Midterm Solutions - Math 132/3

14 May 2012

1. (20pts) Compute all values of the following complex powers: (a) $(1+i)^{1/4}$, (b) i^i, (c) $3^{1/5}$, (d) $(1+i)^{1+i}$.

Solution 1a) $(1+i)^{1/4}$ takes on the values $2^{1/8}e^{i\pi/16}e^{i\pi k/2}$ for $k = 0, 1, 2,$ or 3, where $2^{1/8}$ is the real 8^{th} root of 2.

1b) i^i takes on the values $e^{-\pi/2-2\pi k}$ where $k \in \mathbb{Z}$.

1c) $3^{1/5}$ takes on the values $3^{1/5}e^{i2k\pi/5}$ for $k = 0, 1, 2, 3$ or 4, where $3^{1/5}$ in this answer is the real 5^{th} root of 3.

1d) $(1+i)^{1+i}$ takes on the values $(1+i)e^{i\frac{\log 2}{2}}e^{-\pi/4-2\pi k}$ where $k \in \mathbb{Z}$

2. (20pts) Verify that the following functions are harmonic and compute their harmonic conjugates on the given domains: (a) $u(x, y) = \frac{y}{x^2+y^2}$ on $\mathbb{C} - \{0\}$, (b) $u(x, y) = e^x(x \cos x - y \cos y)$ on $\mathbb{C}$.

Solution 2a) $\frac{x}{x^2+y^2}$

2b) $e^x(x \sin y + y \cos y)$

3. (20pts) We showed in class that linear fractional transformations are 3-transitive. That is, for any 2 sets of 3 distinct points in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, say $\{x_0, x_1, x_2\}$ and $\{y_0, y_1, y_2\}$, there is a linear fractional transformation $f(z) =$

$\frac{az+b}{cz+d}$ such that $f(x_i) = y_i$ for $i = 0, 1, 2$. Are linear fractional transformations 4-transitive? That is, given 2 sets of 4 distinct points $\{x_0, x_1, x_2, x_3\}$ and $\{y_0, y_1, y_2, y_3\}$, is there a linear fractional transformation $f(z)$ such that $f(x_i) = y_i$ for $i = 0, 1, 2, 3$? If yes, prove it. If no, provide a counterexample with proof.

Solution Möbius transformations are not 4-transitive. There are many ways to show this; we give one. There is no fractional linear transformation taking the points $1, 2, 3$, and 4 respectively to $1, 2, 3$, and i. For a fractional linear transformation takes lines to either lines or circles, and therefore, if such a fractional linear transformation existed, it would take the line passing through $1, 2, 3$ to a line or circle passing through $1, 2, 3$ – and of course this is just same line as before, comprised of real complex numbers. But then 4 must be taken to a point on this same lane, but i does not lie on this line. (Alternatively, with some calculation, one can easily show that the only map taking $1, 2, 3$ to $1, 2, 3$ is $z \mapsto z$, which of course does not take 4 to i.) Hence there does not exists such a fractional linear transformation, and the collection of fractional linear transformations is not 4-transitive.

4. (10pts) Determine the linear fractional transformation $f = \frac{az+b}{cz+d}$ that satisfies $f(0) = 1$, $f(1) = 5$, and $f(\infty) = 3$.

Solution Most everyone got

$$f(z) = \frac{6z-1}{2z-1} = \frac{-6z+1}{-2z+1} = \frac{3z-\frac{1}{2}}{z-\frac{1}{2}}.$$

5. (20pts) Compute the following line integrals: (a) $\int_\gamma x dy$, where γ is the semicircle in the upper half-plane from R to $-R$ and R is a positive real number, (b) $\int_\gamma xy^4 dx$ where γ is the right half of the circle $|z| = 4$, in the counterclockwise direction.

Solution There are two natural ways to do this problem. First, you can parameterize the curves and compute from the definitions. Second, you can apply Green's theorem. However, note that Green's theorem applies only to closed curves; neither curve appearing in this problem is closed. If you lost points on the problem, it is most likely because you didn't take this into account.

Here is the correct way to apply Green's theorem in part (a). Let σ be the path from $-R$ to R on the x-axis. Let D be the interior of the upper-half disk of radius R. The boundary of D is γ followed by σ. So, by Green's theorem,

$$\int_\gamma x\,\mathrm{d}y + \int_\sigma x\,\mathrm{d}y = \int_{\partial D} x\,\mathrm{d}y = \int\int_D \mathrm{d}x\,\mathrm{d}y.$$

Now, observe that $\int_\sigma x\,\mathrm{d}y = 0$ because y is constant on the path so that $\mathrm{d}y = 0$. Parameterizing the half disk by $x = r\cos\theta$ and $y = r\sin\theta$ for $0 \le r \le R$ and $0 \le \theta \le \pi$, we get $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$, and we compute

$$\int_\gamma x\,\mathrm{d}y = \int\int_D \mathrm{d}x\,\mathrm{d}y = \int_0^\pi \int_0^R r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^\pi R^2/2\,\mathrm{d}\theta = \pi R^2/2.$$

Completing part (b) with Green's theorem involves similar reasoning. To do it by parameterizing the path, let $x = 4\cos\theta$ and $y = 4\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then,

$$\begin{aligned}
\int_\gamma xy^4\,\mathrm{d}x &= \int_{-\pi/2}^{\pi/2} (4\cos\theta)(4\sin\theta)^4(-4\sin\theta\,\mathrm{d}\theta) \\
&= -4^6 \int_{-\pi/2}^{\pi/2} \cos\theta\sin^5\theta\,\mathrm{d}\theta \\
&= -4^6 \left(\frac{\sin^6\theta}{6}\right)\Big|_{-\pi/2}^{\pi/2} \\
&= -4^6 \left(\frac{1^6}{6} - \frac{(-1)^6}{6}\right) \\
&= 0.
\end{aligned}$$

6. (20pts) Show that if $f = u + iv$ and $f = u - iv$ are both analytic, where u and v are real-valued functions, then f is a constant function.

Solution Recall my e-mail correcting this problem:

> `In problem 6, ignore f. Suppose that u(x,y) and v(x,y) are real-valued function such that u+iv and u-iv are both analytic. Then, u(x,y) and v(x,y) are constant.`
>
> `In the language of that problem, it is assuming that f(z) and the complex conjugate of f(z) are both analytic. Then, show that f(z) is constant.`

Let $f = u + iv$ and let $\overline{f} = u - iv$ be the complex conjugate. The Cauchy-Riemann equations for f and $\overline{f}$ say that

$$f : \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{partial v}{\partial x} \end{cases}$$
$$\overline{f} : \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial(-v)}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial(-v)}{\partial x} \end{cases}$$

Together, these imply that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{\partial v}{\partial x},$$

and hence that $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. Similarly, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. Since u and v are differentiable, this implies that u and v are both constant functions, and hence so is f.

7. (30pts) A polynomial $P(x, y)$ is called harmonic if it satisfies Laplace's equation. Determine all harmonic polynomials of the form $P(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, and find their harmonic conjugates. Show that a harmonic polynomial has a harmonic conjugate on $\mathbb{C}$, and that any such harmonic conjugate is a harmonic polynomial in x and y. Then, show that for any harmonic conjugate $Q(x, y)$ of $P(x, y)$, the analytic function $f(x + iy) = P(x, y) + iQ(x, y)$ is a complex polynomial in z.

Solution The computational part of this problem, to which we just supply the answer, is that a polynomial $P(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$ is harmonic iff $c = -3a$ and $b = -3d$, and that in this case the harmonic conjugate is $Q(x, y) = dx^3 + 3ax^2y - 3dxy^2 - ay^3 + A$, for some constant A.

The next component of the problem is to show that for a general harmonic polynomial, it's harmonic conjugate is also a polynomial in x and y.

If $P(x, y) = \sum_{j,k\leq N} a_{jk}x^jy^k$ harmonic, then it's harmonic conjugate Q is given up to a constant by a path integral

$$\begin{aligned} Q(x_0, y_0) &= \int_0^{(x_0,y_0)} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ &= \sum_{\substack{j,k\leq N \\ k\neq 0}} -ka_{jk} \int_0^{(x_0,y_0)} x^jy^{k-1}dx + \sum_{\substack{j,k\leq N \\ j\neq 0}} ja_{jk} \int_0^{(x_0,y_0)} x^{j-1}y^kdy \end{aligned}$$

where we have excluded $k = 0$ and $j = 0$ from the respective sums because the coefficients attached to these terms are 0. But clearly these path integrals produce polynomials in x_0, y_0, and since a finite sum of polynomials is still a polynomial, we are done.

In fact, though it's not necessary to perform the computation, breaking the path from 0 to (x_0, y_0) into two, traveling first from 0 to $(x_0, 0)$ on a straight line, and then from $(x_0, 0)$ to (x_0, y_0),

$$\int_0^{(x_0,y_0)} x^jy^{k-1}dx = \int_0^{(x_0,0)} x^jy^{k-1}dx + \int_{(x_0,0)}^{(x_0,y_0)} x^jy^{k-1}dx = 0 + 0$$

unless $k - 1 = 0$ in which case this integral is $x_0^{j+1}/(j + 1)$. Likewise,

$$\int_0^{(x_0,y_0)} x^{j-1}y^kdy = \int_0^{(x_0,0)} x^{j-1}y^kdy + \int_{(x_0,0)}^{(x_0,y_0)} x^{j-1}y^kdy = 0 + \frac{x_0^{j-1}y_0^{k+1}}{k+1},$$

so we can explicitly compute up to additive constant,

$$Q(x_0, y_0) = \sum_{\substack{j,k\leq N \\ j\neq 0}} \frac{ja_{jk}}{k+1}x_0^{j-1}y_0^{k+1} - \sum_{j\leq N} \frac{a_{j1}}{j+1}x_0^{j+1}$$

[If you took a different path integral, you would get an answer which is symbolically different, but actually the same; P being harmonic forces relationships between the coefficients a_{jk}.]

The problem finally asks us to show that for $f(x+iy) = P(x,y)+iQ(x,y)$, f is a polynomial in z. There was some confusion about what this meant; this means that for some constants α_j

$$f(z) = \sum \alpha_j z^j,$$

where the sum is finite. In fact, perhaps again because of confusion, no one showed this for general P and Q, so we graded to see if you had done it for the cubic P and Q above. In this case, algebra verifies that $P(x,y) + iQ(x,y) = (a+id)(x+iy)^3 + A$, and full credit (5 points) was given for this observation.

It is interesting to see this more generally. We have for general harmonic P that f as defined above is an analytic function. But then, by the same process of reasoning that gets us the Cauchy-Riemann equations,

$$f'(z) = \frac{\partial f}{\partial x}(x+iy),$$

and likewise $f^{(n)}(z) = \frac{\partial^n f}{\partial x^n}(x+iy)$.

But because $f(x+iy)$ is a polynomial in x and y (we don't know this yet for z, though!), for some large enough N, $f^{(N)}(z) = 0$. But then, in effect integrating, $f^{(N-1)}(z) = \beta_N$ for some constant β_N. Repeating the process, $f^{(N-2)}(z) = \beta_N z + \beta_{N-1}$ for some constant β_{N-1}. (Here we are using the Theorem at the bottom of page 49 of Gamelin which ensures us that having found one antiderivative, we've found all antiderivates up to a constant.) Repeating this process, we have

$$f(z) = \sum_{k=0}^{N} \frac{\beta_k}{k!} z^k,$$

where β_k are constants. This shows f is a polynomial in z.

8. (30pts) Show that on the punctured unit disk $D = \{z : 0 < |z| < 1\}$ there is a non-exact closed differential $Pdx + Qdy$ such that if $Sdx + Tdy$ is a closed differential on D, then $Sdx + Tdy = \alpha(Pdx + Qdy) + dh$ for some real number α and some function h. Show that on the domain

$$E = \{z : 0 < |z| < 2 \text{ and } z \neq 0, i\}$$

there are two closed differentials $P_0dx + Q_0dy$ and $P_1dx + Q_1dy$ such that every non-exact closed differential on D is equal to $\alpha(P_0dx+Q_0dy)+\beta(P_1dx+Q_1dy) + dh$ for real numbers α and β and some function h. You make take the first differential to be $\frac{-ydx+xdy}{x^2+y^2}$. (All functions h, P_i, and Q_i are assumed to have continuous second-order partial derivatives.)

Solution This problem is rather difficult, and I gave out very, very little credit on it. Let's start by proving the bit about the punctured unit disk $D = \{z : 0 < |z| < 1\}$.

Recall that a differential $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is exact on D if and only if all path integrals depend only on the endpoints, or, equivalently, if

$$\int_\gamma P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$$

for all closed paths γ on D.

We saw in class that

$$\int_{|z|=1} \frac{-y\,\mathrm{dx} + x\,\mathrm{dy}}{x^2+y^2} = 2\pi.$$

It follows that $\frac{-y\,\mathrm{dx}+x\,\mathrm{dy}}{x^2+y^2}$ is not exact, and it is trivial to see that it is closed. Now, suppose that $S\,\mathrm{dx} + T\,\mathrm{dy}$ is any closed differential on D, and let

$$\int_{|z|=1} P\,\mathrm{dx} + Q\,\mathrm{dy} = C.$$

Consider the differential form

$$\omega = P\,\mathrm{dx} + Q\,\mathrm{dy} - \frac{C}{2\pi}\frac{-y\,\mathrm{dx} + x\,\mathrm{dy}}{x^2+y^2}.$$

Then, by construction, we know that ω is closed and

$$\int_{|z|=1} \omega = 0.$$

It follows that $\int_\gamma \omega = 0$ for *any* closed path γ in D, and this shows that $\omega = dh$ for some function h, as desired.

Remark: to make this argument truly precise, one needs to argue that any closed path in D can be deformed into travelling around the circle $|z| = 1$

some integral number n of times. Positive numbers correspond to taking the path counter-clockwise, and negative numbers correspond to going clockwise. Similarly, in the next domain, every path can be deformed to some multiple m of loops around 0 and n loops around i taken in various orders. For instance, go around 0 clockwise 10 times, then go around i counterclockwise 4 times, then go around 0 counterclockwise 5 times. So, in the next part, to test for exactness of a differential form ω, it suffices to check that

$$\int_{|z|=.5} \omega = 0 = \int_{|z-i|=.5} \omega.$$

What are our 2 differential forms? Let the two differential forms be

$$P_0 \, \mathrm{dx} + Q_0 \, \mathrm{dy} = \frac{-y \, \mathrm{dx} + x \, \mathrm{dy}}{x^2 + y^2}$$
$$P_1 \, \mathrm{dx} + Q_1 \, \mathrm{dy} = \frac{-(y-1) \, \mathrm{dx} + x \, \mathrm{dy}}{x^2 + (y-1)^2}.$$

Then, it is easy to check that both are closed, and neither is exact. We have

$$\int_{|z|=.5} \frac{-y \, \mathrm{dx} + x \, \mathrm{dy}}{x^2 + y^2} = 2\pi = \int_{|z-i|=.5} \frac{-(y-1) \, \mathrm{dx} + x \, \mathrm{dy}}{x^2 + (y-1)^2},$$
$$\int_{|z-i|=.5} \frac{-y \, \mathrm{dx} + x \, \mathrm{dy}}{x^2 + y^2} = 0 = \int_{|z|=.5} \frac{-(y-1) \, \mathrm{dx} + x \, \mathrm{dy}}{x^2 + (y-1)^2}$$

Let $S \, \mathrm{dx} + T \, \mathrm{dy}$ be an arbitrary closed differential form, and define numbers C and D by

$$\int_{|z|=.5} S \, \mathrm{dx} + T \, \mathrm{dy} = C,$$
$$\int_{|z-i|=.5} S \, \mathrm{dx} + T \, \mathrm{dy} = D.$$

Let

$$\omega = S \, \mathrm{dx} + T \, \mathrm{dy} - \frac{C}{2\pi} \frac{-y \, \mathrm{dx} + x \, \mathrm{dy}}{x^2 + y^2} - \frac{D}{2\pi} \frac{-(y-1) \, \mathrm{dx} + x \, \mathrm{dy}}{x^2 + (y-1)^2}.$$

Then, by construction,

$$\int_{|z|=.5} \omega = 0 = \int_{|z-i|=.5} \omega.$$

Thus, as remarked above, $\int_\gamma \omega = 0$ for any closed path γ in D. Thus, $\omega = dh$ for some function h.