

Problem set 1 for 131 A/3 - Fall 2012

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1. Show that $\sqrt[3]{5}$ is irrational.
2. Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all natural numbers.
3. Show that $\sqrt{2 + 4 + \cdots + 2n}$ is irrational for all $n \geq 2$.
4. If $r \neq 0$ is rational and x is irrational, then $r + x$ and rx are irrational [Rud87].
5. [Ros80, Exercise 3.8]. Let $a, b \in \mathbb{R}$. Show that if $a \leq c$ for every $c > b$, then $a \leq b$.
6. [Ros80, Exercise 4.5]. Let S be a nonempty subset of \mathbb{R} that is bounded above. Prove that if $\sup S$ belongs to S , then $\sup S = \max S$.
7. [Ros80, Exercise 4.6]. Let S be a nonempty bounded subset of \mathbb{R} . Prove that $\inf S \leq \sup S$. When is $\inf S = \sup S$?
8. [Ros80, Exercise 4.8]. Let S and T be nonempty subsets of \mathbb{R} with the following property: $s \leq t$ for all $s \in S$ and $t \in T$. Show that S is bounded above and T is bounded below. Show that $\sup S \leq \inf T$. Give an example where $S \cap T$ is nonempty. Give an example where $\sup S = \inf T$ and $S \cap T = \emptyset$.
9. Let S be a nonempty subset of \mathbb{R} that has a supremum $M \in \mathbb{R}$. Let $-S = \{-x : x \in S\}$. Show that $\inf -S = -M$.
10. [Ros80, Exercise 4.10]. Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $1/n < a < n$.
11. [Ros80, Exercise 4.11]. Show that if $a < b$ then there are infinitely many rational numbers between a and b .
12. [Ros80, Exercise 4.16]. Show that $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{Q}$.
13. For every real $x > 0$ and every integer $n > 0$ there is one and only one real $y > 0$ such that $y^n = x$ [Rud87].
14. Show that if $b > 1$ and r is rational, there is a well-defined real number $b^r > 0$. Well-defined means that the output b^r should be independent of the representation $r = m/n$. Use this define b^x for any real numbers $b > 1$ and x . Show that $b^{x+y} = b^x b^y$ for any x and y [Rud87].

15. If $\mathbf{x} = (x_1, \dots, x_n)$ is a vector in \mathbb{R}^n , let $|\mathbf{x}| = \sqrt{x_1^2 + \dots + x_n^2}$. This is the length of the vector \mathbf{x} . Prove the **Cauchy-Schwarz inequality**: for any two n -tuples \mathbf{x} and \mathbf{y} ,

$$\left| \sum_{i=1}^n x_i y_i \right| \leq |\mathbf{x}| \cdot |\mathbf{y}|.$$

When does equality hold?

16. Prove the **triangle inequality** for vectors in \mathbb{R}^n : $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$.

17. Let $p \geq 1$ and $q \geq 1$ satisfy $1/p + 1/q = 1$. Prove **Hölder's inequality**:

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \left(\sum_{i=1}^n |y_i|^q \right)^{1/q}$$

for vectors \mathbf{x} and \mathbf{y} in \mathbb{R}^n . Note that the Cauchy-Schwarz inequality is a special case.

18. Fix $p \geq 1$. Use Hölder's inequality to prove the **Minkowski inequality**:

$$\left(\sum_{i=1}^n |x_i + y_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} + \left(\sum_{i=1}^n |y_i|^p \right)^{1/p}$$

for vectors \mathbf{x} and \mathbf{y} in \mathbb{R}^n . Note that the triangle inequality is a special case.

19. Let (s_n) be a sequence of real numbers such that $s_{n+1} \geq s_n$ for all natural numbers n . Prove that if n and m are natural numbers and $n > m$, then $s_n \geq s_m$.

20. [Ros80, Exercise 7.4]. Give an example of a sequence of irrational numbers with limit a rational number, and give another example of a sequence of rational numbers with limit an irrational number.

21. When does $\lim_{n \rightarrow \infty} \frac{a^n}{n^p}$ converge?

22. [Ros80, Exercise 8.1]. Prove that $\lim_{n \rightarrow \infty} \frac{n+6}{n^2-6} = 0$.

23. [Ros80, Exercise 8.2]. Find $\lim_{n \rightarrow \infty} \frac{\sin n}{n}$.

24. [Ros80, Exercise 8.7]. Show that $s_n = \sin n\pi/3$ diverges.

25. [Ros80, Exercise 8.9]. Show that if (s_n) converges and if all but finitely many s_n belong to the interval $[a, b]$, then $\lim s_n$ belongs to $[a, b]$.

26. Let S be a nonempty set of real numbers. A point $x \in \mathbb{R}$ is called an accumulation point of S if it is the limit of a sequence (s_n) where $s_n \in S$ for all natural numbers n . Show that if $\sup S$ exists, then it is an accumulation point of S .

27. Determine whether or not $\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{n}$ converges, and compute the limit if it converges.

28. Show that the closed interval $[a, b]$ is a closed set.

29. Fix $\alpha > 0$, and let $x_1 > \sqrt{\alpha}$. Define x_n inductively by

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right).$$

Prove that (x_n) decreases monotonically and that $\lim x_n = \sqrt{\alpha}$. Let $e_n = x_n - \sqrt{\alpha}$, the n th error term. Show that $e_{n+1} = \frac{e_n^2}{2x_n} < \frac{e_n^2}{2\sqrt{\alpha}}$ so that, setting $\beta = 2\sqrt{\alpha}$,

$$e_{n+1} < \beta \left(\frac{e_1}{\beta} \right)^{2^n}.$$

If $\alpha = 3$ and $x_1 = 2$, show that $e_6 < 4 \cdot 10^{-32}$. Convergence is very rapid [Rud87].

30. Find the upper and lower limits of the sequence (s_n) defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m},$$

[Rud87].

31. Prove that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$.

32. Prove that e is irrational.

33. Can the open interval $(0, 1)$ be the set of subsequential limits for a sequence (s_n) ?

34. Prove that a bounded monotonic sequence converges.

35. Prove that a bounded sequence has a monotonic subsequence.

36. [Ros80, Exercise 12.12]. Let (s_n) be a sequence of nonnegative numbers. For each $n \geq 1$, set $\sigma_n = \frac{s_1 + \dots + s_n}{n}$. Show that

$$\liminf s_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup s_n.$$

Show that if $\lim s_n$ exists, then so does $\lim \sigma_n$ and the two limits are equal.

37. Prove that if $a_n \geq 0$ and if $\sum a_n$ converges, then so does $\sum \frac{\sqrt{a_n}}{n}$ [Rud87].

38. If $\sum a_n$ converges and if (b_n) is monotonic and bounded, then $\sum a_n b_n$ converges [Rud87].

39. If $\sum a_n$ and $\sum b_n$ are two series, then their Cauchy product is defined to be $\sum c_n$, where $c_n = \sum_{k=0}^n a_k b_{n-k}$. Prove that the Cauchy product of two absolutely convergent series is absolutely convergent [Rud87].

40. Prove that, in the situation of the last problem, $\sum c_n$ converges to the “correct” value.

41. Prove that every rearrangement of an absolutely convergent series converges to the same sum. Give an example to show that this is not the case if the series is not absolutely convergent.

42. [Ros80, Exercise 14.2]. Do all of them.

43. [Ros80, Exercise 14.7]. Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.
44. Prove that $\sum \frac{1}{n}$ diverges.
45. [Ros80, Exercise 15.1]. Determine which of $\sum \frac{(-1)^n}{n}$ and $\sum \frac{(-1)^n n!}{2^n}$ converge.
46. [Ros80, Exercise 15.2]. Show that $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.
47. If f is a real continuous function defined on a closed interval $[a, b]$, show that there exist continuous real functions g on \mathbb{R} such that $g(x) = f(x)$ for all $x \in [a, b]$. Thus, f extends to all of \mathbb{R} . Show that the result is false on the open interval (a, b) , if $a < b$ [Rud87].
48. Construct a function $\mathbb{R} \rightarrow \mathbb{R}$ that is continuous at every irrational number and has a simple discontinuity at every rational number.
49. Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{otherwise} \end{cases}$$

is continuous at 0 and discontinuous elsewhere.

50. [Ros80, Exercise 17.3].
51. [Ros80, Exercise 17.9].
52. [Ros80, Exercise 17.10].
53. Prove the Archimedean principle.
54. Show that $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.
55. Let (s_n) be a bounded sequence in \mathbb{R} . Show that the set of subsequential limits of (s_n) is closed.
56. Let (s_n) be a sequence in which every rational number appears. Show rigorously that the set of subsequential limits of (s_n) is all of \mathbb{R} .

References

- [KF75] A. N. Kolmogorov and S. V. Fomīn, *Introductory real analysis*, Dover Publications Inc., New York, 1975. Translated from the second Russian edition and edited by Richard A. Silverman; Corrected reprinting.
- [Nat55] I. P. Natanson, *Theory of functions of a real variable*, Frederick Ungar Publishing Co., New York, 1955. Translated by Leo F. Boron with the collaboration of Edwin Hewitt.
- [Ros80] K. A. Ross, *Elementary analysis: the theory of calculus*, Springer-Verlag, New York, 1980. Undergraduate Texts in Mathematics.
- [Rud87] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987.