309 A, 309 B

309A-C - Midterm

21 October 2013

Name:
Student ID #:

This is a closed-book, closed-notes exam. Calculators are not allowed.

Show all work.

If you need more room, write on the back, and make a note on the front.

There are 5 problems of 20 points each for a total of 100 points.

POINTS:

1.

2.

3.

4.

5.

TOTAL:

1.a. (10 points). Can the following set of vector-valued functions form a fundamental set of solutions of a homogeneous system of linear differential equations with constant coefficients $\mathbf{x}' = A\mathbf{x}$? Circle: (yes) or (no).

$$\mathbf{x}^{(1)} = \begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}, \mathbf{x}^{(2)} = \begin{pmatrix} e^{5t} \\ e^{2t} \end{pmatrix}$$

Since I asked for them to be solutions to an equation with constant coefficients, they would be solutions on the entire real line. But, the Wronskian

$$W(t) = \begin{vmatrix} e^t & e^{5t} \\ e^{-t} & e^{2t} \end{vmatrix} = e^{3t} - e^{4t}$$

vanishes at $t=0$.

No partial credit.

1.b. (10 points). Which of the following pictures depicts the phase portrait of a homogeneous system of linear differential equations with constant coefficients having 0 as an eigenvalue? Circle: (a) or (b).

No partial credit.

2. (20 points). Find $exp(At)$ when $A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

Diagonalize, exponentiate, return.

Characteristic polynomial: $f(\lambda) = \lambda^2 - 1$.

Eigenvalues: ± 1.

$\underline{r_1 = 1}$

$A - I_2 = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$

eigenvector: $\vec{\omega}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{r_2 = -1}$

$A + I_2 = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$

eigenvector: $\vec{\omega}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Set $T = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \vec{\omega}^{(1)} & \vec{\omega}^{(2)} \end{pmatrix}$.

$T^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$.

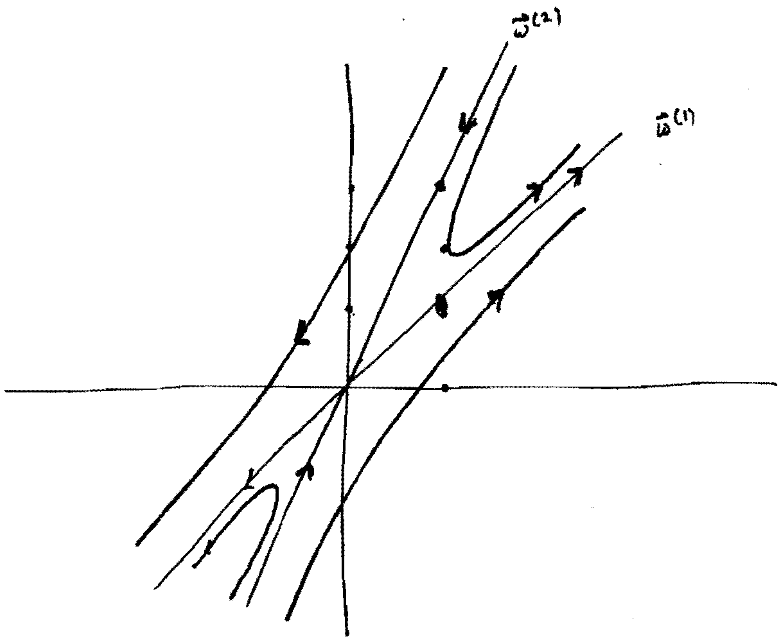
$T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$T^{-1} \exp(At)\, T = \exp\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} t \right) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.

$$\exp(At) = T \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} T^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3e^t & -e^t \\ -e^{-t} & e^{-t} \end{pmatrix} = \boxed{\frac{1}{2} \begin{pmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{pmatrix}}.$$

3. (20 points). Sketch the trajectories of several solutions of $\mathbf{x}' = A\mathbf{x}$, where A is the matrix in the previous problem. You should label your axes x_1 and x_2, and your graph should include sketches of at least 6 distinct trajectories, as well as arrows indicating the direction of the trajectory.



309 A, 309 B

4. (20 points). Find the real-valued solution $\mathbf{x}(t)$ of $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$ satisfying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Characteristic polynomial: $f(\lambda) = \lambda^2 + 2\lambda + 5$.

Eigenvalues: $\lambda = \dfrac{-2 \pm \sqrt{4-20}}{2}$

$= -1 \pm 2i$.

$r = -1-2i$

$A - (-1-2i) = \begin{pmatrix} 2i & -4 \\ 1 & 2i \end{pmatrix}$

Eigenvector for $-1-2i$:

$\vec{w} = \begin{pmatrix} 2i \\ -1 \end{pmatrix}$.

$\vec{w} e^{(-1-2i)t} = \begin{pmatrix} 2i \\ -1 \end{pmatrix} e^{-t} e^{-2it}$

$= \begin{pmatrix} 2i \\ -1 \end{pmatrix} e^{-t} \left(\cos(-2t) + i \sin(-2t) \right)$

$= e^{-t} \left(\begin{pmatrix} -2\sin(-2t) \\ -\cos(-2t) \end{pmatrix} + i \begin{pmatrix} 2\cos(-2t) \\ -\sin(-2t) \end{pmatrix} \right)$.

General solution: $\vec{x}(t) = e^{-t} \left(c_1 \begin{pmatrix} -2\sin(-2t) \\ -\cos(-2t) \end{pmatrix} + c_2 \begin{pmatrix} 2\cos(-2t) \\ -\sin(-2t) \end{pmatrix} \right)$.

$\vec{x}(0) = \begin{pmatrix} 0 \\ -c_1 \end{pmatrix} + \begin{pmatrix} 2c_2 \\ 0 \end{pmatrix}$

Hence, $\boxed{c_1 = 0, \ c_2 = \tfrac{1}{2}.}$

There are many correct ways to do this problem. One of them is to find that $\begin{pmatrix} -2i \\ -1 \end{pmatrix}$ is an eigenvector for $-1+2i$ and to use $c_1 \begin{pmatrix} 2i \\ -1 \end{pmatrix} e^{(-1-2i)t} + c_2 \begin{pmatrix} -2i \\ -1 \end{pmatrix} e^{(-1+2i)t}$.

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5. (20 points). Find a solution of $\mathbf{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 3 \\ e^{-2t} \end{pmatrix}$. Note that the matrix is the same as in problem (2).

From Problem (2):

$r_1 = 1$: $\vec{w}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$r_2 = -1$: $\vec{w}^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

We get the fundamental matrix

$$\Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix},$$

with inverse

$$\Psi^{-1}(t) = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix}.$$

We use variation of parameters:

$$\vec{x}(t) = \Psi(t) \int_0^t \Psi^{-1}(s) \vec{g}(s)\,ds + \Psi(t)\vec{c}.$$

$$\Psi^{-1}(s)\vec{g}(s) = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix} \begin{pmatrix} 3 \\ e^{-2t} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 9e^{-t} - e^{-3t} \\ -3e^t + e^{-t} \end{pmatrix}.$$

$$\int_0^t \Psi^{-1}(s)\vec{g}(s)\,ds = \int_0^t \frac{1}{2} \begin{pmatrix} 9e^{-s} - e^{-3s} \\ -3e^{s} + e^{-s} \end{pmatrix} ds$$

$$= \frac{1}{2} \begin{pmatrix} -9e^{-t} + \frac{1}{3}e^{-3t} + 9 - \frac{1}{3} \\ -3e^t - e^{-t} + 3 - 1 \end{pmatrix}.$$

We can ignore the constants as they get absorbed by $\Psi(t)\vec{c}$.

Thus, one solution to the nonhomogeneous system is

$$\vec{x}(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} -9e^{-t} + \frac{1}{3}e^{-3t} \\ -3e^t - e^{-t} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -9 + \frac{1}{3}e^{-2t} - 3 - e^{-2t} \\ -9 + \frac{1}{3}e^{-2t} - 9 - 3e^{-2t} \end{pmatrix}$$

$$\boxed{= \frac{1}{2} \left(\begin{pmatrix} -\frac{2}{3} \\ -\frac{8}{3} \end{pmatrix} e^{-2t} - \begin{pmatrix} 12 \\ 18 \end{pmatrix} \right).}$$

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1.b. (10 points). Which of the following pictures depicts the phase portrait of a homogeneous system of linear differential equations with constant coefficients having 0 as an eigenvalue? Circle: (a) or (b).

2. (20 points). Find $exp(At)$ when $A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$.

Diagonalize, exponentiate, return.

Characteristic polynomial: $f(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$.

$\underline{r_1 = -2}$

$A - (-2I_2) = \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}$

eigenvector: $\vec{\omega}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\underline{r_2 = -1}$

$A - (-I_2) = \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}$

eigenvector: $\vec{\omega}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Set $T = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} \vec{\omega}^{(1)} & \vec{\omega}^{(2)} \end{pmatrix}$.

$$T^{-1} = -\begin{pmatrix} 1 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$$

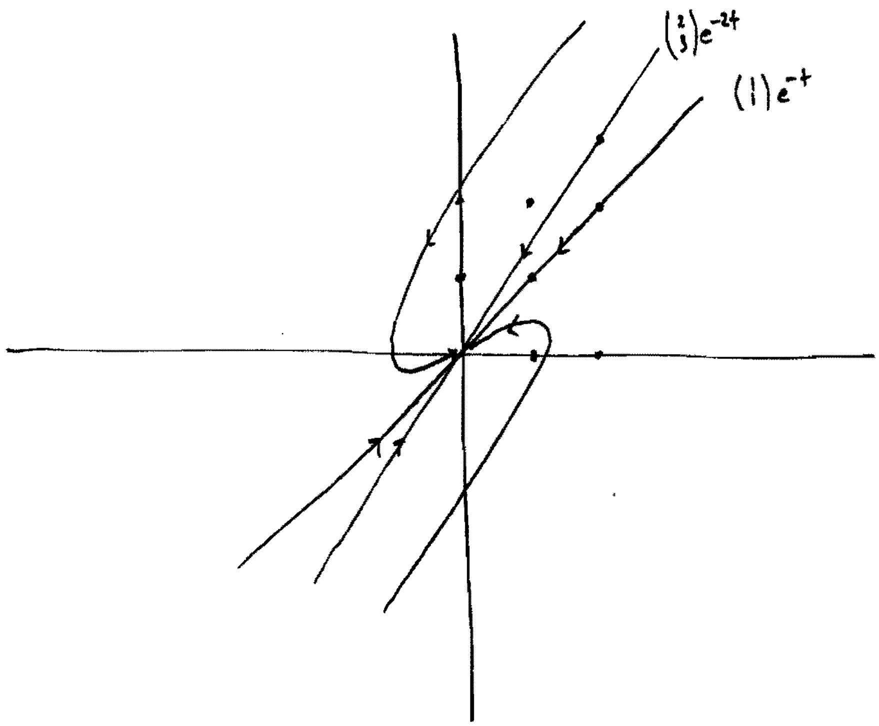
$$T^{-1}AT = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T^{-1}\exp(At)T = \exp\left(\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} t \right) = \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

$$\exp(At) = T\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix}T^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -e^{-2t} & e^{-2t} \\ 3e^{-t} & -2e^{-t} \end{pmatrix} = \boxed{\begin{pmatrix} -2e^{-2t} + 3e^{-t} & 2e^{-2t} - 2e^{-t} \\ -3e^{-2t} + 3e^{-t} & 3e^{-2t} - 2e^{-t} \end{pmatrix}}.$$

3. (20 points). Sketch the trajectories of several solutions of $\mathbf{x}' = A\mathbf{x}$, where A is the matrix in the previous problem. You should label your axes x_1 and x_2, and your graph should include sketches of at least 6 distinct trajectories, as well as arrows indicating the direction of the trajectory.



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4. (20 points). Find the real-valued solution $\mathbf{x}(t)$ of $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$ satisfying the initial condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Characteristic polynomial: $F(\lambda) = \lambda^2 + 2\lambda + 2$.

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i.$$

$r = -1-i$.

$$A - (-1-i)I_2 = \begin{pmatrix} 2+i & -1 \\ 5 & -2+i \end{pmatrix}.$$

$$\vec{w} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix}.$$

$$\vec{w}\, e^{(-1-i)t} = \begin{pmatrix} 1 \\ 2+i \end{pmatrix} e^{-t} e^{-it}$$

$$= e^{-t} \left(\begin{pmatrix} 1 \\ 2+i \end{pmatrix} \left(\cos(-t) + i \sin(-t) \right) \right)$$

$$= e^{-t} \left(\begin{pmatrix} \cos(-t) \\ 2\cos(-t) - \sin(-t) \end{pmatrix} + i \begin{pmatrix} \sin(-t) \\ \cos(-t) + 2\sin(-t) \end{pmatrix} \right)$$

General solution: $\vec{x}(t) = e^{-t} \left(c_1 \begin{pmatrix} \cos(-t) \\ 2\cos(-t) - \sin(-t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(-t) \\ \cos(-t) + 2\sin(-t) \end{pmatrix} \right).$

$$\vec{x}(0) = \begin{pmatrix} c_1 \\ 2c_1 \end{pmatrix} + \begin{pmatrix} 0 \\ c_2 \end{pmatrix}.$$

So, $\boxed{c_1 = 1, \quad c_2 = \text{[redacted]}\ -2.}$

As mentioned in the other version, there were many different approaches.

5. (20 points). Find a solution of $\mathbf{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ e^{-2t} \end{pmatrix}$. Note that the matrix is the same as in problem (2).

From Problem (2):

$r_1 = -2: \quad \vec{v}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$

$r_2 = -1: \quad \vec{v}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Fundamental matrix:

$$\Psi(t) = \begin{pmatrix} 2e^{-2t} & e^{-t} \\ 3e^{-2t} & e^{-t} \end{pmatrix}.$$

Inverse

$$\Psi^{-1}(t) = \frac{1}{2e^{-3t} - 3e^{-3t}} \begin{pmatrix} e^{-t} & -e^{-t} \\ -3e^{-2t} & 2e^{-2t} \end{pmatrix}$$

$$= -e^{+3t} \begin{pmatrix} e^{-t} & -e^{-t} \\ -3e^{-2t} & 2e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} -e^{2t} & e^{2t} \\ 3e^{t} & -2e^{t} \end{pmatrix}.$$

Use $\vec{x}(t) = \Psi(t)\int_0^t \Psi^{-1}(s)\,\vec{g}(s)\,ds + \Psi(t)\vec{c}$.

Constants from integration can be absorbed in $\Psi(t)\vec{c}$ term. So a solution is

$$\vec{x}(t) = \Psi(t)\int \Psi^{-1}(s)\,\vec{g}(s)\,ds.$$

$$\Psi^{-1}(s)\,\vec{g}(s) = \begin{pmatrix} -e^{2s} & e^{2s} \\ 3e^{s} & -2e^{s} \end{pmatrix}\begin{pmatrix} 3 \\ e^{-2s} \end{pmatrix} = \begin{pmatrix} -3e^{2s} + 1 \\ 3e^{s} - 2e^{-s} \end{pmatrix}.$$

$$\int \begin{pmatrix} -3e^{2s} + 1 \\ 3e^{s} - 2e^{-s} \end{pmatrix} ds = \begin{pmatrix} -\frac{3}{2}e^{2t} + t \\ 3e^{t} + 2e^{-t} \end{pmatrix}.$$

Hence,

$$\vec{x}(t) = \Psi(t)\int \Psi^{-1}(s)\,\vec{g}(s)\,ds$$

$$= \begin{pmatrix} 2e^{-2t} & e^{-t} \\ 3e^{-2t} & e^{-t} \end{pmatrix}\begin{pmatrix} -\frac{3}{2}e^{2t} + t \\ 3e^{t} + 2e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} -3 + 2te^{-2t} + 3 + 2e^{-2t} \\ -\frac{9}{2} + 3te^{-2t} + 3 + 2e^{-2t} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2 \\ 3 \end{pmatrix} te^{-2t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix} e^{-2t} + \begin{pmatrix} 0 \\ -\frac{3}{2} \end{pmatrix}.}$$