Final - Math 132/3

Due 2:30pm on 14 June 2012

This is the final for the course. You are not permitted to work on this exam with others. Your work should be your own. You may consult the course textbook and your notes. Questions may be sent to Brad or me by 5pm on Tuesday 12 June to guarantee a timely response. We will post responses on Piazza for everyone to see, where applicable. Show all work and write as neatly as possible.

1. (20pts) Compute all values of the following complex logarithms: (a) $\log 2i$, (b) $\log(1+i)$, and (c) $\log_2 2i$. Note that the last is a logarithm in base 2. You may have to give some thought as to what this means.

2. (30pts) Consider the power series in two real variables

$$u(x,y) = \sum_{k=0}^{\infty} \sum_{\substack{0 \le j \le k \\ j \text{ even}}} (-1)^{\frac{j}{2}} \binom{k}{j} x^{k-j} y^j .$$

Find the radius of convergence R of the series. Show that the function $u(x,y)$ is harmonic where it is defined. Find a harmonic conjugate $v(x,y)$ for $u(x,y)$ such that $v(0,0) = 0$. Write the resulting analytic function $f(x+iy) = u(x,y) + iv(x,y)$ as a rational function; that is, as a quotient of two polynomials in the variable $z = x+iy$.

3. (20pts) State and prove the polar version of the Cauchy-Riemann equations. Derive the polar form of the Laplace equation, and show that $f(re^{i\theta})$ satisfies this equation when f is analytic.

4. (20pts) Give a proof of the fundamental theorem of algebra about polynomials in one variable over the complex numbers using techniques developed in class. State the theorem clearly, and give a proof in your own words.

5. (20pts) Compute

$$\int_{|z|=100000} \frac{e^z}{z^5}\,\mathrm{d}z \qquad \text{and} \qquad \int_{|z|=2} \frac{1}{z^2+1}\,\mathrm{d}z.$$

6. (20pts) Find the value of the integral

$$\int_{|z|=1} \frac{\tan z}{(z-\frac{\pi}{4})^2}\,\mathrm{d}z.$$

7. (20pts) Compute the following integrals:

$$\int_{|z|=1} \frac{\sin^{10}(5z)}{z^9}\,\mathrm{d}z \qquad \text{and} \qquad \int_{|z|=1} e^{\frac{3}{z}}\,\mathrm{d}z.$$

8. (20pts) Compute all zeros and poles of $f(z) = \frac{\sin z}{z^3 \cos z}$ together with their orders. Then, compute

$$\int_{|z|=10} f(z)\,\mathrm{d}z.$$

9. (30pts) Find

$$\int_0^\infty \frac{x^2}{x^4+5x^2+6}\,\mathrm{d}x \qquad \text{and} \qquad \int_0^\infty \frac{x\sin x}{x^2+a^2}\,\mathrm{d}x, \text{ when } a>0.$$

10. (65pts) Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

using the following steps.

(a) Show that for $n \in \mathbb{Z}$, if $\mathrm{Re}(z) = \pi(n+\frac{1}{2})$, then $|\sin(z)| \geq 1$, and if $\mathrm{Im}(z) = \pi(n+\frac{1}{2})$, then $|\sin(z)| \geq \sinh(\pi/2)$. [Hint: You may use, without proof, the identity $\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$. (In fact, this identity is easy to prove by expanding into exponentials, but you don't need to run through this.)]

b) Show that for points z on the rectangular path $\mathcal{C}_n$ connecting the points $\pm\pi(n+\frac{1}{2}) \pm i\pi(n+\frac{1}{2})$ with straight lines one has $|\frac{\cos z}{\sin z}| \leq \sqrt{2}$. [Hint: $(\frac{\cos z}{\sin z})^2 = (\frac{1}{\sin z})^2 - 1$, and $\sinh(\pi/2) \geq 1$. You don't need to prove these facts to use them.]

c) Suppose $f(z)$ is analytic on $\mathbb{C}$ except at the points $z_1, z_2, \ldots$ where $f(z)$ has simple poles with residues $b_1, b_2, \ldots$, and in addition there is an absolute constant K (independent of n) so that for all n and all $\zeta \in \mathcal{C}_n$, $|f(\zeta)| \leq K$. Suppose also, for convenience, that $0 \notin \{z_1, z_2, \ldots\}$. Show that for $z \notin \{z_1, z_2, \ldots\}$,

$$f(z) = f(0) + \sum_{k=1}^{\infty} b_k\Big(\frac{1}{z-z_k} + \frac{1}{z_k}\Big).$$

[Hint: Fix z. Prove using partial fractions that

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_n} \frac{f(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}\oint_{\mathcal{C}_n} \frac{f(\zeta)}{\zeta}d\zeta + \frac{1}{2\pi i}\oint_{\mathcal{C}_n} \frac{zf(\zeta)}{\zeta(\zeta-z)}d\zeta.$$

Use this.]

(d) Show that the Laurent expansion for $\frac{\cos(z)}{\sin(z)}$ in the annulus $0 < |z| < \pi$ is

$$\frac{\cos z}{\sin z} = \frac{1}{z} - \frac{z}{3} + \text{higher order terms.}$$

(e) Show that for $z \notin \{k\pi : k \in \mathbb{Z}\}$,

$$\frac{\cos(z)}{\sin(z)} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{2z}{\pi^2 n^2 - z^2}.$$

[Hint: Consider $f(z) = \frac{\cos z}{\sin z} - \frac{1}{z}$ and use (c).]

(f) Deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. [Hint: Use the second coefficient in the Laurent series expansion in (d), and use (e) as well.]

In the event you are not able to establish a given claim in parts (a)-(f), please assume it and use it in the remainder of the argument to get as many points as possible.