where $a_0, \dots, a_n$ are complex numbers, and $a_n \neq 0$. A key property of the complex numbers, not enjoyed by the real numbers, is that any polynomial with complex coefficients can be factored as a product of linear factors.

Fundamental Theorem of Algebra. *Every complex polynomial $p(z)$ of degree $n \geq 1$ has a factorization*

$$p(z) = c(z-z_1)^{m_1} \cdots (z-z_k)^{m_k},$$

where the z_j's are distinct and $m_j \geq 1$. This factorization is unique, up to a permutation of the factors.

We will not prove this theorem now, but we will give several proofs later. Some remarks are in order.

The uniqueness of the factorization is easy to establish. The points $z_1, \dots, z_k$ are uniquely characterized as the **roots** of $p(z)$, or the **zeros** of $p(z)$. These are the points where $p(z) = 0$. The integer m_j is characterized as the unique integer m with the property that $p(z)$ can be factored as $(z-z_j)^m q(z)$ where $q(z)$ is a polynomial satisfying $q(z_j) \neq 0$.

For the proof of the existence of the factorization, one proceeds by induction on the degree n of the polynomial. The crux of the matter is to find a point z_1 such that $p(z_1) = 0$. With a root z_1 in hand, one easily factors $p(z)$ as a product $(z-z_1)q(z)$, where $q(z)$ is a polynomial of degree $n-1$. (See the exercises.) The induction hypothesis allows one to factor $q(z)$ as a product of linear factors, and this yields the factorization of $p(z)$. Thus the fundamental theorem of algebra is equivalent to the statement that every complex polynomial of degree $n \geq 1$ has a zero.

Example. The polynomial $p(x) = x^2 + 1$ with real coefficients cannot be factored as a product of linear polynomials with real coefficients, since it does not have any real roots. However, the complex polynomial $p(z) = z^2 + 1$ has the factorization

$$z^2 + 1 = (z-i)(z+i),$$

corresponding to the two complex roots $\pm i$ of $z^2 + 1$.

Exercises for I.1

1. Identify and sketch the set of points satisfying:
   (a) $|z-1-i| = 1$
   (b) $1 < |2z-6| < 2$
   (c) $|z-1|^2 + |z+1|^2 < 8$
   (d) $|z-1| + |z+1| \leq 2$
   (e) $|z-1| < |z|$
   (f) $0 < \operatorname{Im} z < \pi$
   (g) $-\pi < \operatorname{Re} z < \pi$
   (h) $|\operatorname{Re} z| < |z|$
   (i) $\operatorname{Re}(iz+2) > 0$
   (j) $|z-i|^2 + |z+i|^2 < 2$

2. Verify from the definitions each of the identities (a) $\overline{z+w} = \bar{z} + \bar{w}$, (b) $\overline{zw} = \bar{z}\bar{w}$, (c) $|\bar{z}| = |z|$, (d) $|z|^2 = z\bar{z}$. Draw sketches to illustrate (a) and (c).

3. Show that the equation $|z|^2 - 2\operatorname{Re}(\bar{a}z) + |a|^2 = \rho^2$ represents a circle centered at a with radius ρ.

4. Show that $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$, and sketch the set of points for which equality holds.

5. Show that $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$. Show that
$$|z+w|^2 = |z|^2 + |w|^2 + 2\operatorname{Re}(z\bar{w}).$$
Use this to prove the triangle inequality $|z+w| \leq |z| + |w|$.

6. For fixed $a \in \mathbb{C}$, show that $|z-a|/|1-\bar{a}z| = 1$ if $|z| = 1$ and $1 - \bar{a}z \neq 0$.

7. Fix $\rho > 0$, $\rho \neq 1$, and fix $z_0, z_1 \in \mathbb{C}$. Show that the set of z satisfying $|z - z_0| = \rho|z - z_1|$ is a circle. Sketch it for $\rho = \frac{1}{2}$ and $\rho = 2$, with $z_0 = 0$ and $z_1 = 1$. What happens when $\rho = 1$?

8. Let $p(z)$ be a polynomial of degree $n \geq 1$ and let $z_0 \in \mathbb{C}$. Show that there is a polynomial $h(z)$ of degree $n-1$ such that $p(z) = (z - z_0)h(z) + p(z_0)$. In particular, if $p(z_0) = 0$, then $p(z) = (z-z_0)h(z)$.

9. Find the polynomial $h(z)$ in the preceding exercise for the following choices of $p(z)$ and z_0: (a) $p(z) = z^2$ and $z_0 = i$, (b) $p(z) = z^3+z^2+z$ and $z_0 = -1$, (c) $p(z) = 1 + z + z^2 + \cdots + z^m$ and $z_0 = -1$.

10. Let $q(z)$ be a polynomial of degree $m \geq 1$. Show that any polynomial $p(z)$ can be expressed in the form
$$p(z) = h(z)q(z) + r(z),$$
where $h(z)$ and $r(z)$ are polynomials and the degree of the remainder $r(z)$ is strictly less than m. *Hint.* Proceed by induction on the degree of $p(z)$. The resulting method is called the **division algorithm**.

11. Find the polynomials $h(z)$ and $r(z)$ in the preceding exercise for $p(z) = z^n$ and $q(z) = z^2 - 1$.

2. Polar Representation

Any point $(x, y) \neq (0,0)$ in the plane can be described by polar coordinates r and θ, where $r = \sqrt{x^2+y^2}$ and θ is the angle subtended by (x,y) and the x-axis. The angle θ is determined only up to adding an integral multiple of 2π. The Cartesian coordinates x, y are recovered from the polar coordinates r, θ by

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

where each formula is understood to hold modulo adding integral multiples of 2π. To establish (2.8) and (2.9), note that if the polar representation of z is $re^{i\theta}$, then the polar representation of $\bar{z}$ is $re^{-i\theta}$, and that of $1/z$ is $(1/r)e^{-i\theta}$. For (2.10), write $z_1 = r_1e^{i\varphi_1}$, $z_2 = r_2e^{i\theta_2}$, and use the addition formula to obtain the polar form of z_1z_2,

$$z_1z_2 \;=\; r_1r_2e^{i\theta_1}e^{i\theta_2} \;=\; r_1r_2e^{i(\theta_1+\theta_2)}.$$

The addition formula (2.6) can be used to derive formulae for $\cos(n\theta)$ and $\sin(n\theta)$ in terms of $\cos\theta$ and $\sin\theta$. Write

$$\cos(n\theta) + i\sin(n\theta) \;=\; e^{in\theta} \;=\; (e^{i\theta})^n \;=\; (\cos\theta + i\sin\theta)^n,$$

expand the right-hand side, and equate real and imaginary parts. This yields expressions for $\cos(n\theta)$ and $\sin(n\theta)$ that are polynomials in $\cos\theta$ and $\sin\theta$. These identities are known as **de Moivre's formulae**. For instance, by equating $\cos(3\theta) + i\sin(3\theta)$ to

$$(\cos\theta + i\sin\theta)^3 \;=\; \cos^3\theta - 3\cos\theta\sin\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$$

and taking real and imaginary parts, we obtain

$$\begin{aligned}\cos(3\theta) &= \operatorname{Re}(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta,\\ \sin(3\theta) &= \operatorname{Im}(\cos\theta + i\sin\theta)^3 = 3\cos^2\theta\sin\theta - \sin^3\theta.\end{aligned}$$

A complex number z is an ***n*th root** of w if $z^n = w$. Thus the nth roots of w are precisely the zeros of the polynomial $z^n - w$ of degree n. Since this polynomial has degree n, w has at most n nth roots. If $w \neq 0$, then w has exactly n nth roots, and these are determined as follows. First express w in polar form,

$$w \;=\; \rho e^{i\varphi}.$$

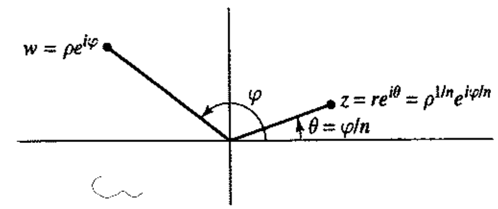
The equation $z^n = w$ becomes

$$r^n e^{in\theta} \;=\; \rho e^{i\varphi}.$$

Thus $r^n = \rho$ and $n\theta = \varphi + 2\pi k$ for some integer k. This leads to the explicit solutions

$$\begin{aligned} r &= \rho^{1/n},\\ \theta &= \frac{\varphi}{n} + \frac{2\pi k}{n}, \qquad k = 0, 1, 2, \ldots, n-1,\end{aligned}$$

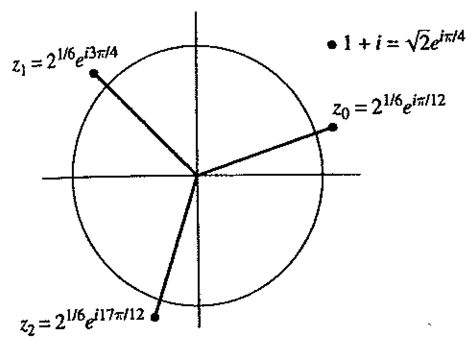
where we take the usual positive root of ρ. Since these n roots are distinct, and there are at most n nth roots, this list includes all the nth roots of w. Other values of k do not give different roots, since any other integer k leads to a value of θ that is obtained from the above list by adding an integral multiple of 2π. Graphically, the roots are distributed in equal arcs on the circle centered at 0 of radius $|w|^{1/n}$.



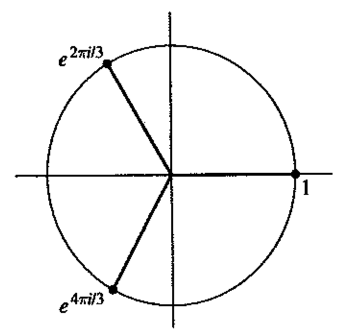
Example. To find and plot the square roots of $4i$, first express $4i$ in polar form $\rho e^{i\varphi}$. Here $\rho = |4i| = 4$ and $\varphi = \arg(4i) = \pi/2$. One root is given by $\sqrt{\rho}\, e^{i\varphi/2} = 2e^{i\pi/4}$. The other is $2e^{i(\pi/4+\pi)} = -2e^{i\pi/4}$. In Cartesian form, the roots are $\sqrt{4i} = \pm(\sqrt{2} + \sqrt{2}i)$.

Example. To find and plot the cube roots of $1+i$, express $1+i$ in polar form as $\sqrt{2}\, e^{i\pi/4}$. The polar form of the three cube roots is given by

$$2^{1/6}e^{i(\pi/12+2k\pi/3)}, \qquad k = 0, 1, 2.$$



cube roots of $1+i$



cube roots of unity

The nth roots of 1 are also called the ***n*th roots of unity**. They are given explicitly by

$$\omega_k \;=\; e^{2\pi ik/n}, \qquad 0 \le k \le n-1.$$

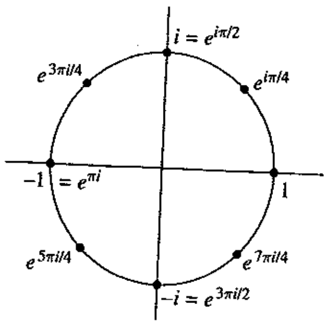
Graphically, they are situated at equal intervals around the unit circle in the complex plane. Thus the two square roots of unity are $e^0 = 1$ and $e^{i\pi} = -1$.

The procedure for finding the nth roots of $w \neq 0$ can be rephrased in terms of the nth roots of unity. We express $w = \rho e^{i\varphi/n}$ in polar form as above. One root is given by $z_0 = \rho^{1/n}e^{i\varphi/n}$. The others are found by multiplying z_0 by the nth roots of unity:

$$z_k \;=\; z_0\omega_k \;=\; \rho^{1/n}e^{i\varphi/n}e^{2\pi ik/n}, \qquad 0 \le k \le n-1.$$

Exercises for I.2

1. Express all values of the following expressions in both polar and cartesian coordinates, and plot them.



The eight eighth roots of unity

(a) $\sqrt{i}$ (b) $\sqrt{i-1}$ (c) $\sqrt[4]{-1}$ (d) $\sqrt[4]{i}$ (e) $(-8)^{1/3}$ (f) $(3-4i)^{1/8}$ (g) $(1+i)^8$ (h) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{25}$

2. Sketch the following sets:
   (a) $|\arg z| < \pi/4$ (c) $|z| = \arg z$
   (b) $0 < \arg(z-1-i) < \pi/3$ (d) $\log|z| = -2\arg z$

3. For a fixed complex number b, sketch the curve $\{e^{i\theta} + be^{-i\theta} : 0 \le \theta \le 2\pi\}$. Differentiate between the cases $|b| < 1$, $|b| = 1$ and $|b| > 1$. *Hint.* First consider the case $b > 0$, and then reduce the general case to this case by a rotation.

4. For which n is i an nth root of unity?

5. For $n \ge 1$, show that
   (a) $1 + z + z^2 + \cdots + z^n = (1 - z^{n+1})/(1-z), \qquad z \ne 1,$
   (b) $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \dfrac{1}{2} + \dfrac{\sin(n+\frac{1}{2})\theta}{2\sin\theta/2}.$

6. Fix $n \ge 1$. Show that the nth roots of unity $\omega_0, \ldots, \omega_{n-1}$ satisfy:
   (a) $(z-\omega_0)(z-\omega_1)\cdots(z-\omega_{n-1}) = z^n - 1$,
   (b) $\omega_0 + \cdots + \omega_{n-1} = 0$ if $n \ge 2$,
   (c) $\omega_0 \cdots \omega_{n-1} = (-1)^{n-1}$,
   (d) $\displaystyle\sum_{j=0}^{n-1} \omega_j^k = \begin{cases} 0, & 1 \le k \le n-1, \\ n, & k = n. \end{cases}$

7. Fix $R > 1$ and $n \ge 1$, $m \ge 0$. Show that

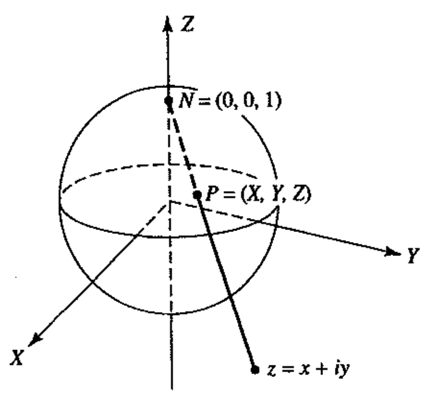
$$\left|\frac{z^m}{z^n+1}\right| \le \frac{R^m}{R^n-1}, \qquad |z| = R.$$

Sketch the set where equality holds. *Hint.* See (1.1).

8. Show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\cos\theta\sin\theta$ using de Moivre's formulae. Find formulae for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

3. Stereographic Projection

The **extended complex plane** is the complex plane together with the point at infinity. We denote the extended complex plane by $\mathbb{C}^*$, so that $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. One way to visualize the extended complex plane is through stereographic projection. This is a function, or map, from the unit sphere in three-dimensional Euclidean space $\mathbb{R}^3$ to the extended complex plane, which is defined as follows. If $P = (X, Y, Z)$ is any point of the unit sphere other than the north pole $N = (0,0,1)$, we draw a straight line through N and P, and we define the **stereographic projection** of P to be the point $z = x + iy \sim (x, y, 0)$ where the straight line meets the coordinate plane $Z = 0$. The stereographic projection of the north pole N is defined to be ∞, the point at infinity.



An explicit formula for the stereographic projection is derived as follows. We represent the line through P and N parametrically by $N + t(P - N)$, $-\infty < t < \infty$. The line meets the (x,y)-plane at a point $(x, y, 0)$ that satisfies

$$\begin{aligned}(x, y, 0) &= (0,0,1) + t[(X,Y,Z) - (0,0,1)] \\ &= (tX, tY, 1 + t(Z-1))\end{aligned}$$

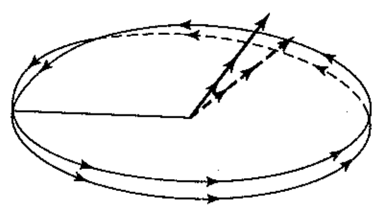
for some parameter value t. Equating the third components, we obtain $0 = 1 + t(Z-1)$, which allows us to solve for the parameter value t,

$$t = 1/(1-Z).$$

and as w approaches $-r$ from below, the values $f_2(w)$ approach the value $+i\sqrt{r}$ on the positive imaginary axis. Again we think of the slit as having two edges, though on this sheet the top edge is mapped to the negative imaginary axis and the bottom edge is mapped to the positive imaginary axis. Further, we have

$$f_1(-r+i0) = i\sqrt{r} = f_2(-r-i0), \quad f_1(-r-i0) = -i\sqrt{r} = f_2(-r+i0).$$

This leads us to the idea of constructing a surface to represent the inverse function by gluing together the edges where the functions $f_1(w)$ and $f_2(w)$ coincide. We glue the top edge of the branch cut on the sheet corresponding to $f_1(w)$ to the bottom edge of the branch cut on the sheet corresponding to $f_2(w)$, and similarly for the remaining two edges, to obtain a two-sheeted surface. Since the values of $f_1(w)$ and $f_2(w)$ coincide on the edges we have glued together, they determine a function $f(w)$ defined on the two-sheeted surface, with values in the z-plane that move continuously with w.



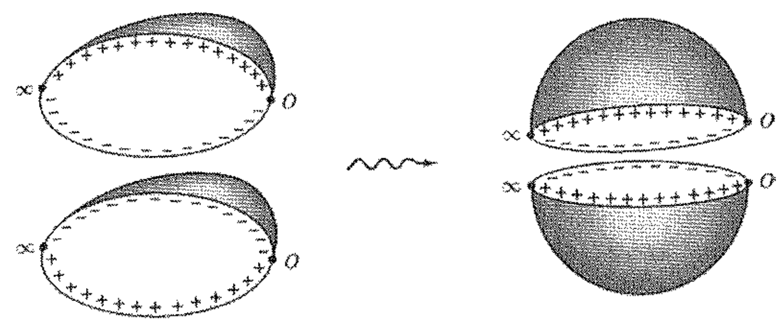
Since each sheet of the surface is a copy of the slit w-plane, we may think of the sheets as "lying over" the w-plane. Each $w \in \mathbb{C}\backslash\{0\}$ corresponds to exactly two points on the surface. The function $f(w)$ on the surface represents the multivalued function $\sqrt{w}$ in the sense that the values of $\sqrt{w}$ are precisely the values assumed by $f(w)$ at the points of the surface lying over w.

The surface we have constructed is called the **Riemann surface of** $\sqrt{w}$. The surface is essentially a sphere with two punctures corresponding to 0 and ∞. One way to see this is to note that the function $f(w)$ maps the surface one-to-one onto the z-plane punctured at 0. Another way to see this is to deform the surface by prying open each sheet at the slit, opening it to a hemisphere, and then joining the two hemispheres along the slit edges to form a sphere with two punctures corresponding to the endpoints 0 and ∞ of the slits.

Exercises for I.4

1. Sketch each curve in the z-plane, and sketch its image under $w = z^2$.
   (a) $|z-1| = 1$ (c) $y = 1$ (e) $y^2 = x^2 - 1,\ x > 0$
   (b) $x = 1$ (d) $y = x + 1$ (f) $y = 1/x,\ x \neq 0$

2. Sketch the image of each curve in the preceding problem under the principal branch of $w = \sqrt{z}$, and also sketch, on the same grid but



in a different color, the image of each curve under the other branch of $\sqrt{z}$.

3. (a) Give a brief description of the function $z \mapsto w = z^3$, considered as a mapping from the z-plane to the w-plane. (Describe what happens to w as z traverses a ray emanating from the origin, and as z traverses a circle centered at the origin.) (b) Make branch cuts and define explicitly three branches of the inverse mapping. (c) Describe the construction of the Riemann surface of $z^{1/3}$.

4. Describe how to construct the Riemann surfaces for the following functions: (a) $w = z^{1/4}$, (b) $w = \sqrt{z-i}$, (c) $w = (z-1)^{2/5}$. *Remark.* To describe the Riemann surface of a multivalued function, begin with one sheet for each branch of the function, make branch cuts so that the branches are defined continuously on each sheet, and identify each edge of a cut on one sheet to another appropriate edge so that the function values match up continuously.

5. The Exponential Function

We extend the definition of the exponential function to all complex numbers z by defining

$$e^z = e^x \cos y + ie^x \sin y, \qquad z = x + iy \in \mathbb{C}.$$

Since $e^{iy} = \cos y + i \sin y$, this is equivalent to

$$e^z = e^x e^{iy}, \qquad z = x + iy.$$

This identity is simply the polar representation of e^z,

$$|e^z| = e^x, \tag{5.1}$$

$$\arg e^z = y. \tag{5.2}$$

If z is real ($y = 0$), the definition of e^z agrees with the usual exponential function e^x. If z is imaginary ($x = 0$), the definition agrees with the definition of $e^{i\theta}$ given in Section 2.

A fundamental property of the exponential function is that it is periodic. The complex number λ is a **period** of the function $f(z)$ if $f(z+\lambda) = f(z)$ for all z for which $f(z)$ and $f(z+\lambda)$ are defined. The function $f(z)$ is **periodic** if it has a nonzero period. Since $\sin x$ and $\cos y$ are periodic functions with period 2π, the function e^z is periodic with period $2\pi i$:

$$e^{z+2\pi i} = e^z, \qquad z \in \mathbb{C}.$$

In fact, $2\pi i k$ is a period of e^z for any integer k.

Another fundamental property of the exponential function is the **addition formula**

(5.3) $$e^{z+w} = e^z e^w, \qquad z, w \in \mathbb{C}.$$

To check this, let $z = x + iy$ and $w = u + iv$. Then

$$e^{z+w} = e^{x+u}e^{i(y+v)} = e^x e^u e^{iy} e^{iv} = e^z e^w,$$

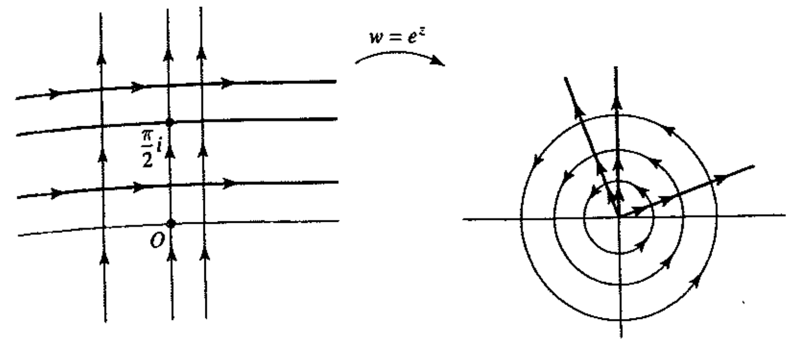
where we have used the addition formulae for e^x and $e^{i\theta}$.

From the addition formula (5.3) we have $e^z e^{-z} = e^0 = 1$. Consequently, the inverse of e^z is e^{-z},

$$1/e^z = e^{-z}, \qquad z \in \mathbb{C}.$$

To understand the exponential function better, we view $w = e^z$ as a mapping from the z-plane to the w-plane. If we restrict the exponential function to the real line $\mathbb{R}$, we obtain the usual exponential function $x \mapsto e^x$, $-\infty < x < \infty$, which maps the real line $\mathbb{R}$ to the positive real axis $(0,\infty)$. The equation (5.2) shows that an arbitrary horizontal line $x + iy_0$, $-\infty < x < \infty$, is mapped to the curve $e^x e^{iy_0}$, $-\infty < x < \infty$, which is a ray issuing from the origin at angle y_0. If we move the horizontal line up, the angle subtended by the ray increases, and the image ray is rotated in the positive (counterclockwise) direction. As we move the horizontal line upwards from the x-axis at $y_0 = 0$ to height $y_0 = \pi/2$, the image rays sweep out the first quadrant in the w-plane. The horizontal line at height $y_0 = \pi/2$ is mapped to the positive imaginary axis, the horizontal line of height $y_0 = \pi$ is mapped to the negative real axis, and when we reach the horizontal line of height $y_0 = 2\pi$, the image rays have swept out the full w-plane and returned to the positive real axis. The picture then repeats itself periodically. Each point in the w-plane, except $w = 0$, is hit infinitely often, by a sequence of z-values spaced at equal intervals of length 2π along a vertical line.

While the images of horizontal lines are rays issuing from the origin, the images of vertical lines are circles centered at the origin. The equation (5.1) shows that the image of the vertical line $x_0 + iy$, $-\infty < y < \infty$, is a circle in the w-plane of radius e^{x_0}. As z traverses the vertical line, the value w wraps infinitely often around the circle, completing one turn each time $y = \operatorname{Im} z$ increases by 2π.



horizontal lines ⟶ rays
vertical lines ⟶ circles

Exercises for I.5

1. Calculate and plot e^z for the following points z:
   (a) 0 (b) $\pi i + 1$ (c) $\pi(i-1)/3$ (d) $37\pi i$ (e) $\pi i/m$, $m = 1,2,3,\ldots$ (f) $m(i-1)$, $m = 1,2,3,\ldots$

2. Sketch each of the following figures and its image under the exponential map $w = e^z$. Indicate the images of horizontal and vertical lines in your sketch.
   (a) the vertical strip $0 < \operatorname{Re} z < 1$,
   (b) the horizontal strip $5\pi/3 < \operatorname{Im} z < 8\pi/3$,
   (c) the rectangle $0 < x < 1$, $0 < y < \pi/4$,
   (d) the disk $|z| \le \pi/2$,
   (e) the disk $|z| \le \pi$,
   (f) the disk $|z| \le 3\pi/2$.

3. Show that $e^{\bar z} = \overline{e^z}$.

4. Show that the only periods of e^z are the integral multiples of $2\pi i$, that is, if $e^{z+\lambda} = e^z$ for all z, then λ is an integer times $2\pi i$.

6. The Logarithm Function

For $z \ne 0$ we define $\log z$ to be the multivalued function

$$\begin{aligned}\log z &= \log|z| + i \arg z \\ &= \log|z| + i \operatorname{Arg} z + 2\pi i m, \qquad m = 0, \pm 1, \pm 2, \ldots.\end{aligned}$$

The values of $\log z$ are precisely the complex numbers w such that $e^w = z$. To see this, we plug in and compute. If $w = \log|z| + i\operatorname{Arg} z + 2\pi i m$, then

$$e^w = e^{\log|z|}e^{i\operatorname{Arg} z}e^{2\pi i m} = |z|e^{i\operatorname{Arg} z} = z,$$

Exercises for I.6

1. Find and plot $\log z$ for the following complex numbers z. Specify the principal value. (a) 2, (b) i, (c) $1+i$, (d) $(1+i\sqrt{3})/2$.
2. Sketch the image under the map $w = \operatorname{Log} z$ of each of the following figures.
   (a) the right half-plane $\operatorname{Re} z > 0$,
   (b) the half-disk $|z| < 1$, $\operatorname{Re} z > 0$,
   (c) the unit circle $|z| = 1$,
   (d) the slit annulus $\sqrt{e} < |z| < e^2$, $z \notin (-e^2, -\sqrt{e})$,
   (e) the horizontal line $y = e$,
   (f) the vertical line $x = e$.
3. Define explicitly a continuous branch of $\log z$ in the complex plane slit along the negative imaginary axis, $\mathbb{C}\backslash[0, -i\infty)$.
4. How would you make a branch cut to define a single-valued branch of the function $\log(z + i - 1)$? How about $\log(z - z_0)$?

7. Power Functions and Phase Factors

Let α be an arbitrary complex number. We define the power function z^α to be the multivalued function

$$z^\alpha = e^{\alpha \log z}, \qquad z \neq 0.$$

Thus the values of z^α are given by

$$\begin{aligned} z^\alpha &= e^{\alpha[\log|z| + i \operatorname{Arg} z + 2\pi i m]} \\ &= e^{\alpha \operatorname{Log} z} e^{2\pi i \alpha m}, \qquad m = 0, \pm 1, \pm 2, \ldots . \end{aligned}$$

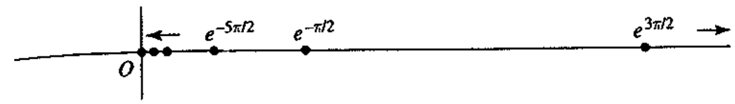
The various values of z^α are obtained from the principal value $e^{\alpha \operatorname{Log} z}$ by multiplying by the integral powers $(e^{2\pi i \alpha})^m$ of $e^{2\pi i \alpha}$. If α is itself an integer, then $e^{2\pi i \alpha} = 1$, and the function z^α is single-valued, the usual power function. If $\alpha = 1/n$ for some integer n, then the integral powers $e^{2\pi i m/n}$ of $e^{2\pi i/n}$ are exactly the nth roots of unity, and the values of $z^{1/n}$ are the n nth roots of z discussed earlier (Section 2).

Example. The values of i^i are given by

$$e^{i \log i} = e^{-\operatorname{Arg} i - 2\pi m} = e^{-\pi/2} e^{-2\pi m}, \quad m = 0, \pm 1, \pm 2, \ldots .$$

The values form a two-tailed sequence of positive real numbers, accumulating at 0 and at $+\infty$. Similarly, the values of i^{-i} are given by

$$e^{-i \log i} = e^{-\operatorname{Arg}(-i) - 2\pi k} = e^{\pi/2} e^{-2\pi k}, \quad k = 0, \pm 1, \pm 2, \ldots .$$



Danger! If we multiply the values of i^i by those of i^{-i}, we obtain infinitely many values $e^{2\pi n}$, $-\infty < n < \infty$. Thus

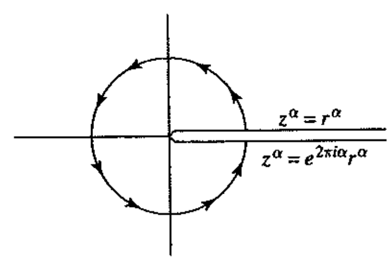
$$(i^i)(i^{-i}) \neq i^0 = 1,$$

and the usual algebraic rules do not apply to power functions when they are multivalued.

If α is not an integer, we cannot define z^α on the entire complex plane in such a way that the values move continuously with z. To define the function continuously, we must again make a branch cut. We could make the cut along the negative real axis, but this time let us make the cut along the positive real axis, from 0 to $+\infty$. We define a continuous branch of z^α on the slit plane $\mathbb{C}\backslash[0, \infty)$ explicitly by

$$w = r^\alpha e^{i\alpha\theta}, \qquad \text{for } z = re^{i\theta},\ 0 < \theta < 2\pi.$$

At the top edge of the slit, corresponding to $\theta = 0$, we have the usual power function $r^\alpha = e^{\alpha \operatorname{Log} r}$. At the bottom edge of the slit, corresponding to $\theta = 2\pi$, we have the function $r^\alpha e^{2\pi i \alpha}$. If we fix r and let θ increase from 0 to 2π, $z = re^{i\theta}$ starts at the top edge of the slit and proceeds around a circle, ending at the bottom edge of the slit. As z describes this circle, the values $w = r^\alpha e^{i\theta\alpha}$ move continuously, starting from r^α at the top edge of the slit and ending at $r^\alpha e^{2\pi i \alpha}$ at the bottom edge. Thus the values of this branch of z^α on the bottom edge are $e^{2\pi i \alpha}$ times the values at the top edge. The multiplier $e^{2\pi i \alpha}$ is called the **phase factor** of z^α at $z = 0$.



If we continue any other choice $w = r^\alpha e^{i\alpha(\theta + 2\pi m)}$ of z^α around the same circle, the values of w move continuously from $r^\alpha e^{2\pi i \alpha m}$ at the top edge of the slit to $r^\alpha e^{i\alpha(2\pi + 2\pi m)} = r^\alpha e^{2\pi i \alpha m} e^{2\pi i \alpha}$ at the bottom edge. Again the final w-value is the phase factor $e^{2\pi i \alpha}$ times the initial w-value.

The same analysis shows that the function $(z - z_0)^\alpha$ has a phase factor of $e^{2\pi i \alpha}$ at $z = z_0$, in the sense that if any branch of $w = (z - z_0)^\alpha$ is continued around a full circle centered at z_0 in the positive direction, the final w-value is $e^{2\pi i \alpha}$ times the initial w-value. This can be seen by making a change of variable $\zeta = z - z_0$. Further, this result does not change if we multiply $(z-z_0)^\alpha$ by any (single-valued) function. We state the result formally for emphasis.

Phase Change Lemma. *Let $g(z)$ be a (single-valued) function that is defined and continuous near z_0. For any continuously varying branch of $(z-z_0)^\alpha$ the function $f(z) = (z-z_0)^\alpha g(z)$ is multiplied by the phase factor $e^{2\pi i\alpha}$ when z traverses a complete circle about z_0 in the positive direction.*

Example. If α is an integer, the phase factor of z^α at 0 is $e^{2\pi i\alpha} = 1$, in accord with the fact that z^α is single-valued.

Example. The phase factor of $\sqrt{z-z_0}$ at z_0 is $e^{\pi i} = -1$. As z traverses a circle about z_0, the values of $f(z) = \sqrt{z-z_0}$ return to $-f(z)$. The phase factor of $1/\sqrt{z_0-z} = i/\sqrt{z-z_0}$ at z_0 is also -1.

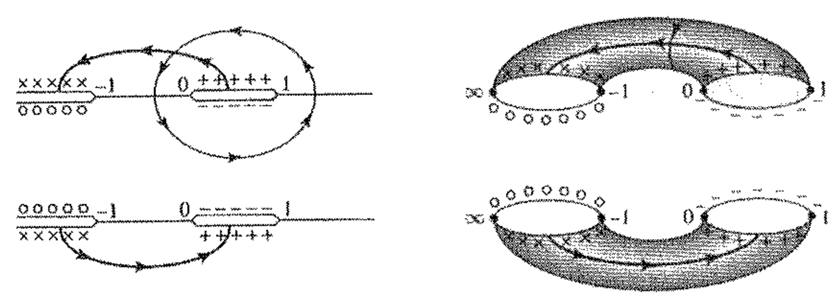
Example. The function $\sqrt{z(1-z)}$ has two branch points, at 0 and at 1. At $z=0$, each branch of $\sqrt{1-z}$ is single-valued, so the phase factor of each branch of $\sqrt{z(1-z)}$ at $z=0$ is the same as that of $\sqrt{z}$, which is -1. Similarly, the phase factor of $\sqrt{z(1-z)}$ at $z=1$ is the same as that of $\sqrt{1-z}$, which is -1. Now suppose we draw a branch cut from 0 to 1 and consider the branch $f(z)$ of $\sqrt{z(1-z)}$ that is positive on the top edge of the slit. As z traverses a small circle around 0, the values of $f(z)$ return to $-f(z)$ on the bottom edge of the slit, corresponding to the phase factor -1 at $z=0$. As z traverses the bottom edge of the slit and returns to the top edge around a small circle at $z=1$, the values of $-f(z)$ are again multiplied by the phase factor -1. Thus the values of $f(z)$ return to the original positive value on the top edge of the slit when z traverses a dogbone path encircling both branch points. It follows that the branch $f(z)$ is a continuous single-valued function in the slit plane $\mathbb{C}\backslash[0,1]$. Now we may proceed, in analogy with $\sqrt{z}$ and $\log z$, to define a Riemann surface for the function $\sqrt{z(1-z)}$ that captures both branches of the function. We require two sheets, since there are two choices of branches for the function $\sqrt{z(1-z)}$. On each sheet we make the same cut, to form two copies of $\mathbb{C}\backslash[0,1]$. On one sheet we define $F(z)$ to be the branch $f(z)$ of $\sqrt{z(1-z)}$ specified above, and on the other sheet we define $F(z)$ to be the other branch $-f(z)$ of $\sqrt{z(1-z)}$. The sheets are then joined by identifying edges of the slits in such a way that $F(z)$ extends continuously to the surface. In this case, the top edge of the slit $[0,1]$ on one sheet is identified to the bottom edge of the slit on the other sheet, and the remaining two edges are identified, to form the two-sheeted Riemann surface of $\sqrt{z(1-z)}$.

In constructing the Riemann surface of a multivalued function, the number of sheets always coincides with the number of branches of the function. However, the branch cuts can be made in many ways, as long as there are

surface with closed path

enough branch cuts so that each branch of the function can be defined continuously in the slit plane. For instance, the branch cuts for the function $f(x) = \sqrt{z(1-z)}$ could as well be made from $-\infty$ to 0 along the negative real axis and from $+1$ to $+\infty$ along the positive real axis. The branch cuts could also be made along more complicated paths from 0 to 1.

Example. Consider $\sqrt{z-1/z}$. We rewrite this as $\sqrt{z-1}\sqrt{z+1}/\sqrt{z}$. The function has three finite branch points, at 0 and ± 1. We must also consider ∞ as a branch point, since there is a phase change corresponding to a phase factor -1 as z traverses a very large circle centered at 0. Each branch point has phase factor -1, so any branch of the function returns to its original values when z traverse a path encircling two of the branch points. Thus it suffices to make two cuts, say $(-\infty,-1]$ and $[0,1]$. Each branch of the function is continuous on $\mathbb{C}\backslash((-\infty,-1]\cup[0,1])$. Again top edges of slits on one sheet are identified to bottom edges of the others. The resulting surface is a torus (doughnut, or inner tube), with punctures corresponding to the branch points. What would happen if we were to make initially an additional branch cut along $[-1,0]$, in addition to the two branch cuts above? The values of each branch at the top edge of the new cut would agree with the values of the same branch on the bottom edge. Consequently, we would identify the top and bottom edges of the slit $[-1,0]$ on the same sheet, thereby effectively erasing the slits and arriving at the same doughnut surface.



Exercises for I.7

1. Find all values and plot: (a) $(1+i)^i$, (b) $(-i)^{1+i}$, (c) $2^{-1/2}$, (d) $(1+i\sqrt{3})^{(1-i)}$.

2. Compute and plot $\log\left[(1+i)^{2i}\right]$.

3. Sketch the image of the sector $\{0 < \arg z < \pi/6\}$ under the map $w = z^a$ for (a) $a = \frac{3}{2}$, (b) $a = i$, (c) $a = i + 2$. Use only the principal branch of z^a.

4. Show that $(zw)^a = z^a w^a$, where on the right we take all possible products.

5. Find i^{i^i}. Show that it does not coincide with $i^{i \cdot i} = i^{-1}$.

6. Determine the phase factors of the function $z^a(1-z)^b$ at the branch points $z = 0$ and $z = 1$. What conditions on a and b guarantee that $z^a(1-z)^b$ can be defined as a (continuous) single-valued function on $\mathbb{C}\backslash[0,1]$?

7. Let $x_1 < x_2 < \cdots < x_n$ be n consecutive points on the real axis. Describe the Riemann surface of $\sqrt{(z-x_1)\cdots(z-x_n)}$. Show that for $n = 1$ and $n = 2$ the surface is topologically a sphere with certain punctures corresponding to the branch points and ∞. What is it when $n = 3$ or $n = 4$? Can you say anything for general n? (Any compact Riemann surface is topologically a sphere with handles. Thus a torus is topologically a sphere with one handle. For a given n, how many handles are there, and where do they come from?)

8. Show that $\sqrt{z^2 - 1/z}$ can be defined as a (single-valued) continuous function outside the unit disk, that is, for $|z| > 1$. Draw branch cuts so that the function can be defined continuously off the branch cuts. Describe the Riemann surface of the function.

9. Consider the branch of the function $\sqrt{z(z^3-1)(z+1)^3}$ that is positive at $z = 2$. Draw branch cuts so that this branch of the function can be defined continuously off the branch cuts. Describe the Riemann surface of the function. To what value at $z = 2$ does this branch return if it is continued continuously once counterclockwise around the circle $\{|z| = 2\}$?

10. Consider the branch of the function $\sqrt{z(z^3-1)(z+1)^3(z-1)}$ that is positive at $z = 2$. Draw branch cuts so that this branch of the function can be defined continuously off the branch cuts. Describe the Riemann surface of the function. To what value at $z = 2$ does this branch return if it is continued continuously once counterclockwise around the circle $\{|z| = 2\}$?

11. Find the branch points of $\sqrt[3]{z^3 - 1}$ and describe the Riemann surface of the function.

8. Trigonometric and Hyperbolic Functions

If we solve the equations

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta, \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}$$

for $\cos\theta$ and $\sin\theta$, we obtain

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned}$$

This motivates us to extend the definition of $\cos z$ and $\sin z$ to complex numbers z by

$$\begin{aligned} \cos z &= \frac{e^{iz} + e^{-iz}}{2}, & z \in \mathbb{C}, \\ \sin z &= \frac{e^{iz} - e^{-iz}}{2i}, & z \in \mathbb{C}. \end{aligned}$$

This definition agrees with the usual definition when z is real. Evidently, $\cos z$ is an even function,

$$\cos(-z) = \cos z, \qquad z \in \mathbb{C},$$

while $\sin z$ is an odd function,

$$\sin(-z) = -\sin z, \qquad z \in \mathbb{C}.$$

As functions of a complex variable, $\cos z$ and $\sin z$ are periodic, with period 2π,

$$\begin{aligned} \cos(z + 2\pi) &= \cos z, & z \in \mathbb{C}, \\ \sin(z + 2\pi) &= \sin z, & z \in \mathbb{C}. \end{aligned}$$

After some algebraic manipulation, one checks (Exercise 1) that the addition formulae for $\cos z$ and $\sin z$ remain valid,

$$\begin{aligned} \cos(z + w) &= \cos z \cos w - \sin z \sin w, & z, w \in \mathbb{C}, \\ \sin(z + w) &= \sin z \cos w + \cos z \sin w, & z, w \in \mathbb{C}. \end{aligned}$$

If we substitute $w = -z$ in the addition formula for cosine, we obtain the familiar identity

$$\cos^2 z + \sin^2 z = 1, \qquad z \in \mathbb{C}.$$

We shall see, in fact, that any reasonable identity that holds for analytic functions of a real variable, such as $\cos x$ and $\sin x$, also holds when the functions are extended to be functions of a complex variable. This will be a special case of the **principle of permanence of functional equations**, proved in Chapter V.

The hyperbolic functions $\cosh x = (e^x+e^{-x})/2$ and $\sinh x = (e^x-e^{-x})/2$ are also extended to the complex plane in the obvious way, by

$$\cosh z = \frac{e^z+e^{-z}}{2}, \qquad z \in \mathbb{C},$$
$$\sinh z = \frac{e^z-e^{-z}}{2}, \qquad z \in \mathbb{C}.$$

Both $\cosh z$ and $\sinh z$ are periodic, with period $2\pi i$,

$$\cosh(z+2\pi i) = \cosh z, \qquad z \in \mathbb{C},$$
$$\sinh(z+2\pi i) = \sinh z, \qquad z \in \mathbb{C}.$$

Evidently, $\cosh z$ is an even function and $\sinh z$ is an odd function. There are addition formulae for $\cosh z$ and $\sinh z$, derived easily from the addition formulae for $\cos z$ and $\sin z$ (Exercise 1).

When viewed as functions of a complex variable, the trigonometric and the hyperbolic functions exhibit a close relationship. They are obtained from each other by rotating the domain space by $\pi/2$,

$$\cosh(iz) = \cos z, \qquad \cos(iz) = \cosh z,$$
$$\sinh(iz) = i\sin z, \qquad \sin(iz) = i\sinh z.$$

If we use these equations and the addition formula

$$\sin(x+iy) = \sin x\cos(iy) + \cos x\sin(iy),$$

we obtain the Cartesian representation for $\sin z$,

$$\sin z = \sin x\cosh y + i\cos x\sinh y, \qquad z = x+iy \in \mathbb{C}.$$

Thus

$$|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y.$$

Using $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 y = 1+\sinh^2 y$, we obtain

$$|\sin z|^2 = \sin^2 x + \sinh^2 y.$$

From this formula it is clear where the zeros of $\sin z$ are located; $\sin z = 0$ only when $\sin x = 0$ and $\sinh y = 0$, and this occurs only on the real axis $y = 0$, at the usual zeros $0, \pm\pi, \pm 2\pi, \ldots$ of $\sin x$. Similarly, the only zeros of $\cos z$ are the usual zeros of $\cos x$ on the real axis (Exercise 2).

Other trigonometric and hyperbolic functions are defined by the usual formulae, such as

$$\tan z = \frac{\sin z}{\cos z}, \quad \tanh z = \frac{\sinh z}{\cosh z}, \qquad z \in \mathbb{C}.$$

Thus $\tan z$ and $\tanh z$ are odd functions, and $\tanh(iz) = i\tan z$.

The inverse trigonometric functions are multivalued functions, which can be expressed in terms of the logarithm function. Suppose $w = \sin^{-1} z$, that is,

$$\sin w = \frac{e^{iw}-e^{-iw}}{2i} = z.$$

Then $e^{2iw} - 2ize^{iw} - 1 = 0$. This is a quadratic equation in e^{iw}, which can be solved by the usual quadratic formula. The solutions are given by

$$e^{iw} = iz \pm \sqrt{1-z^2}.$$

Taking logarithms we obtain

$$\sin^{-1} z = -i\log\left(iz \pm \sqrt{1-z^2}\right).$$

This identity is to be understood as a set identity, in the sense that w satisfies $\sin w = z$ if and only if w is one of the values of $-i\log\left(iz \pm \sqrt{1-z^2}\right)$. To obtain a genuine function, we must restrict the domain and specify the branch. One way to do this is to draw two branch cuts, from $-\infty$ to -1 and from $+1$ to $+\infty$ along the real axis, and to specify the branch of $\sqrt{1-z^2}$ that is positive on the interval $(-1,1)$. With this branch of $\sqrt{1-z^2}$, we obtain a continuous branch $-i\,\mathrm{Log}\left(iz + \sqrt{1-z^2}\right)$ of $\sin^{-1} z$.

Exercises for I.8

1. Establish the following addition formulae:
   (a) $\cos(z+w) = \cos z\cos w - \sin z\sin w$,
   (b) $\sin(z+w) = \sin z\cos w + \cos z\sin w$,
   (c) $\cosh(z+w) = \cosh z\cosh w + \sinh z\sinh w$,
   (d) $\sinh(z+w) = \sinh z\cosh w + \cosh z\sinh w$,

2. Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$, where $z = x+iy$. Find all zeros and periods of $\cos z$.

3. Find all zeros and periods of $\cosh z$ and $\sinh z$.

4. Show that

$$\tan^{-1} z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right),$$

   where both sides of the identity are to be interpreted as subsets of the complex plane. In other words, show that $\tan w = z$ if and only if $2iw$ is one of the values of the logarithm featured on the right.

5. Let S denote the two slits along the imaginary axis in the complex plane, one running from i to $+i\infty$, the other from $-i$ to $-i\infty$. Show that $(1+iz)/(1-iz)$ lies on the negative real axis $(-\infty, 0]$ if and

only if $z \in S$. Show that the principal branch

$$\mathrm{Tan}^{-1} z = \frac{1}{2i} \operatorname{Log}\left(\frac{1+iz}{1-iz}\right)$$

maps the slit plane $\mathbb{C}\backslash S$ one-to-one onto the vertical strip $\{|\operatorname{Re} w| < \pi/2\}$.

6. Describe the Riemann surface for $\tan^{-1} z$.

7. Set $w = \cos z$ and $\zeta = e^{iz}$. Show that $\zeta = w \pm \sqrt{w^2-1}$. Show that

$$\cos^{-1} w = -i \log\left[w \pm \sqrt{w^2-1}\right],$$

where both sides of the identity are to be interpreted as subsets of the complex plane.

8. Show that the vertical strip $|\operatorname{Re}(w)| < \pi/2$ is mapped by the function $z(w) = \sin w$ one-to-one onto the complex z-plane with two slits $(-\infty, -1]$ and $[+1, +\infty)$ on the real axis. Show that the inverse function is the branch of $\sin^{-1} z = -i \operatorname{Log}\left(iz + \sqrt{1-z^2}\right)$ obtained by taking the principal value of the square root. *Hint.* First show that the function $1 - z^2$ on the slit plane omits the negative real axis, so that the principal value of the square root is defined and continuous on the slit plane, with argument in the open interval between $-\pi/2$ and $\pi/2$.

II

Analytic Functions

In this chapter we take up the complex differential calculus. After reviewing some basic analysis in Section 1, we introduce complex derivatives and analytic functions in Section 2 and we show that the rules for complex differentiation are the same as the usual rules for differentiation. In Section 3 we characterize analytic functions in terms of the Cauchy-Riemann equations. In Sections 4 and 5 we give several applications of the Cauchy-Riemann equations, to inverses of analytic functions and to harmonic functions. In Section 6 we discuss conformality, which is a direct consequence of complex differentiability. We close in Section 7 with a discussion of fractional linear transformations, which form an important class of analytic functions.

1. Review of Basic Analysis

We begin by reviewing the background material in analysis that will (eventually) be called upon, and we say something about the language of formal mathematics. For the most part, we will not phrase our arguments completely formally, though any bilingual person will be able to translate easily to the language of formal mathematics in such a way that our development becomes completely rigorous.

Since the complex derivative is defined as a limit, we require some background material on limits and continuity. To be able to define and work with analytic functions, we also require some basic topological concepts, including open and closed sets, and domains. The confident reader may pass directly to the definitions of complex derivative and analytic function in the next section, and refer back to the material in this section only when needed.

We begin with the notion of a convergent sequence. For this we have two definitions.

Informal Definition. A sequence $\{s_n\}$ **converges to** s if the sequence eventually lies in any disk centered at s.