}{x}$

41. $\int \frac{x \, dx}{\sqrt{x+1}}$

42. $\int x^2 (x^3 + 9)^{15} \, dx$

43. $\int \cos x \ln(\sin x) \, dx$

44. $\int \sin \sqrt{x} \, dx$

45. $\int \sqrt{x} e^{\sqrt{x}} \, dx$

46. $\int \frac{\tan \sqrt{x} \, dx}{\sqrt{x}}$

47. $\int \frac{\ln(\ln x) \ln x \, dx}{x}$

48. $\int \sin(\ln x) \, dx$

In Exercises 49–54, compute the definite integral.

49. $\int_0^3 x e^{4x} \, dx$

50. $\int_0^{\pi/4} x \sin 2x \, dx$

51. $\int_1^2 x \ln x \, dx$

52. $\int_1^e \frac{\ln x \, dx}{x^2}$

53. $\int_0^{\pi} e^x \sin x \, dx$

54. $\int_0^1 \tan^{-1} x \, dx$

55. Use Eq. (5) to evaluate $\int x^4 e^x \, dx$.

56. Use substitution and then Eq. (5) to evaluate $\int x^4 e^{7x} \, dx$.

57. Find a reduction formula for $\int x^n e^{-x} \, dx$ similar to Eq. (5).

58. Evaluate $\int x^n \ln x \, dx$ for $n \neq -1$. Which method should be used to evaluate $\int x^{-1} \ln x \, dx$?

In Exercises 59–66, indicate a good method for evaluating the integral (but do not evaluate). Your choices are algebraic manipulation, substitution (specify u and du), and Integration by Parts (specify u and v'). If it appears that the techniques you have learned thus far are not sufficient, state this.

59. $\int \sqrt{x} \ln x \, dx$

60. $\int \frac{x^2 - \sqrt{x}}{2x} \, dx$

61. $\int \frac{x^3 \, dx}{\sqrt{4 - x^2}}$

62. $\int \frac{dx}{\sqrt{4 - x^2}}$

63. $\int \frac{x + 2}{x^2 + 4x + 3} \, dx$

64. $\int \frac{dx}{(x+2)(x^2 + 4x + 3)}$

65. $\int x \sin(3x + 4) \, dx$

66. $\int x \cos(9x^2) \, dx$

67. Evaluate $\int (\sin^{-1} x)^2 \, dx$. *Hint:* Use Integration by Parts first and then substitution.

68. Evaluate $\int \frac{(\ln x)^2 \, dx}{x^2}$. *Hint:* Use substitution first and then Integration by Parts.

69. Evaluate $\int x^7 \cos(x^4) \, dx$.

70. Find $f(x)$, assuming that

$$\int f(x) e^x \, dx = f(x) e^x - \int x^{-1} e^x \, dx$$

71. Find the volume of the solid obtained by revolving the region under $y = e^x$ for $0 \le x \le 2$ about the y-axis.

72. Find the area enclosed by $y = \ln x$ and $y = (\ln x)^2$.

73. Recall that the *present value* (PV) of an investment that pays out income continuously at a rate $R(t)$ for T years is $\int_0^T R(t) e^{-rt} \, dt$, where r is the interest rate. Find the PV if $R(t) = 5000 + 100t$ \$/year, $r = 0.05$ and $T = 10$ years.

74. Derive the reduction formula

$$\int (\ln x)^k \, dx = x(\ln x)^k - k \int (\ln x)^{k-1} \, dx \qquad \boxed{6}$$

75. Use Eq. (6) to calculate $\int (\ln x)^k \, dx$ for $k = 2, 3$.

76. Derive the reduction formulas

$$\int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$$

$$\int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

77. Prove that $\int x b^x \, dx = b^x \left(\frac{x}{\ln b} - \frac{1}{\ln^2 b} \right) + C$.

78. Define $P_n(x)$ by

$$\int x^n e^x \, dx = P_n(x) \, e^x + C$$

Use Eq. (5) to prove that $P_n(x) = x^n - n P_{n-1}(x)$. Use this recursion relation to find $P_n(x)$ for $n = 1, 2, 3, 4$. Note that $P_0(x) = 1$.

Further Insights and Challenges

79. The Integration by Parts formula can be written

$$\int u(x)v(x)\,dx = u(x)V(x) - \int u'(x)V(x)\,dx \qquad \boxed{7}$$

where $V(x)$ satisfies $V'(x) = v(x)$.

(a) Show directly that the right-hand side of Eq. (7) does not change if $V(x)$ is replaced by $V(x) + C$, where C is a constant.

(b) Use $u = \tan^{-1} x$ and $v = x$ in Eq. (7) to calculate $\int x \tan^{-1} x\,dx$, but carry out the calculation twice: first with $V(x) = \frac{1}{2}x^2$ and then with $V(x) = \frac{1}{2}x^2 + \frac{1}{2}$. Which choice of $V(x)$ results in a simpler calculation?

80. Prove in two ways that

$$\int_0^a f(x)\,dx = af(a) - \int_0^a xf'(x)\,dx \qquad \boxed{8}$$

First use Integration by Parts. Then assume $f(x)$ is increasing. Use the substitution $u = f(x)$ to prove that $\int_0^a xf'(x)\,dx$ is equal to the area of the shaded region in Figure 1 and derive Eq. (8) a second time.

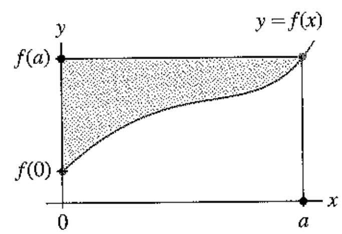

FIGURE 1

81. Assume that $f(0) = f(1) = 0$ and that f'' exists. Prove

$$\int_0^1 f''(x)f(x)\,dx = -\int_0^1 f'(x)^2\,dx \qquad \boxed{9}$$

Use this to prove that if $f(0) = f(1) = 0$ and $f''(x) = \lambda f(x)$ for some constant λ, then $\lambda < 0$. Can you think of a function satisfying these conditions for some λ?

82. Set $I(a, b) = \int_0^1 x^a(1 - x)^b\,dx$, where a, b are whole numbers.

(a) Use substitution to show that $I(a, b) = I(b, a)$.

(b) Show that $I(a, 0) = I(0, a) = \dfrac{1}{a+1}$.

(c) Prove that for $a \ge 1$ and $b \ge 0$,

$$I(a, b) = \frac{a}{b+1} I(a - 1, b + 1)$$

(d) Use (b) and (c) to calculate $I(1, 1)$ and $I(3, 2)$.

(e) Show that $I(a, b) = \dfrac{a!\,b!}{(a + b + 1)!}$.

83. Let $I_n = \int x^n \cos(x^2)\,dx$ and $J_n = \int x^n \sin(x^2)\,dx$.

(a) Find a reduction formula that expresses I_n in terms of J_{n-2}. *Hint:* Write $x^n \cos(x^2)$ as $x^{n-1}(x \cos(x^2))$.

(b) Use the result of (a) to show that I_n can be evaluated explicitly if n is odd.

(c) Evaluate I_3.

8.2 Trigonometric Integrals

Many trigonometric functions can be integrated by combining substitution and Integration by Parts with the appropriate trigonometric identities. First, consider

$$\int \sin^m x \cos^n x\,dx$$

where m, n are whole numbers. The easier case is when at least one of m, n is *odd*.

EXAMPLE 1 Odd Power of $\sin x$ Evaluate $\int \sin^3 x\,dx$.

Solution Because $\sin^3 x$ is an odd power, the identity $\sin^2 x = 1 - \cos^2 x$ allows us to split off a factor of $\sin x\,dx$:

$$\sin^3 x\,dx = \sin^2 x(\sin x\,dx) = (1 - \cos^2 x)\sin x\,dx$$

and use the substitution $u = \cos x$, $du = -\sin x\,dx$:

$$\int \sin^3 x\,dx = \int (1 - \cos^2 x)\sin x\,dx = -\int (1 - u^2)\,du$$

$$= \frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C$$